Rutgers University
Math 152

Section 10.3: The Integral Test - Worksheet Solutions

1. For each sequence $\{a_n\}_{n=n_0}^{\infty}$ given below, determine

   (i) whether the **sequence** $\{a_n\}_{n=n_0}^{\infty}$ converges or diverges. If the sequence converges, find its limit.

   (ii) whether the **series** $\sum\limits_{n=n_0}^{\infty} a_n$ converges or diverges. If the series converges, find its sum if possible.

   Note: the integral test is not possible/necessary for all the series.

   (a) $\{n5^{-n}\}_{n=0}^{\infty}$

   Solution. (i) The limit of this sequence is an indeterminate form $\infty \cdot 0$, which can be resolved by rewriting the expression as a fraction and using L'Hôpital's Rule. This gives

$$\begin{aligned}\lim_{n\to\infty} n5^{-n} &= \lim_{x\to\infty} \frac{x}{5^x}\\ &\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{1}{\ln(5)5^x}\\ &= 0.\end{aligned}$$

   So $\boxed{\text{the sequence } \left\{n5^{-n}\right\}_n \text{ converges to the limit } 0}$.

   (ii) We use the Integral Test. Put $f(x) = x5^{-x}$. Then f is continuous and positive on $[0, \infty)$. We have

$$f'(x) = 5^{-x} - \ln(5)x5^{-x} = -5^{-x}(\ln(5)x - 1),$$

   which is negative when $x > \frac{1}{\ln(5)}$. So f is decreasing on $\left[\frac{1}{\ln(5)}, \infty\right)$. Therefore, the Integral Test applies.

   We now need to determine if the improper integral $\displaystyle\int_0^{\infty} x5^{-x}dx$ converges or diverges, which we can do by evaluating it. Let us start by calculating an antiderivative using an IBP with parts

$$\begin{aligned}u = x &\;\Rightarrow\; du = dx,\\ dv = 5^{-x}dx &\;\Rightarrow\; v = -\frac{5^{-x}}{\ln(5)}.\end{aligned}$$

   This gives

$$\begin{aligned}\int x5^{-x}dx &= -\frac{x5^{-x}}{\ln(5)} - \int -\frac{5^{-x}}{\ln(5)}dx\\ &= -\frac{x5^{-x}}{\ln(5)} + \frac{1}{\ln(5)}\int 5^{-x}dx\\ &= -\frac{x5^{-x}}{\ln(5)} - \frac{5^{-x}}{\ln(5)^2} + C.\end{aligned}$$

With this at hand, we can evaluate the improper integral.

$$\begin{aligned}
\int_0^\infty x5^{-x}dx &= \lim_{b\to\infty}\int_0^b x5^{-x}dx \\
&= \lim_{b\to\infty}\left[-\frac{x5^{-x}}{\ln(5)} - \frac{5^{-x}}{\ln(5)^2}\right]_0^b \\
&= \lim_{b\to\infty}\left(\frac{1}{\ln(5)^2} - \frac{b5^{-b}}{\ln(5)} - \frac{5^{-b}}{\ln(5)^2}\right) \\
&= \frac{1}{\ln(5)^2}.
\end{aligned}$$

So $\displaystyle\int_0^\infty x5^{-x}dx$ converges. It follows that

$$\boxed{\sum_{n=0}^{\infty} n5^{-n} \text{ converges}}.$$

(b) $\displaystyle\left\{\frac{1}{n\left(1+\ln(n)^2\right)}\right\}_{n=2}^{\infty}$

Solution. (i) Since $n, \ln(n) \to \infty$ when $n \to \infty$, we have

$$\lim_{n\to\infty}\frac{1}{n\left(1+\ln(n)^2\right)} = 0.$$

So $\boxed{\text{the sequence } \left\{\frac{1}{n\left(1+\ln(n)^2\right)}\right\}_n \text{ converges to the limit } 0}$.

(ii) We use the Integral Test. Put $f(x) = \frac{1}{x(1+\ln(x)^2)}$. Then f is positive and continuous on $[2,\infty)$. Furthermore, x and $\ln(x)$ are increasing on $[2,\infty)$, so f is decreasing on $[2,\infty)$ as the reciprocal of an increasing function. Therefore, the assumptions of the Integral Test are met.

