Rutgers University
Math 152

Section 10.10: Applications of Taylor Series - Worksheet Solutions

1. Use Maclaurin series to compute the following limits.

(a) $\lim_{x\to 0} \dfrac{e^{-2x^2} - \cos(2x)}{x^2\ln(1+5x) - 5x^3}$.

Solution. We find the first few terms of the Maclaurin series of the numerator and denominator.

$$\begin{aligned} e^{-2x^2} - \cos(2x) &= \left(1 - 2x^2 + \frac{(-2x^2)^2}{2!} + \frac{(-2x^2)^3}{3!} \cdots\right) - \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots\right) \\ &= \frac{4}{3}x^4 - \frac{56}{45}x^6 + \cdots \end{aligned}$$

$$\begin{aligned} x^2\ln(1+5x) - 5x^3 &= x^2\left(5x - \frac{(5x)^2}{2} + \frac{(5x)^3}{3} + \cdots\right) - 5x^3 \\ &= -\frac{25x^4}{2} + \frac{125x^5}{3} + \cdots \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{x\to 0} \frac{e^{-2x^2} - \cos(2x)}{x^2\ln(1+5x) - 5x^3} &= \lim_{x\to 0} \frac{\frac{4}{3}x^4 - \frac{56}{45}x^6 + \cdots}{-\frac{25x^4}{2} + \frac{125x^5}{3} + \cdots} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}} \\ &= \lim_{x\to 0} \frac{\frac{4}{3} - \frac{56}{45}x^2 + \cdots}{-\frac{25}{2} + \frac{125x}{3} + \cdots} \\ &= \frac{\frac{4}{3}}{-\frac{25}{2}} \\ &= \boxed{-\frac{8}{75}}. \end{aligned}$$

(b) $\lim_{x\to\infty} x^3\left(\tan^{-1}\left(\frac{4}{x}\right) - 2\sin\left(\frac{2}{x}\right)\right)$.

Solution. We have

$$\begin{aligned} \tan^{-1}\left(\frac{4}{x}\right) - 2\sin\left(\frac{2}{x}\right) &= \left(\frac{4}{x} - \frac{1}{3}\left(\frac{4}{x}\right)^3 + \frac{1}{5}\left(\frac{4}{x}\right)^5 + \cdots\right) - 2\left(\frac{2}{x} - \frac{1}{3!}\left(\frac{2}{x}\right)^3 + \frac{1}{5!}\left(\frac{2}{x}\right)^5 + \cdots\right) \\ &= -\frac{56}{3x^3} + \frac{3064}{15x^5} + \cdots. \end{aligned}$$

So

$$\lim_{x\to\infty} x^3\left(\tan^{-1}\left(\frac{4}{x}\right) - 2\sin\left(\frac{2}{x}\right)\right) = \lim_{x\to\infty} x^3\left(-\frac{56}{3x^3} + \frac{3064}{15x^5} + \cdots\right)$$

$$= \lim_{x\to\infty} -\frac{56}{3} + \frac{3064}{15x^2} + \cdots$$
$$= \boxed{-\frac{56}{3}}.$$

(c) $\displaystyle\lim_{x\to 0} \frac{\sin(x^6)}{\cos(x^3) - 1}$.

Solution. We have

$$\cos(x^3) - 1 = \left(1 - \frac{(x^3)^2}{2!} + \frac{(x^3)^4}{4!} + \cdots\right) - 1$$
$$= -\frac{x^6}{2} + \frac{x^{12}}{24} + \cdots$$

$$\sin(x^6) = x^6 - \frac{(x^6)^3}{3!} + \cdots$$
$$= x^6 - \frac{x^{18}}{6} + \cdots$$

Thus

$$\lim_{x\to 0} \frac{\sin(x^6)}{\cos(x^3) - 1} = \lim_{x\to 0} \frac{x^6 - \frac{x^{18}}{6} + \cdots}{-\frac{x^6}{2} + \frac{x^{12}}{24} + \cdots} \cdot \frac{\frac{1}{x^6}}{\frac{1}{x^6}}$$
$$= \lim_{x\to 0} \frac{1 - \frac{x^{12}}{6} + \cdots}{-\frac{1}{2} + \frac{x^6}{24} + \cdots}$$
$$= \boxed{-2}.$$

(d) $\displaystyle\lim_{x\to\infty} x^2 \left(5\ln\left(1 + \frac{3}{x}\right) - 3\ln\left(1 + \frac{5}{x}\right)\right)$.

Solution. We have

$$5\ln\left(1 + \frac{3}{x}\right) - 3\ln\left(1 + \frac{5}{x}\right) = 5\left(\frac{3}{x} - \frac{1}{2}\left(\frac{3}{x}\right)^2 + \frac{1}{3}\left(\frac{3}{x}\right)^3 + \cdots\right) - 3\left(\frac{5}{x} - \frac{1}{2}\left(\frac{5}{x}\right)^2 + \frac{1}{3}\left(\frac{5}{x}\right)^3 + \cdots\right)$$
$$= \frac{15}{x^2} - \frac{80}{x^3} + \cdots$$

So

$$\lim_{x\to\infty} x^2 \left(5\ln\left(1 + \frac{3}{x}\right) - 3\ln\left(1 + \frac{5}{x}\right)\right) = \lim_{x\to\infty} x^2 \left(\frac{15}{x^2} - \frac{80}{x^3} + \cdots\right)$$
$$= \lim_{x\to\infty} 15 - \frac{80}{x} + \cdots$$
$$= \boxed{15}.$$

2. Use Maclaurin series to write each integral below as the sum of an infinite series of numbers (your series should not contain x).

