Section 10.3

The Integral Test

Learning Goals

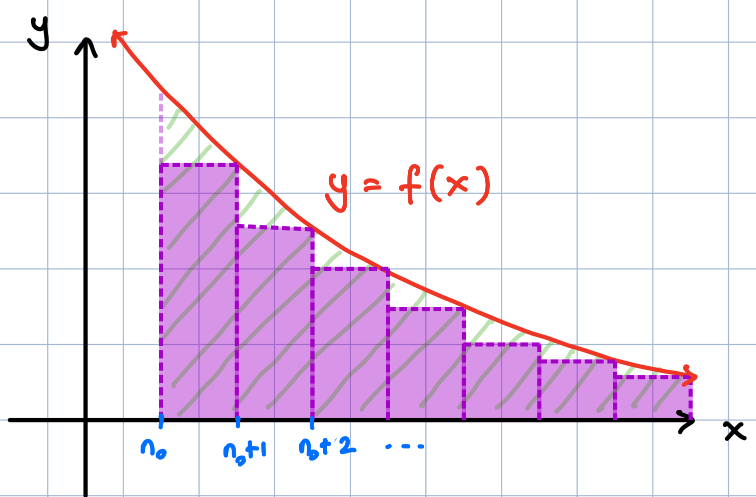
Learning Goal	*Homework Problems*
10.3.1 Determine whether a series with all positive terms diverges or converges using the integral test	10.3: 5,6,7,8,15,21,25,37,43,61,62

<u>Conceptual introduction</u> : it is in general very difficult to calculate the sum of a series $\sum_{n=n_0}^{\infty} a_n$. We have learned how to do it for telescoping and geometric series. In most other cases, we will just try to determine if the series converges or diverges. The <u>Integral Test</u> helps determining convergence for some series.

<u>Integral Test</u>: suppose that $a_n = f(n)$, where f is a <u>positive</u>, <u>continuous</u>, <u>decreasing</u> function. Then:

If $\int_{n_0}^{\infty} f(x)\,dx$ converges, then $\sum_{n=n_0}^{\infty} a_n$ converges.
If $\int_{n_0}^{\infty} f(x)\,dx$ diverges, then $\sum_{n=n_0}^{\infty} a_n$ diverges.

<u>Intuition</u> : $\sum_{n=n_0}^{\infty} a_n$ represents the area of rectangles of width 1 under the graph of f:



We have the inequality:

$$\sum_{n=n_0+1}^{\infty} a_n \leq \int_{n_0}^{\infty} f(x)\,dx$$

So if the integral is finite, then so is the series.

A similar reasoning with left-handpoint sums would show that if the integral is infinite, then so is the series.

Examples: 1) Determine if $\sum_{n=1}^{\infty} ne^{-n^2}$ converges or diverges.

Integral Test: $f(x) = xe^{-x^2}$ is positive, continuous on $[1, \infty)$

$f'(x) = e^{-x^2} - 2x^2e^{-x^2} = (1-2x^2)e^{-x^2} < 0$ on $[1,\infty)$

so f is decreasing on $[1,\infty)$

$\Rightarrow$ we can use the integral test.

$$\int_1^{\infty} xe^{-x^2}dx = \lim_{b\to\infty} \int_1^b xe^{-x^2}dx = \lim_{b\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_1^b$$

$$= \lim_{b\to\infty}\left(-\frac{e^{-b^2}}{2} + \frac{e}{2}\right) = \frac{e}{2}$$ so $\int_1^{\infty} xe^{-x^2}dx$ converges.

Therefore, $\boxed{\sum_{n=1}^{\infty} ne^{-n^2} \text{ converges}}$ as well.

2) Determine if $\sum_{n=3}^{\infty} \frac{1}{n\ln(n)}$ converges or diverges.

Integral Test: $f(x) = \frac{1}{x\ln(x)}$ is positive, continuous on $[3,\infty)$

$f'(x) = -\frac{\ln(x)+1}{(x\ln(x))^2} < 0$ on $[3,\infty)$

so f is decreasing on $[3,\infty)$.

