Section 10.2

# Infinite Series

## Learning Goals

| *Learning Goal* | *Homework Problems* |
|---|---|
| 10.2.1 Find the first terms of a series | 10.2: 7,10,11,14 |
| 10.2.2 Determine whether a series converges or diverges when the term is function of a term of another series | 10.2: 94 |
| 10.2.3 Determine whether a series converges or diverges using the sequence of partial sums | 10.2: 39,45,69 |
| 10.2.4 Evaluate the sum of a telescopic series | 10.2: 39,41,45,49 |
| 10.2.5 Determine if a geometric series converges and if so find its sum | 10.2: 53,59,61,67,71,100 |
| 10.2.6 Express repeating decimals as fractions using geometric series | 10.2: 23,29 |
| 10.2.7 Show that a series diverges by the Term Divergence Theorem | 10.2: 33,35,38 |
| 10.2.8 Determine the value of x for which a geometric series converges (preview of power series) | 10.2: 77,81,97 |

## Conceptual introduction: an infinite series is the sum of all terms in a sequence.

Sequence : $a_1, a_2, a_3, a_4, \ldots$ (list of numbers)

Series : $a_1 + a_2 + a_3 + a_4 + \cdots$ (infinite sum)

Notation for series: $\boxed{\sum_{n=1}^{\infty} a_n}$

could start at any index - does not have to be 1.

this refers to both:
- the series itself - the idea of summing all terms,
- the value of the sum if it exists.

## How do we compute the sum of a series?

We consider the partial sums

$$\boxed{S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_{N-1} + a_N}$$

Then, we take the limit as $N \to \infty$:

$$\boxed{\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=1}^{N} a_n}$$

We say that the series converges if the limit exists and is finite, diverges otherwise.

**Examples:** 1) For $a_n = \frac{1}{n}$, write out the first 4 terms of the sequence $\{a_n\}_{n=1}^{\infty}$ and the first 4 partial sums of the series $\sum_{n=1}^{\infty} a_n$.

Sequence: $a_1 = \frac{1}{1} = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{3}, \quad a_4 = \frac{1}{4}$

Partial sums: $S_1 = a_1 = 1$

$S_2 = a_1 + a_2 = 1 + \frac{1}{2} = \frac{3}{2}$

$S_3 = \underbrace{a_1 + a_2}_{S_2} + a_3 = \underbrace{1 + \frac{1}{2}}_{S_2} + \frac{1}{3} = \frac{11}{6}$

$S_4 = \underbrace{a_1 + a_2 + a_3}_{S_3} + a_4 = \underbrace{1 + \frac{1}{2} + \frac{1}{3}}_{S_3} + \frac{1}{4} = \frac{25}{12}$

In the previous section, we saw that the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ converges: the numbers $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ approach $0$.
In the next section, we will explain that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges: if we sum more and more terms $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, the sum will eventually surpass any number.
So we have $\boxed{\sum_{n=1}^{\infty} \frac{1}{n} = \infty}$

2) Evaluate $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$
We look at the partial sums: $S_N = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$S_1 = \overbrace{\left(\frac{1}{1} - \frac{1}{2}\right)}^{n=1}$

$S_2 = \overbrace{\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right)}^{n=1} + \overbrace{\left(\cancel{\frac{1}{2}} - \frac{1}{3}\right)}^{n=2} = 1 - \frac{1}{3}$

$S_3 = \overbrace{\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right)}^{n=1} + \overbrace{\left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right)}^{n=2} + \overbrace{\left(\cancel{\frac{1}{3}} - \frac{1}{4}\right)}^{n=3} = 1 - \frac{1}{4}$

the negative part of each term cancels out with the positive part of the next term.

In general, we get the cancellation:

$S_N = \overbrace{\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right)}^{n=1} + \overbrace{\left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right)}^{n=2} + \overbrace{\left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right)}^{n=3} + \cdots + \overbrace{\left(\cancel{\frac{1}{N-1}} - \cancel{\frac{1}{N}}\right)}^{n=N-1} + \overbrace{\left(\cancel{\frac{1}{N}} - \frac{1}{N+1}\right)}^{n=N}$

$S_N = 1 - \frac{1}{N+1}$ so $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(1 - \frac{1}{N+1}\right) = \boxed{1}$
This is called a telescoping series/sum.

3) Evaluate $\sum_{n=1}^{\infty} \left( \frac{1}{2n} - \frac{1}{2n+4} \right)$.

We still get cancellation, but less overlap.

