Rutgers University
Math 152

**Section 10.2: Infinite Series - Worksheet Solutions**

1. Each of the series $\sum_{n=n_0}^{\infty} a_n$ below is either geometric or telescoping. For each series, find a formula for the partial sum $S_N = \sum_{n=n_0}^{N} a_n$, then determine if the series converges or diverges, and compute its sum if it does.

   (a) $\sum_{n=4}^{\infty} 2^n 3^{-n}$

   *Solution.* This is a geometric series of common ratio $r = \frac{2}{3}$. The partial sum is

$$S_N = \frac{\text{(first term)} - \text{(term after last)}}{1 - \text{(common ratio)}} = \frac{\left(\frac{2}{3}\right)^4 - \left(\frac{2}{3}\right)^{N+1}}{1 - \frac{2}{3}} = \boxed{\frac{16}{27} - \frac{2^{N+1}}{3^N}}.$$

   Since $|r| < 1$, $\boxed{\text{the series converges}}$. We can compute the sum two ways: either taking the limit of $S_N$ when $N \to \infty$ or using the formula for the sum of a convergent geometric series. Either way, we get

$$\boxed{\sum_{n=4}^{\infty} 2^n 3^{-n} = \frac{16}{27}}.$$

   (b) $\sum_{n=0}^{\infty} \left( \frac{4}{2n+1} - \frac{4}{2n+5} \right)$

   *Solution.* This is a telescoping series. We have

$$\begin{aligned} S_N &= \left(\frac{4}{1} - \frac{4}{5}\right) + \left(\frac{4}{3} - \frac{4}{7}\right) + \left(\frac{4}{5} - \frac{4}{9}\right) + \cdots + \left(\frac{4}{2N-1} - \frac{4}{2N+3}\right) + \left(\frac{4}{2N+1} - \frac{4}{2N+5}\right) \\ &= \frac{4}{1} + \frac{4}{3} - \frac{4}{2N+3} - \frac{4}{2N+5} \\ &= \boxed{\frac{16}{3} - \frac{4}{2N+3} - \frac{4}{2N+5}}. \end{aligned}$$

   So

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \left( \frac{16}{3} - \frac{4}{2N+3} - \frac{4}{2N+5} \right) = \frac{16}{3} - 0 - 0 = \frac{16}{3}.$$

   Therefore, $\boxed{\text{the series converges}}$ and

$$\boxed{\sum_{n=0}^{\infty} \left( \frac{4}{2n+1} - \frac{4}{2n+5} \right) = \frac{16}{3}}.$$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{1-3\cdot 4^{2n}}{5^{n-1}}$

*Solution.* We have

$$\frac{1-3\cdot 4^{2n}}{5^{n-1}} = 5\left(\frac{1}{5}\right)^n - 15\left(\frac{16}{5}\right)^n.$$

Thus, using the formula for geometric sums, we get

$$\begin{aligned} S_N &= 5\sum_{n=0}^{N}\left(\frac{1}{5}\right)^n - 15\sum_{n=0}^{N}\left(\frac{16}{5}\right)^n \\ &= 5\frac{1-\frac{1}{5^{N+1}}}{1-\frac{1}{5}} - 15\frac{1-\left(\frac{16}{5}\right)^{N+1}}{1-\frac{16}{5}} \\ &= \boxed{\frac{25}{4}\left(1-\frac{1}{5^{N+1}}\right) + \frac{75}{11}\left(1-\left(\frac{16}{5}\right)^{N+1}\right)}. \end{aligned}$$

We have $\frac{1}{5^{N+1}} \to 0$ and $\left(\frac{16}{5}\right)^{N+1} \to \infty$ when $N \to \infty$. Therefore $S_N \to -\infty$ as $N \to \infty$, so

$$\boxed{\sum_{n=0}^{\infty} \frac{1-3\cdot 4^{2n}}{5^{n-1}} \text{ diverges}}.$$

(d) $\displaystyle\sum_{n=3}^{\infty} \ln\left(\frac{3n+1}{3n+4}\right)$

*Solution.* After rewriting the general term as

$$\ln\left(\frac{3n+1}{3n+4}\right) = \ln(3n+1) - \ln(3n+4)$$

we see that this a telescoping series. We have

$$\begin{aligned} S_N &= \sum_{n=3}^{N}\left(\ln(3n+1) - \ln(3n+4)\right) \\ &= (\ln(10)-\ln(13)) + (\ln(13)-\ln(16)) + \cdots + (\ln(3N-2)-\ln(3N+1)) + (\ln(3N+1)-\ln(3N+4)) \\ &= \boxed{\ln(10)-\ln(3N+4)}. \end{aligned}$$