$\Rightarrow$ we can use the integral test.

$$\int_3^{\infty}\frac{dx}{x\ln(x)} = \lim_{b\to\infty}\int_3^b \frac{\overbrace{dx}^{du}}{x\underbrace{\ln(x)}_{u}}$$

u-sub

$u = \ln(x)$	$x=3 \Rightarrow u=\ln(3)$
$du = \frac{dx}{x}$	$x=b \Rightarrow u=\ln(b)$

$$= \lim_{b\to\infty}\int_{\ln(3)}^{\ln(b)}\frac{du}{u}$$

$$= \int_{\ln(3)}^{\infty}\frac{du}{u}$$ diverges (p- integral with $p=1$).

Therefore, $\boxed{\sum_{n=3}^{\infty}\frac{1}{n\ln(n)} \text{ diverges}}$ as well.

3) Determine if $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges or diverges.

Integral Test: $f(x) = \frac{1}{x^2}$ is <u>positive</u>, <u>continuous</u> on $[1, \infty)$

$f'(x) = -\frac{2}{x^3} < 0$ <0 on $[1,\infty)$

so f is <u>decreasing</u> on $[1,\infty)$

$\Rightarrow$ we can use the integral test.

$\int_1^{\infty} \frac{dx}{x^2}$ converges (p-integral with $p = 2 > 1$).

Therefore, $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges}}$ as well.

⚠ The Integral Test does not say that $\int_1^{\infty} \frac{dx}{x^2}$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ are equal, just that they both converge.

$$\int_1^{\infty} \frac{dx}{x^2} = 1 \quad \text{but} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

More generally, we have the following results for <u>p-series</u>

$$\boxed{\sum_{n=n_0}^{\infty} \frac{1}{n^p} \quad \begin{array}{ll} \text{converges} & \text{if } p>1 \\ \text{diverges} & \text{if } p \le 1 \end{array}}$$ (Integral Test + p-integrals).

So $\sum_{n=1}^{\infty} n^{-37/36} = \sum_{n=1}^{\infty} \frac{1}{n^{37/36}}$ converges since $\frac{37}{36} > 1$.

$\sum_{n=1}^{\infty} \frac{1}{n^{\ln(2)}}$ diverges since $\ln(2) \simeq 0.7 \le 1$.

Practice: determine if the following series converge or diverge, and find the value of the sum when possible.
(The Integral Test is not required / possible for all.)

1) $\displaystyle\sum_{n=0}^{\infty} \frac{e^n}{1+e^{2n}}$ 2) $\displaystyle\sum_{n=1}^{\infty} \frac{1-2^{2n}}{5^n}$ 3) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$

4) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2n+3} - \frac{1}{2n+5}\right)$ 5) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{\sqrt{n^2+1}}$ 6) $\displaystyle\sum_{n=2}^{\infty} ne^{-n}$

Solutions:

1) Integral Test: $f(x) = \dfrac{e^x}{1+e^{2x}}$ is positive, continuous on $[0,\infty)$

$$f'(x) = \frac{e^x(1+e^{2x}) - e^x(2e^{2x})}{(1+e^{2x})^2} = \frac{e^x(1-e^{2x})}{(1+e^{2x})^2} < 0 \text{ on } (0,\infty)$$

so f is decreasing on $[0,\infty)$.
So we can use the integral test.

$u = e^x$

$$\int_0^{\infty} \frac{e^x}{1+e^{2x}}\,dx = \lim_{b\to\infty} \int_0^b \frac{e^x}{1+e^{2x}}\,dx = \lim_{b\to\infty} \int_1^{e^b} \frac{du}{1+u^2}$$

$$= \lim_{b\to\infty} \left[\tan^{-1}(u)\right]_1^{e^b} = \lim_{b\to\infty} \left(\tan^{-1}(\underset{\xrightarrow[b\to\infty]{}\infty}{e^b}) - \frac{\pi}{4}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

So $\displaystyle\int_0^{\infty} \frac{e^x}{1+e^{2x}}\,dx$ converges.

Therefore, $\boxed{\displaystyle\sum_{n=0}^{\infty} \frac{e^n}{1+e^{2n}} \text{ converges}}$ as well.

2) $\displaystyle\sum_{n=1}^{\infty} \frac{1-2^{2n}}{5^n} = \sum_{n=1}^{\infty} \left[\left(\frac{1}{5}\right)^n - \left(\frac{4}{5}\right)^n\right]$ converges (geometric series w/ |common ratios| <1)

$$= \frac{1/5}{1-1/5} - \frac{4/5}{1-4/5}$$

$$= \frac{1}{4} - 4 = \boxed{-\frac{15}{4}}$$

3) Integral test: $f(x) = \frac{1}{x\sqrt{\ln(x)}}$ positive, continuous on $[2,\infty)$.

$$f'(x) = -\frac{\sqrt{\ln(x)} + x\,\frac{1}{x}\cdot\frac{1}{2}\ln(x)^{-1/2}}{x^2\ln(x)} < 0 \text{ on } [2,\infty)$$

$\Rightarrow f$ decreasing on $[2,\infty)$.

