Rutgers University
Math 152

Section 10.1: Sequences - Worksheet Solutions

1. Determine if the sequences below converge or diverge. In case of convergence, find the limit.

(a) $a_n = \dfrac{\sqrt{1+16n^4}}{n^2+1}$

Solution. We can compute the limit by dividing numerator and denominator by n^2 as follows:

$$\begin{aligned}\lim_{n\to\infty} \frac{\sqrt{1+16n^4}}{n^2+1} &= \lim_{n\to\infty} \frac{\sqrt{1+16n^4}}{n^2+1} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \\ &= \lim_{n\to\infty} \frac{\sqrt{\frac{1}{n^4}+16}}{1+\frac{1}{n^2}} \\ &= \frac{\sqrt{0+16}}{1+0} \\ &= 4.\end{aligned}$$

Since the limit exists and is finite, we conclude that $\boxed{\text{the sequence } \{a_n\} \text{ converges to the limit } 4}$.

(b) $a_n = \dfrac{5n+4}{2\cos(n)^2+3n}$

Solution. We can start the limit computation by dividing numerator and denominator by n as follows:

$$\begin{aligned}\lim_{n\to\infty} \frac{5n+4}{2\cos(n)^2+3n} &= \lim_{n\to\infty} \frac{5n+4}{2\cos(n)^2+3n} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{5+\frac{4}{n}}{\frac{2\cos(n)^2}{n}+3}.\end{aligned}$$

Now we have

$$\begin{aligned} 0 \leqslant 2\cos(n)^2 \leqslant 2 \\ \Rightarrow\ 0 \leqslant \frac{2\cos(n)^2}{n} \leqslant \frac{2}{n}\end{aligned}$$

and $\lim_{n\to\infty} \frac{2}{n} = 0$. Therefore, $\lim_{n\to\infty} \frac{2\cos(n)^2}{n} = 0$ by the Sandwich Theorem. It follows that

$$\begin{aligned}\lim_{n\to\infty} \frac{5n+4}{2\cos(n)^2+3n} &= \lim_{n\to\infty} \frac{5+\frac{4}{n}}{\frac{2\cos(n)^2}{n}+3} \\ &= \frac{5+0}{0+3} \\ &= \frac{5}{3}.\end{aligned}$$

Since the limit exists and is finite, we conclude that $\boxed{\text{the sequence } \{a_n\} \text{ converges to the limit } \frac{5}{3}}$.

(c) $a_n = \tan^{-1}(1-\sqrt{n})$

Solution. We have $\lim_{n\to\infty} 1-\sqrt{n} = -\infty$. Therefore,

$$\lim_{n\to\infty} \tan^{-1}\left(1-\sqrt{n}\right) = \text{“}\tan^{-1}(-\infty)\text{”} = -\frac{\pi}{2}.$$

Since the limit exists and is finite, we conclude that $\boxed{\text{the sequence } \{a_n\} \text{ converges to the limit } -\frac{\pi}{2}}$.

(d) $a_n = \dfrac{n+(-1)^n}{n^3+1}$

Solution. If $n \geqslant 1$, we have

$$0 \leqslant n+(-1)^n \leqslant n+1 \text{ and } n^3+1 \geqslant n^3 > 0$$
$$\Rightarrow\ 0 \leqslant \frac{n+(-1)^n}{n^3+1} \leqslant \frac{n+1}{n^3} = \frac{1}{n^2}+\frac{1}{n^3}$$

and $\lim_{n\to\infty} \frac{1}{n^2}+\frac{1}{n^3} = 0$. Therefore by the Sandwich Theorem we have

$$\lim_{n\to\infty} \frac{n+(-1)^n}{n^3+1} = 0.$$

So $\boxed{\text{the sequence } \{a_n\} \text{ converges to the limit } 0}$.

(e) $a_n = \dfrac{4^n - 5^{2n}}{7^n}$

Solution. We have

$$\frac{4^n-5^{2n}}{7^n} = \left(\frac{4}{7}\right)^n - \left(\frac{25}{7}\right)^n$$

Since a geometric sequence converges to 0 if its common ratio is in $(-1,1)$ and diverges to ∞ if its common ratio is > 1, we have

$$\lim_{n\to\infty} \frac{4^n-5^{2n}}{7^n} = \text{“}0-\infty\text{”} = -\infty,$$

so the sequence $\boxed{\text{diverges}}$.