Since $\ln(3N+4) \to \infty$ when $N \to \infty$, we deduce that $S_N \to -\infty$ when $N \to \infty$. Thus

$$\boxed{\sum_{n=3}^{\infty} \ln\left(\frac{3n+1}{3n+4}\right) \text{ diverges}}.$$

(e) $\displaystyle\sum_{n=1}^{\infty} 5\cdot 3^{1-2n}$

*Solution.* We can rewrite the general term as

$$5 \cdot 3^{1-2n} = \frac{15}{9^n}.$$

So this is a geometric with common ratio $r = \frac{1}{9}$. The partial sum is

$$S_N = \frac{\frac{15}{81} - \frac{15}{9^{N+1}}}{1 - \frac{1}{9}} = \boxed{\frac{5}{24}\left(1 - \frac{1}{9^N}\right)}.$$

Since $|r| < 1$, $\boxed{\text{the series converges}}$. We can compute the sum two ways: either taking the limit of $S_N$ when $N \to \infty$ or using the formula for the sum of a convergent geometric series. Either way, we get

$$\boxed{\sum_{n=1}^{\infty} 5 \cdot 3^{1-2n} = \frac{5}{24}}.$$

(f) $\displaystyle\sum_{n=1}^{\infty} \left(\tan^{-1}(n+1) - \tan^{-1}(n)\right)$

*Solution.* This is a telescoping series. We have

$$\begin{aligned} S_N &= (\tan^{-1}(2) - \tan^{-1}(1)) + (\tan^{-1}(3) - \tan^{-1}(2)) + \cdots + (\tan^{-1}(N+1) - \tan^{-1}(N)) \\ &= -\tan^{-1}(1) + \tan^{-1}(N+1) \\ &= -\frac{\pi}{4} + \tan^{-1}(N+1) \end{aligned}$$

Since $\lim\limits_{N\to\infty} \tan^{-1}(N+1) = \frac{\pi}{2}$, we have

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(-\frac{\pi}{4} + \tan^{-1}(N+1)\right) = -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4}.$$

So $\boxed{\text{the series converges}}$ and

$$\boxed{\sum_{n=1}^{\infty} \left(\tan^{-1}(n+1) - \tan^{-1}(n)\right) = \frac{\pi}{4}}.$$

2. Use geometric series to express the repeating decimals below as a fraction of two integers.

(a) $1.5222\cdots = 1.5\overline{2}$

*Solution.* We have

$$\begin{aligned} 1.5\overline{2} &= 1.5 + 0.02 + 0.002 + \cdots \\ &= \frac{15}{10} + \frac{2}{100} + \frac{2}{1000} + \cdots \\ &= \frac{3}{2} + \sum_{n=2}^{\infty} \frac{2}{10^n} \end{aligned}$$

$$
\begin{aligned}
&= \frac{3}{2} + \frac{\frac{2}{100}}{1-\frac{1}{10}} \\
&= \frac{3}{2} + \frac{10}{9}\cdot\frac{1}{50} \\
&= \boxed{\frac{137}{90}}.
\end{aligned}
$$

(b) $0.126126\cdots = 0.\overline{126}$

*Solution.* We have

$$
\begin{aligned}
0.\overline{126} &= 0.126 + 0.000126 + \cdots \\
&= \frac{126}{1000} + \frac{126}{1000000} + \cdots \\
&= \sum_{n=1}^{\infty} \frac{126}{1000^n} \\
&= \frac{\frac{126}{1000}}{1-\frac{1}{1000}} \\
&= \frac{1000}{999}\cdot\frac{126}{1000} \\
&= \boxed{\frac{14}{111}}.
\end{aligned}
$$

3. For each sequence $\{a_n\}_{n=n_0}^{\infty}$ given below, determine

   (i) whether the **sequence** $\{a_n\}_{n=n_0}^{\infty}$ converges or diverges. If the sequence converges, find its limit.

   (ii) whether the **series** $\sum\limits_{n=n_0}^{\infty} a_n$ converges or diverges. If the series converges, find its sum if possible.

   (a) $\left\{\left(1+\frac{4}{n}\right)^n\right\}_{n=1}^{\infty}$

   *Solution.* (i) The limit of this sequence is an indeterminate power $1^{\infty}$. We can write it in exponential form

$$
\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^n = \lim_{n\to\infty} e^{n\ln\left(1+\frac{4}{n}\right)}.
$$

   We can now compute the limit of the exponent using L'Hôpital's Rule as follows:

$$
\begin{aligned}
\lim_{n\to\infty} n\ln\left(1+\frac{4}{n}\right) &= \lim_{x\to\infty}\frac{\ln\left(1+\frac{4}{x}\right)}{\frac{1}{x}} \\
&\overset{\mathrm{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty}\frac{-\frac{4}{x^2}\cdot\frac{1}{1+\frac{4}{x}}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty}\frac{4}{1+\frac{4}{x}} \\
&= 4.
\end{aligned}
$$

Therefore

$$\lim_{n\to\infty} e^{n\ln\left(1+\frac{4}{n}\right)} = e^4,$$

so $\boxed{\text{the sequence } \left\{\left(1+\frac{4}{n}\right)^n\right\}_n \text{ converges to the limit } e^4}$.