We now compute the improper integral.

$$\begin{aligned}
\int_2^\infty \frac{dx}{x\left(1+\ln(x)^2\right)} &= \lim_{b\to\infty}\int_2^b \frac{dx}{x\left(1+\ln(x)^2\right)} \\
&= \lim_{b\to\infty}\left[\arctan\left(\ln(x)\right)\right]_2^b \\
&= \lim_{b\to\infty}\left(\arctan\left(\ln(b)\right) - \arctan(\ln(2))\right) \\
&= \frac{\pi}{2} - \arctan(\ln(2)).
\end{aligned}$$

Therefore, $\displaystyle\int_2^\infty \frac{dx}{x\left(1+\ln(x)^2\right)}$ converges. Thus,

$$\boxed{\sum_{n=2}^{\infty}\frac{1}{n\left(1+\ln(n)^2\right)} \text{ converges}}.$$

(c) $\left\{\dfrac{1}{n^{\log_5(3)}}\right\}_{n=1}^{\infty}$

Solution. (i) Since 3¿1, we have $\log_5(3) > 0$. So

$$\lim_{n\to\infty} \frac{1}{n^{\log_5(3)}} = 0,$$

and $\boxed{\text{the sequence } \left\{\dfrac{1}{n^{\log_5(3)}}\right\}_n \text{ converges to the limit } 0}$.

(ii) Since 3¡5, we have $\log_5(3) < 1$. Therefore, by the *p*-series test,

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^{\log_5(3)}} \text{ diverges}}.$$

(d) $\left\{\cos\left(n^{1/n}\right)\right\}_{n=1}^{\infty}$

Solution. (i) We have

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{\ln(n)/n}$$

and

$$\lim_{x\to\infty} \frac{\ln(x)}{x} \overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0.$$

so $\lim_{n\to\infty} n^{1/n} = e^0 = 1$, and by the Continuous Function Theorem,

$$\lim_{n\to\infty} \cos\left(n^{1/n}\right) = \cos(1).$$

Hence, $\boxed{\text{the sequence } \left\{\cos\left(n^{1/n}\right)\right\}_n \text{ converges to the limit } \cos(1)}$.

(ii) Since the limit of the general term is $\cos(1) \neq 0$, the Term Divergence Test implies that

$$\boxed{\sum_{n=1}^{\infty} \cos\left(n^{1/n}\right) \text{ diverges}}.$$

(e) $\left\{\dfrac{1}{(n^2+9)^{3/2}}\right\}_{n=0}^{\infty}$

Solution. (i) We have

$$\lim_{n\to\infty} \frac{1}{(n^2+9)^{3/2}} = 0$$

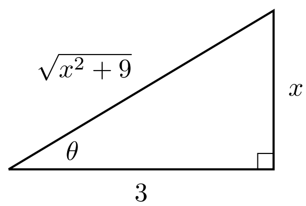
so $\boxed{\text{the sequence } \left\{\dfrac{1}{(n^2+9)^{3/2}}\right\}_n \text{ converges to the limit } 0}$.

(ii) We use the Integral Test. Put $f(x) = \frac{1}{(x^2+9)^{3/2}}$. Then f is positive, continuous and decreasing (because $y = (x^2+9)^{3/2}$ is increasing) on $[1, \infty)$. So the Integral Test applies.

To compute an antiderivative of f, we can use the trigonometric substitution $x = 3\tan(\theta)$, so that $dx = 3\sec(\theta)^2 d\theta$ and $x^2 + 9 = 9\tan(\theta)^2 + 9 = 9\sec(\theta)^2$. This gives

$$\begin{aligned}
\int \frac{dx}{(x^2+9)^{3/2}} &= \int \frac{3\sec(\theta)^2 d\theta}{(9\sec(\theta)^2)^{3/2}} \\
&= \frac{1}{9}\int \cos(\theta) d\theta \\
&= \frac{\sin(\theta)}{9} + C.
\end{aligned}$$

To express this result in terms of x, we use the right triangle for this trigonometric substitution, which has base angle θ so that $\tan(\theta) = \frac{x}{3}$ as shown below.