$S_1 = \left( \frac{1}{2} - \frac{1}{6} \right)$

$S_2 = \left( \frac{1}{2} - \cancel{\frac{1}{6}} \right) + \left( \frac{1}{4} - \frac{1}{8} \right) + \left( \cancel{\frac{1}{6}} - \frac{1}{10} \right) = \frac{1}{2} + \frac{1}{4} - \frac{1}{8} - \frac{1}{10}$

$S_3 = \left( \frac{1}{2} - \cancel{\frac{1}{6}} \right) + \left( \frac{1}{4} - \cancel{\frac{1}{8}} \right) + \left( \cancel{\frac{1}{6}} - \frac{1}{10} \right) + \left( \cancel{\frac{1}{8}} - \frac{1}{12} \right) = \frac{1}{2} + \frac{1}{4} - \frac{1}{10} - \frac{1}{12}$

$\vdots$

negative part of each term cancels out with the positive part of the one after next (skip one)

$$S_N = \left( \frac{1}{2} - \cancel{\frac{1}{6}} \right) + \left( \frac{1}{4} - \cancel{\frac{1}{8}} \right) + \left( \cancel{\frac{1}{6}} - \cancel{\frac{1}{10}} \right) + \left( \cancel{\frac{1}{8}} - \cancel{\frac{1}{12}} \right) + \cdots + \left( \cancel{\frac{1}{2N-2}} - \frac{1}{2N+2} \right) + \left( \cancel{\frac{1}{2N}} - \frac{1}{2N+4} \right)$$

$$= \frac{1}{2} + \frac{1}{4} - \frac{1}{2N} - \frac{1}{2N+4}$$ (first two positives, last two negatives)

$\lim_{N \to \infty} S_N = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ so $\boxed{\sum_{n=1}^{\infty} \left( \frac{1}{2n} - \frac{1}{2n+4} \right) = \frac{3}{4}}$.

4) Does the series $\sum_{n=1}^{\infty} \frac{2n}{5n+3}$ converge or diverge?

Observe that $\lim_{n \to \infty} \frac{2n}{5n+3} = \frac{2}{5}$

So in the sum $\sum_{n=1}^{\infty} \frac{2n}{5n+3}$, we are summing infinitely many terms very close to $\frac{2}{5}$. Therefore, $\sum_{n=1}^{\infty} \frac{2n}{5n+3} = \infty \Rightarrow \boxed{\text{diverges}}$.

This is an example of:

Term Divergence Test:

if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \to \infty} a_n = 0$

or: if $\lim_{n \to \infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges

Intuitively: the only way an infinite sum can be finite is if the terms approach 0.

Note: the converse is false: $\lim_{n \to \infty} \frac{1}{n} = 0$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

This is because $\frac{1}{n}$ does not approach 0 "fast enough".

So if you just know that $\lim_{n \to \infty} a_n = 0$, you cannot conclude anything about $\sum_{n=n_0}^{\infty} a_n$.

## Geometric Series: $\sum_{n=N_0}^{\infty} ar^n$, $r$ = common ratio $(r \neq 1)$.

We have an explicit formula for the partial sums of a geometric series.

$$\boxed{\sum_{n=N_0}^{N} r^n = \frac{r^{N_0} - r^{N+1}}{1-r} = \frac{(\text{first term}) - (\text{term after last})}{1-(\text{common ratio})}}$$

So

$$\boxed{\sum_{n=N_0}^{\infty} r^n = \begin{cases} \dfrac{r^{N_0}}{1-r} = \dfrac{\text{first term}}{1-(\text{common ratio})} & \text{if } |r|<1 \\ \text{diverges} & \text{if } |r|\geq 1 \end{cases}}$$

## Examples:

1) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{128} = ?$

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{128}$ = geometric sum of common ratio $\frac{1}{2}$

$$= \sum_{n=0}^{7} \left(\frac{1}{2}\right)^n$$

$$= \frac{(\text{first term}) - (\text{term after last})}{1-(\text{common ratio})} = \frac{1-\frac{1}{256}}{1-\frac{1}{2}} = \boxed{\frac{255}{128}}$$

2) Evaluate $\sum_{n=3}^{\infty} \frac{1}{2^n}$, $\sum_{n=0}^{\infty} 4^{-n}5^n$ or explain why they diverge.

$\sum_{n=3}^{\infty} \frac{1}{2^n}$ = geometric series of common ratio $\frac{1}{2}$ (so converges)

$$= \frac{\text{first term}}{1-\text{common ratio}} = \frac{1/8}{1-1/2} = \boxed{\frac{1}{4}}.$$

$\sum_{n=0}^{\infty} 4^{-n}5^n = \sum_{n=0}^{\infty} \left(\frac{5}{4}\right)^n$ <u>diverges</u> since common ratio $= \frac{5}{4} \geq 1$.