(ii) Since the limit of the general term $\left(1+\frac{4}{n}\right)^n$ is not zero, the Term Divergence Test tells us that

$$\boxed{\sum_{n=1}^{\infty}\left(1+\frac{4}{n}\right)^n \text{ diverges}}.$$

(b) $\left\{\sqrt{n+1}-\sqrt{n}\right\}_{n=0}^{\infty}$

*Solution.* (i) The limit of this sequence is an indeterminate form $\infty-\infty$. We can resolve the indeterminate by multiplying by the conjugate in the numerator and denominator:

$$\begin{aligned}
\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right) &= \lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \\
&= \lim_{n\to\infty}\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} \\
&= \lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} \\
&= 0.
\end{aligned}$$

So $\boxed{\text{the sequence } \left\{\sqrt{n+1}-\sqrt{n}\right\}_n \text{ converges to the limit } 0}$.

(ii) To determine if the series converges or diverges, we can use the fact that this series is telescoping. The partial sums can be expressed as follows

$$\begin{aligned}
S_N &= (\sqrt{1}-\sqrt{0})+(\sqrt{2}-\sqrt{1})+\cdots+\left(\sqrt{N+1}-\sqrt{N}\right) \\
&= \sqrt{N+1}.
\end{aligned}$$

Therefore, $S_N\to\infty$ as $N\to\infty$, and

$$\boxed{\sum_{n=0}^{\infty}\left(\sqrt{n+1}-\sqrt{n}\right) \text{ diverges}}.$$

(c) $\{e^{-n}\}_{n=0}^{\infty}$

*Solution.* (i) This is a geometric sequence of common ratio $r=e^{-1}$, which satisfies $|r|<1$. So

$$\lim_{n\to\infty} e^{-n} = 0,$$

and $\boxed{\text{the sequence } \left\{e^{-n}\right\}_n \text{ converges to the limit } 0}$.

(ii) Since $|r| = e^{-1} < 1$, the geometric series $\boxed{\sum_{n=0}^{\infty} e^{-n} \text{ converges}}$ and we can evaluate the sum as

$$\boxed{\sum_{n=0}^{\infty} e^{-n} = \frac{1}{1-e^{-1}}}.$$

(d) $\left\{ \frac{e^{5n}}{n^{3/2}} \right\}_{n=1}^{\infty}$

*Solution.* (i) The limit of this sequence is an indeterminate form $\frac{\infty}{\infty}$. We can use L'Hôpital's Rule to compute the limit:

$$\begin{aligned}\lim_{n\to\infty} \frac{e^{5n}}{n^{3/2}} &= \lim_{x\to\infty} \frac{e^{5x}}{x^{3/2}} \\ &\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{5e^{5x}}{\frac{3}{2}x^{1/2}} \\ &\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{25e^{5x}}{\frac{3}{4}x^{-1/2}} \\ &= \lim_{x\to\infty} \frac{100}{3} x^{1/2} e^{5x} \\ &= \infty.\end{aligned}$$

So $\boxed{\text{the sequence } \left\{ \frac{e^{5n}}{n^{3/2}} \right\}_n \text{ diverges}}$.

(ii) Since the limit of the general term $\frac{e^{5n}}{n^{3/2}}$ is not zero, the Term Divergence Test tells us that

$$\boxed{\sum_{n=1}^{\infty} \frac{e^{5n}}{n^{3/2}} \text{ diverges}}.$$

4. Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{2 \cdot 5^{n+1}}$. Find the values of $x$ for which the series converges and find the sum of the series when it converges.

*Solution.* Observe that $f(x)$ is a geometric series of common ratio $r = \frac{x}{5}$. So it will converge when

$$|r| < 1 \;\Rightarrow\; \left|\frac{x}{5}\right| < 1 \;\Rightarrow\; |x| < 5 \;\Rightarrow\; \boxed{-5 < x < 5}.$$

When $-5 < x < 5$, the sum of the series is

$$\begin{aligned}f(x) &= \frac{\text{first term}}{1 - (\text{common ratio})} \\ &= \frac{\frac{1}{10}}{1 - \frac{x}{5}} \\ &= \boxed{\frac{1}{10-2x}}.\end{aligned}$$