From this we see that $\sin(\theta) = \frac{x}{\sqrt{x^2+9}}$, so we obtain

$$\int \frac{dx}{(x^2+9)^{3/2}} = \frac{x}{9\sqrt{x^2+9}} + C.$$

We can now use this to compute the improper integral.

$$\begin{aligned}
\int_0^\infty \frac{dx}{(x^2+9)^{3/2}} &= \lim_{b\to\infty} \int_0^b \frac{dx}{(x^2+9)^{3/2}} \\
&= \lim_{b\to\infty} \left[\frac{x}{9\sqrt{x^2+9}}\right]_0^b \\
&= \lim_{b\to\infty} \frac{b}{9\sqrt{b^2+9}} \cdot \frac{\frac{1}{b}}{\frac{1}{b}} \\
&= \lim_{b\to\infty} \frac{1}{9\sqrt{1+9/b^2}} \\
&= \frac{1}{9}.
\end{aligned}$$

So the improper integral $\int_0^\infty \frac{dx}{(x^2+9)^{3/2}}$ converges. It follows that

$$\boxed{\sum_{n=0}^{\infty} \frac{1}{(n^2+9)^{3/2}} \text{ converges}}.$$

(f) $\left\{\sec\left(\frac{\pi}{n}\right) - \sec\left(\frac{\pi}{n+1}\right)\right\}_{n=3}^{\infty}$

Solution. (i) Since $\frac{\pi}{n}, \frac{\pi}{n+1} \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty}\left(\sec\left(\frac{\pi}{n}\right) - \sec\left(\frac{\pi}{n+1}\right)\right) = \sec(0) - \sec(0) = 0.$$

So $\boxed{\text{the sequence } \left\{\sec\left(\frac{\pi}{n}\right) - \sec\left(\frac{\pi}{n+1}\right)\right\}_n \text{ converges to the limit } 0}$.

(ii) To determine if the series converges or diverges, we can use the fact that this series is telescoping. The partial sums can be expressed as follows

$$\begin{aligned}
S_N &= \sum_{n=3}^{N} \left(\sec\left(\frac{\pi}{n}\right) - \sec\left(\frac{\pi}{n+1}\right)\right) \\
&= \left(\left(\sec\left(\frac{\pi}{3}\right) - \sec\left(\frac{\pi}{4}\right)\right) + \left(\sec\left(\frac{\pi}{4}\right) - \sec\left(\frac{\pi}{5}\right)\right) + \cdots + \left(\sec\left(\frac{\pi}{N}\right) - \sec\left(\frac{\pi}{N+1}\right)\right)\right) \\
&= \sec\left(\frac{\pi}{3}\right) - \sec\left(\frac{\pi}{N+1}\right) \\
&= 2 - \sec\left(\frac{\pi}{N+1}\right).
\end{aligned}$$

Therefore,

$$\sum_{n=3}^{\infty} \left(\sec\left(\frac{\pi}{n}\right) - \sec\left(\frac{\pi}{n+1}\right)\right) = \lim_{N\to\infty} \left(2 - \sec\left(\frac{\pi}{N+1}\right)\right) = 2 - \sec(0) = 2 - 1 = \boxed{1},$$

and in particular

$$\boxed{\sum_{n=3}^{\infty} \left(\sec\left(\frac{\pi}{n}\right) - \sec\left(\frac{\pi}{n+1}\right)\right) \text{ converges}}.$$

(g) $\left\{2^{2n+1}5^{-n}\right\}_{n=0}^{\infty}$

Solution. (i) Observe that we have

$$2^{2n+1}5^{-n} = 2\left(\frac{4}{5}\right)^n.$$

So we have a geometric sequence of common ratio $r = \frac{4}{5}$, which satisfies $|r| < 1$. So

$$\lim_{n\to\infty} 2^{2n+1}5^{-n} = 0,$$

and $\boxed{\text{the sequence } \left\{2^{2n+1}5^{-n}\right\}_n \text{ converges to the limit } 0}$.