3) Evaluate $\sum_{n=1}^{\infty} \frac{3 - 2^{2n}}{7^{n+1}}$ using the rules for series:

> If $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ converge, then
>
> - $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$
> - $\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$ (c constant)

$$\sum_{n=1}^{\infty} \frac{3 - 2^{2n}}{7^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{7} \cdot \frac{3 - 4^n}{7^n} = \sum_{n=1}^{\infty} \left( \frac{3}{7}\left(\frac{1}{7}\right)^n - \frac{1}{7}\left(\frac{4}{7}\right)^n \right)$$

$$= \frac{3}{7} \sum_{n=1}^{\infty} \left(\frac{1}{7}\right)^n - \frac{1}{7} \sum_{n=1}^{\infty} \left(\frac{4}{7}\right)^n = \frac{3}{7} \cdot \frac{1/7}{1 - 1/7} - \frac{1}{7} \cdot \frac{4/7}{1 - 4/7} = \boxed{-\frac{5}{42}}.$$

4) Write the number $3.16161616\ldots = 3.\overline{16}$ as a ratio of two integers.

$$3.16161616\ldots = 3 + 0.16 + 0.0016 + 0.000016 + 0.00000016 + \cdots$$

$$= 3 + \frac{16}{100} + \frac{16}{100^2} + \frac{16}{100^3} + \frac{16}{100^4} + \cdots$$

$$= 3 + \sum_{n=1}^{\infty} \frac{16}{100^n} = 3 + 16 \sum_{n=1}^{\infty} \left(\frac{1}{100}\right)^n = 3 + 16 \cdot \frac{\frac{1}{100}}{1 - \frac{1}{100}} = 3 + \frac{16}{99}$$

$$= \boxed{\frac{313}{99}}.$$

5) Let $f(x) = \sum_{n=1}^{\infty} \frac{3x^n}{2^n}$. Find the values of $x$ for which $f(x)$ converges and find a formula for the sum when it converges.

$$f(x) = \sum_{n=1}^{\infty} 3\left(\frac{x}{2}\right)^n$$

geometric sum with common ratio $r = \frac{x}{2}$ and first term $\frac{3x}{2}$.

$f(x)$ converges when $|r| < 1 \Rightarrow \left|\frac{x}{2}\right| < 1 \Rightarrow \boxed{-2 < x < 2}$.

When $-2 < x < 2$, $f(x) = \dfrac{\text{first term}}{1-\text{common ratio}}$

$$= \frac{\frac{3x}{2}}{1-\frac{x}{2}} = \boxed{\frac{3x}{2-x}}.$$

Practice: Evaluate the following series or explain why they diverge.

a) $\displaystyle\sum_{n=2}^{\infty} \frac{2^{3-n}}{3^{n+2}}$ b) $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$ c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{3n+2}-\frac{1}{3n+8}\right)$ d) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{3n}}{5^{n+4}}$

Solutions: a) $\displaystyle\sum_{n=2}^{\infty} \frac{2^{3-n}}{3^{n+2}} = \sum_{n=2}^{\infty} \frac{2^3 2^{-n}}{3^n 3^2} = \frac{2^3}{3^2}\sum_{n=2}^{\infty}\frac{1}{3^n \cdot 2^n} = \frac{8}{9}\sum_{n=2}^{\infty}\left(\frac{1}{6}\right)^n$

$$= \frac{8}{9}\cdot\frac{1/36}{1-1/6} = \frac{8}{9}\cdot\frac{1}{30} = \boxed{\frac{4}{135}}$$

b) Term Divergence Test: $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} e^{n\ln(1+\frac{1}{n})}$

and $\displaystyle\lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) = \lim_{x\to\infty}\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}} \overset{L'H}{\underset{0/0}{=}} \lim_{x\to\infty}\frac{\frac{1}{1+1/x}\cdot\frac{-1}{\cancel{x^2}}}{-\frac{1}{\cancel{x^2}}} = 1.$

So $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e^1 = e \neq 0.$

Therefore, $\boxed{\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n \text{ diverges}}$.

c) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{3n+2}-\frac{1}{3n+8}\right)$ is a telescopic series.

$$S_N = \sum_{n=1}^{N} \left( \frac{1}{3n+2} - \frac{1}{3n+8} \right)$$

$$= \left(\frac{1}{5} - \cancel{\frac{1}{11}}\right) + \left(\frac{1}{8} - \cancel{\frac{1}{14}}\right) + \left(\cancel{\frac{1}{11}} - \cancel{\frac{1}{17}}\right) + \cdots + \left(\cancel{\frac{1}{3N-1}} - \frac{1}{3N+5}\right) + \left(\cancel{\frac{1}{3N+2}} - \frac{1}{3N+8}\right)$$

$$= \frac{1}{5} + \frac{1}{8} - \frac{1}{3N+5} - \frac{1}{3N+8} \xrightarrow[N\to\infty]{} \frac{1}{5} + \frac{1}{8} = \frac{13}{40}$$

So $\boxed{\sum_{n=1}^{\infty} \left( \frac{1}{3n+2} - \frac{1}{3n+8} \right) = \frac{13}{40}}$

d) $\sum_{n=1}^{\infty} \frac{2^{3n}}{5^{n+4}} = \frac{1}{5^4} \sum_{n=1}^{\infty} \frac{8^n}{5^n} = \frac{1}{5^4} \sum_{n=1}^{\infty} \left(\frac{8}{5}\right)^n$ is a geometric series

with common ratio $\frac{8}{5} > 1$, so it <u>diverges</u>.