(a) $\displaystyle\int_0^{1/2} \cos(5x^2)dx.$

Solution. The Maclaurin series for the integrand is

$$\cos(5x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n(5x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n} x^{4n}}{(2n)!}.$$

Integrating term-by-term gives

$$\begin{aligned}
\int_0^{1/2} \cos(x^2)dx &= \sum_{n=0}^{\infty} \int_0^{1/2} \frac{(-1)^n 5^{2n} x^{4n}}{(2n)!} dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n}}{(2n)!} \int_0^{1/2} x^{4n} dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n}}{(2n)!} \left[\frac{x^{4n+1}}{4n+1} \right]_0^{1/2} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n}}{(2n)!} \left(\frac{1}{(4n+1)2^{4n+1}} - \frac{0^{4n+1}}{4n+1} \right) \\
&= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n}}{(2n)!(4n+1)2^{4n+1}}}.
\end{aligned}$$

(b) $\displaystyle\int_0^{1} x^3 e^{-4x^3} dx.$

Solution. The Maclaurin series for the integrand is

$$x^3 e^{-4x^3} = x^3 \sum_{n=0}^{\infty} \frac{(-4x^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-4)^n x^{3n+3}}{n!}.$$

Integrating term-by-term gives

$$\begin{aligned}
\int_0^{1} x^3 e^{-4x^3} dx &= \sum_{n=0}^{\infty} \int_0^{1} \frac{(-4)^n x^{3n+3}}{n!} dx \\
&= \sum_{n=0}^{\infty} \frac{(-4)^n}{n!} \int_0^{1} x^{3n+3} dx \\
&= \sum_{n=0}^{\infty} \frac{(-4)^n}{n!} \left[\frac{x^{3n+4}}{3n+4} \right]_0^{1} \\
&= \sum_{n=0}^{\infty} \frac{(-4)^n}{n!} \left(\frac{1^{3n+4}}{3n+4} - \frac{0^{3n+4}}{3n+4} \right) \\
&= \boxed{\sum_{n=0}^{\infty} \frac{(-4)^n}{n!(3n+4)}}.
\end{aligned}$$

(c) $\displaystyle\int_0^{1/3} x^7 \sin(2x^5)dx.$

Solution. The Maclaurin series for the integrand is

$$x^7 \sin(2x^5) = x^7 \sum_{n=0}^{\infty} \frac{(-1)^n (2x^5)^{2n+1}}{(2n+1)!} = x^7 \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{10n+5}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{10n+12}}{(2n+1)!}.$$

Integrating term-by-term gives

$$\begin{aligned}
\int_0^{1/3} x^7 \sin(2x^5)dx &= \sum_{n=0}^{\infty} \int_0^{1/3} \frac{(-1)^n 2^{2n+1} x^{10n+12}}{(2n+1)!} dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} \int_0^{1/3} x^{10n+12} dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} \left[\frac{x^{10n+13}}{10n+13} \right]_0^{1/3} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} \left(\frac{1}{(10n+13)3^{10n+13}} - \frac{0^{10n+13}}{10n+13} \right) \\
&= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!(10n+13)3^{10n+13}}}.
\end{aligned}$$

3. (a) Use Maclaurin series to write the integral $I = \displaystyle\int_0^1 e^{-x^2} dx$ as the sum of an infinite series of numbers.

Solution. The Maclaurin series for the integrand is

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}.$$

Integrating term-by-term gives

$$\begin{aligned}
\int_0^1 e^{-x^2} dx &= \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n x^{2n}}{n!} dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^{2n} dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[\frac{x^{2n+1}}{2n+1} \right]_0^1 \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{1^{2n+1}}{2n+1} - \frac{0^{2n+1}}{2n+1} \right) \\
&= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)}}.
\end{aligned}$$

(b) Use the Alternating Series Estimation Theorem to find how many terms of the series found in (a) need to be summed in order to obtain an approximation of I with an error of less than 10^{-5}.

Solution. The alternating series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)}$$

meets the assumptions of the Alternating Series Estimation Theorem since $a_n = \frac{1}{n!(2n+1)}$ is positive, decreasing and converges to 0. Therefore, the error made approximating I with the partial $S_N = \sum_{n=0}^{N} \frac{(-1)^n}{n!(2n+1)}$ is

$$|I - S_N| \leqslant a_{N+1} = \frac{1}{(N+1)!(2N+3)}.$$

Let us find the smallest value of N giving us an error of less than 10^{-5}. We want $\frac{1}{(N+1)!(2N+3)} < 10^{-5}$, or $(N+1)!(2N+3) > 10^5$. Solving this with a calculator gives the smallest value as $N = 6$. The corresponding partial sum

$$S_6 = \sum_{n=0}^{6} \frac{(-1)^n}{n!(2n+1)}$$

has $\boxed{\text{7 terms}}$.