(ii) Since $|r| = \frac{4}{5} < 1$, the geometric series $\boxed{\sum_{n=0}^{\infty} 2^{2n+1}5^{-n} \text{ converges}}$ and we can evaluate the sum as

$$\boxed{\sum_{n=0}^{\infty} 2^{2n+1}5^{-n} = \frac{2}{1-\frac{4}{5}} = 10}.$$

(h) $\left\{\left(1+\frac{1}{2n}\right)^n\right\}_{n=1}^{\infty}$

Solution. (i) The limit of this sequence is an indeterminate power 1^∞. We can write it in exponential form

$$\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^n = \lim_{n\to\infty} e^{n\ln\left(1+\frac{1}{2n}\right)}.$$

We can now compute the limit of the exponent using L'Hôpital's Rule as follows:

$$\begin{aligned}\lim_{n\to\infty} n\ln\left(1+\frac{1}{2n}\right) &= \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{2x}\right)}{\frac{1}{x}}\\ &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty}\frac{-\frac{1}{2x^2}\cdot\frac{1}{1+\frac{1}{2x}}}{-\frac{1}{x^2}}\\ &= \lim_{x\to\infty}\frac{1}{2\left(1+\frac{4}{x}\right)}\\ &= \frac{1}{2}.\end{aligned}$$

Therefore

$$\lim_{n\to\infty} e^{n\ln\left(1+\frac{1}{2n}\right)} = e^{1/2},$$

so $\boxed{\text{the sequence } \left\{\left(1+\frac{1}{2n}\right)^n\right\}_n \text{ converges to the limit } e^{1/2}}$.

(ii) Since the limit of the general term $\left(1+\frac{1}{2n}\right)^n$ is not zero, the Term Divergence Test tells us that

$$\boxed{\sum_{n=1}^{\infty}\left(1+\frac{1}{2n}\right)^n \text{ diverges}}.$$

(i) $\left\{\frac{1}{n\ln(n)\ln(\ln(n))}\right\}_{n=4}^{\infty}$

Solution. We have $n\ln(n)\ln(\ln(n))\to\infty$ as $n\to\infty$, so

$$\lim_{n\to\infty}\frac{1}{n\ln(n)\ln(\ln(n))} = 0,$$

and $\boxed{\text{the sequence } \left\{\frac{1}{n\ln(n)\ln(\ln(n))}\right\}_n \text{ converges to the limit } 0}$.

To determine whether the series converges or not, we can use the Integral Test with $f(x) = \frac{1}{x\ln(x)\ln(\ln(x))}$ which is continuous, positive and decreasing (because $y = x\ln(x)\ln(\ln(x))$ is increasing) on $[4, \infty)$. We now compute the improper integral.

$$\begin{aligned}
\int_4^\infty \frac{dx}{x\ln(x)\ln(\ln(x))} &= \lim_{b\to\infty}\int_4^b \frac{dx}{x\ln(x)\ln(\ln(x))} \\
&= \lim_{b\to\infty}\int_{\ln(\ln(4))}^{\ln(\ln(b))} \frac{du}{u} \qquad \left(u = \ln(\ln(x)), du = \frac{dx}{x\ln(x)}\right) \\
&= \lim_{b\to\infty} \left[\ln|u|\right]_{\ln(\ln(4))}^{\ln(\ln(b))} \\
&= \lim_{b\to\infty} \left(\ln(\ln(\ln(b))) - \ln(\ln(\ln(4)))\right) \\
&= \infty.
\end{aligned}$$

So the improper integral $\displaystyle\int_4^\infty \frac{dx}{x\ln(x)\ln(\ln(x))}$ diverges, from which we deduce that

$$\boxed{\sum_{n=4}^\infty \frac{1}{n\ln(n)\ln(\ln(n))} \text{ diverges}}.$$

2. (a) Determine for which values of p the series $\displaystyle\sum_{n=2}^\infty \frac{1}{n\ln(n)^p}$ converges or diverges.

Solution. We use the Integral Test. The function $f(x) = \dfrac{1}{x\ln(x)^p}$ is continuous, positive and decreasing (because $x\ln(x)^p$ is increasing) on $[2, \infty)$. Therefore, the Integral Test applies and we can test for convergence of the series by testing for convergence of the corresponding improper integral.