So we can use the integral test.

$$\int_2^\infty \frac{dx}{x\sqrt{\ln(x)}} = \lim_{b\to\infty}\int_2^b \frac{dx}{x\sqrt{\ln(x)}} \overset{u=\ln(x)}{=} \lim_{b\to\infty}\int_{\ln(2)}^{\ln(b)} \frac{du}{\sqrt{u}}$$

$$= \int_{\ln(2)}^\infty \frac{du}{\sqrt{u}}$$, which diverges (p-integral, $p=\frac{1}{2}<1$).

Therefore, $\boxed{\sum_{n=2}^\infty \frac{1}{n\sqrt{\ln(n)}} \text{ diverges}}$ as well.

4) $\sum_{n=0}^\infty \left(\frac{1}{2n+3} - \frac{1}{2n+5}\right)$ is a telescoping series.

$$S_N = \sum_{n=0}^N \left(\frac{1}{2n+3} - \frac{1}{2n+5}\right)$$

$$= \overbrace{\left(\frac{1}{3} - \cancel{\frac{1}{5}}\right)}^{n=0} + \overbrace{\left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right)}^{n=1} + \overbrace{\left(\cancel{\frac{1}{7}} - \cancel{\frac{1}{9}}\right)}^{n=2} + \cdots + \overbrace{\left(\cancel{\frac{1}{2N+1}} - \cancel{\frac{1}{2N+3}}\right)}^{n=N-1} + \overbrace{\left(\cancel{\frac{1}{2N+3}} - \frac{1}{2N+5}\right)}^{n=N}$$

$$= \frac{1}{3} - \frac{1}{2N+5}$$

We have $\sum_{n=0}^\infty \left(\frac{1}{2n+3} - \frac{1}{2n+5}\right) = \lim_{N\to\infty} S_N = \boxed{\frac{1}{3}}$, so the series converges.

5) Observe that $\lim\limits_{n\to\infty} \dfrac{2n+1}{\sqrt{n^2+1}} \cdot \dfrac{\frac{1}{n}}{\frac{1}{n}} = \lim\limits_{n\to\infty} \dfrac{2+1/n}{\sqrt{1+1/n^2}} = 2$

Since the general term does not converge to 0, we deduce that $\boxed{\sum\limits_{n=1}^{\infty} \frac{2n+1}{\sqrt{n^2+1}} \text{ diverges}}$ by the Term Divergence Test.

6) Integral test: $f(x) = xe^{-x}$ positive, continuous on $[2,\infty)$.

$f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x} < 0$ on $[2,\infty)$

$\Rightarrow f$ decreasing on $[2,\infty)$.

So we can use the integral test.

$$\int_2^{\infty} xe^{-x}\,dx = \lim_{b\to\infty} \int_2^b \underbrace{x}_{u}\underbrace{e^{-x}\,dx}_{dv}$$

IBP

$u = x$	$dv = e^{-x}dx$
$du = dx$	$v = -e^{-x}$

$$= \lim_{b\to\infty} \left(\left[-xe^{-x}\right]_2^b + \int_2^b e^{-x}\,dx \right)$$

$$= \lim_{b\to\infty} \left(-be^{-b} + 2e^{-2} + \left[-e^{-x}\right]_2^b \right)$$

$$= \lim_{b\to\infty} \left(\underbrace{-be^{-b}}_{?} + 2e^{-2} - \underbrace{e^{-b}}_{\to 0} + e^{-2} \right)$$

We have $\lim\limits_{b\to\infty} be^{-b} = \lim\limits_{b\to\infty} \dfrac{b}{e^b} \underset{\frac{\infty}{\infty}}{\overset{L'H}{=}} \lim\limits_{b\to\infty} \dfrac{1}{e^b} = 0.$

So $\int_2^{\infty} xe^{-x}\,dx = 2e^{-2} + e^{-2}$, so it converges.

Therefore, $\boxed{\sum\limits_{n=2}^{\infty} ne^{-n} \text{ converges}}$ as well.