(f) $a_n = \cos\left(\dfrac{5}{\sqrt{n}}\right)^n$

Solution. When computing the limit of this sequence, we have an indeterminate power 1^∞. We can resolve the indeterminate form by writing the power in exponential form

$$\lim_{n\to\infty} \cos\left(\frac{5}{\sqrt{n}}\right)^n = \lim_{n\to\infty} e^{n\ln\left(\cos\left(\frac{5}{\sqrt{n}}\right)\right)}.$$

We now compute the limit of the exponent using L'Hôpital's Rule twice:

$$
\begin{aligned}
\lim_{n\to\infty} n \ln\left(\cos\left(\frac{5}{\sqrt{n}}\right)\right) &= \lim_{x\to\infty} \frac{\ln\left(\cos\left(\frac{5}{\sqrt{x}}\right)\right)}{\frac{1}{x}} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty} \frac{\frac{5}{2x^{3/2}}\tan\left(\frac{5}{\sqrt{x}}\right)}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} -\frac{5\tan\left(\frac{5}{\sqrt{x}}\right)}{\frac{2}{\sqrt{x}}} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty} -\frac{-\frac{25}{2x^{3/2}}\sec\left(\frac{5}{\sqrt{x}}\right)^2}{-\frac{1}{x^{3/2}}} \\
&= \lim_{x\to\infty} -\frac{25}{2}\sec\left(\frac{5}{\sqrt{x}}\right)^2 \\
&= -\frac{25}{2}\sec(0)^2 \\
&= -\frac{25}{2}.
\end{aligned}
$$

So

$$\lim_{n\to\infty} e^{n\ln\left(\cos\left(\frac{5}{\sqrt{n}}\right)\right)} = e^{-25/2},$$

and $\boxed{\text{the sequence } \{a_n\} \text{ converges to the limit } e^{-25/2}}$.

(g) $a_n = \left(\frac{n+5}{n+3}\right)^{4n}$

Solution. When computing the limit of this sequence, we have an indeterminate power 1^∞. We can resolve the indeterminate form by writing the power in exponential form

$$\lim_{n\to\infty}\left(\frac{n+5}{n+3}\right)^{4n} = \lim_{n\to\infty} e^{4n\ln\left(\frac{n+5}{n+3}\right)}.$$

We now compute the limit of the exponent using L'Hôpital's Rule:

$$
\begin{aligned}
\lim_{n\to\infty} 4n\ln\left(\frac{n+5}{n+3}\right) &= \lim_{x\to\infty} 4\frac{\ln(x+5)-\ln(x+3)}{\frac{1}{x}} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} 4\frac{\frac{1}{x+5}-\frac{1}{x+3}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} -4x^2\frac{(x+3)-(x+5)}{(x+5)(x+3)} \\
&= \lim_{x\to\infty} \frac{8x^2}{(x+5)(x+3)}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{8}{(1+5/x)(1+3/x)} \\
&= 8.
\end{aligned}
$$

So

$$\lim_{n\to\infty} e^{4n\ln\left(\frac{n+5}{n+3}\right)} = e^8,$$

and $\boxed{\text{the sequence } \{a_n\} \text{ converges to the limit } e^8}$.

(h) $a_n = (2n+1)^{3/n}$.
Solution. This time, the indeterminate power has the form ∞^0. The method stays the same and we write the power in exponential form to obtain

$$\lim_{n\to\infty} (2n+1)^{3/n} = \lim_{n\to\infty} e^{3\ln(2n+1)/n}.$$

We can compute the limit of the exponent using L'Hôpital's Rule as follows:

$$\begin{aligned} \lim_{n\to\infty} \frac{3\ln(2n+1)}{n} &= \lim_{x\to\infty} \frac{3\ln(2x+1)}{x} \\ &\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{3\frac{2}{2x+1}}{1} \\ &= \lim_{x\to\infty} \frac{6}{2x+1} \\ &= 0. \end{aligned}$$

So

$$\lim_{n\to\infty} e^{3\ln(2n+1)/n} = e^0 = 1,$$

and $\boxed{\text{the sequence } \{a_n\} \text{ converges to the limit } 1}$.

(i) $a_n = \sin(n\pi)e^n$

Solution. Observe that $\sin(n\pi) = 0$ if n is an integer. Therefore $a_n = 0$ for all n, and thus

$$\lim_{n\to\infty} \sin(n\pi)e^n = 0.$$

So $\boxed{\text{the sequence } \{a_n\} \text{ converges to the limit } 0}$.

2. Suppose that a_n is a sequence defined inductively by

$$\begin{cases} a_1 = 2, \\ a_{n+1} = \dfrac{5}{a_n + 4} \quad \text{for } n \geqslant 1. \end{cases}$$

(a) Find the first 4 terms of the sequence $\{a_n\}$.

Solution. We have

$$\begin{aligned} a_1 &= \boxed{2}, \\ a_2 &= \frac{5}{a_1+4} = \frac{5}{2+4} = \boxed{\frac{5}{6}}, \\ a_3 &= \frac{5}{a_2+4} = \frac{5}{\frac{5}{6}+4} = \boxed{\frac{35}{29}}, \\ a_4 &= \frac{5}{a_3+4} = \frac{5}{\frac{35}{29}+4} = \boxed{\frac{145}{151}}. \end{aligned}$$

(b) The sequence $\{a_n\}$ converges. Find its limit.

Solution. Call L the limit of the sequence. Taking $n \to \infty$ in the recursion formula gives

$$a_{n+1} = \frac{5}{a_n + 4} \;\Rightarrow\; L = \frac{5}{L+4}.$$

Solving this for L gives the solutions $L = -5, 1$. Since the terms of the sequence are positive, the only possibility is $L = 1$. Hence,

$$\boxed{\lim_{n\to\infty} a_n = 1}.$$