To compute the integral, we use the substitution $u = \ln(x)$, which gives $du = \frac{dx}{x}$. This gives

$$\begin{aligned}
\int_2^\infty \frac{dx}{x\ln(x)^p} &= \lim_{b\to\infty}\int_2^b \frac{dx}{x\ln(x)^p} \\
&= \lim_{b\to\infty}\int_{\ln(2)}^{\ln(b)} \frac{du}{u^p} \\
&= \int_{\ln(2)}^\infty \frac{du}{u^p}.
\end{aligned}$$

This last integral is a type I p-integral, so it converges if $p > 1$ and diverges if $p \leqslant 1$. Therefore,

$$\boxed{\sum_{n=2}^\infty \frac{1}{n\ln(n)^p} \text{ converges if } p > 1, \text{ diverges if } p \leqslant 1}.$$

(b) Determine for which values of p the series $\displaystyle\sum_{n=1}^\infty \frac{\ln(n)}{n^p}$ converges or diverges.

Solution. We use the Integral Test. The function $f(x) = \dfrac{\ln(x)}{x^p}$ is continuous and positive on $[2, \infty)$. We have

$$f'(x) = \frac{1}{x}\cdot\frac{1}{x^p} - \frac{p\ln(x)}{x^{p+1}} = \frac{1 - p\ln(x)}{x^{p+1}}.$$

Observe that $f'(x) < 0$ when $x > e^{1/p}$. So f is decreasing on $[e^{1/p}, \infty)$. Therefore, the Integral Test applies and we can test for convergence of the series by testing for convergence of the corresponding improper integral.

To compute the integral, we will need to distinguish the cases $p = 1$ and $p \neq 1$. When $p = 1$, we have

$$\begin{aligned} \int_1^\infty \frac{\ln(x)}{x} dx &= \lim_{b\to\infty} \int_1^b \frac{\ln(x)}{x} dx \\ &= \lim_{b\to\infty} \left[\frac{\ln(x)^2}{2}\right]_1^b \\ &= \lim_{b\to\infty} \frac{\ln(b)^2}{2} \\ &= \infty, \end{aligned}$$

so we conclude that the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^1}$ diverges.

When $p \neq 1$, we can use an IBP to compute an antiderivative. We will choose the parts

$$\begin{aligned} u = \ln(x) \ &\Rightarrow \ du = \frac{dx}{x}, \\ dv = x^{-p} dx \ &\Rightarrow \ v = \frac{x^{1-p}}{1-p}. \end{aligned}$$

This gives

$$\begin{aligned} \int \frac{\ln(x)}{x^p} dx &= \frac{\ln(x) x^{1-p}}{1-p} - \int \frac{x^{-p}}{1-p} dx \\ &= \frac{\ln(x) x^{1-p}}{1-p} - \frac{x^{1-p}}{(1-p)^2} + C \\ &= \frac{x^{1-p}\left((1-p)\ln(x) - 1\right)}{(1-p)^2} + C. \end{aligned}$$

So the improper integral is

$$\begin{aligned} \int_1^\infty \frac{\ln(x)}{x^p} dx &= \lim_{b\to\infty} \int_1^b \frac{\ln(x)}{x^p} dx \\ &= \lim_{b\to\infty} \left[\frac{x^{1-p}\left((1-p)\ln(x) - 1\right)}{(1-p)^2}\right]_1^b \\ &= \lim_{b\to\infty} \left(\frac{b^{1-p}\left((1-p)\ln(b) - 1\right)}{(1-p)^2} + \frac{1}{(1-p)^2}\right). \end{aligned}$$

In the case $1 - p > 0$, that is $p < 1$, we have

$$\lim_{b\to\infty} b^{1-p}\left((1-p)\ln(b) - 1\right) = \infty,$$

so the improper integral diverges. In the case $1 - p < 0$, that is $p > 1$, we have by L'Hôpital's Rule

$$\lim_{b\to\infty} b^{1-p}\left((1-p)\ln(b) - 1\right) = \lim_{b\to\infty} \frac{(1-p)\ln(b) - 1}{b^{p-1}} = \lim_{b\to\infty} \frac{\frac{(1-p)}{b}}{(p-1)b^{p-2}} = \lim_{b\to\infty} \frac{-1}{b^{p-1}} = 0,$$

so the improper integral converges.

In conclusion,

$$\boxed{\sum_{n=1}^{\infty} \frac{\ln(n)}{n^p} \text{ converges if } p > 1, \text{ diverges if } p \leqslant 1}.$$