Section 10.4

Comparison Tests

Learning Goals

Learning Goal	*Homework Problems*
10.4.1 Determine whether a series diverges or converges using either the Direct or the Limit Comparison Tests with a p-series	10.4: 1,4,5,10, 18,23,24, 34,45,47,56
10.4.2 Determine whether a series diverges or converges using either the Direct or the Limit Comparison Tests with a geometric series	10.4: 6,12,13,19

<u>Conceptual introduction</u>: for improper integrals, we learned the Direct Comparison Test and Limit Comparison Test to test for convergence. We are now going to learn similar tests for series. (They are almost the same.)

⚠ These test only work for series with <u>non-negative terms</u>.

<u>Direct Comparison Test</u>: assume that $0 \leq a_n \leq b_n$ for all n.

$$\boxed{\begin{array}{l} \text{If } \sum_{n=n_0}^{\infty} b_n \text{ converges, then } \sum_{n=n_0}^{\infty} a_n \text{ converges.} \\[2ex] \text{If } \sum_{n=n_0}^{\infty} a_n \text{ diverges, then } \sum_{n=n_0}^{\infty} b_n \text{ diverges.} \end{array}}$$

<u>Examples</u>: 1) Does the series $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ converge or diverge?

Observe that $0 \leq \frac{1}{n2^n} \leq \frac{1}{2^n}$ for $n \geq 1$.

We know that $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges (geometric series with common ratio $r = \frac{1}{2}$ and $|r| < 1$).

By the DCT, $\boxed{\sum_{n=1}^{\infty} \frac{1}{n2^n} \text{ converges}}$.

<u>Remark</u>: we also have $0 \leq \frac{1}{n2^n} \leq \frac{1}{n}$ for $n \geq 1$, but that does not tell us anything since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

2) Does the series $\sum_{n=1}^{\infty} \frac{\cos(3n)^2 + 4n}{n^{4/3}}$ converge or diverge ?

We have $\cos(3n)^2 \geq 0$ so $\frac{\cos(3n)^2+4n}{n^{4/3}} \geq \frac{4n}{n^{4/3}}$

$$\frac{\cos(3n)^2+4n}{n^{4/3}} \geq \frac{4}{n^{1/3}} \geq 0$$

We know that $\sum_{n=1}^{\infty} \frac{4}{n^{1/3}}$ diverges (p- series with $p = \frac{1}{3} \leq 1$).

So by the DCT, $\boxed{\sum_{n=1}^{\infty} \frac{\cos(3n)^2 + 4n}{n^{4/3}} \text{ diverges}}$.

3) Does the series $\sum_{n=2}^{\infty} \frac{n^2}{n^3 - n}$ converge or diverge ?

We have $0 < n^3 - n \leq n^3$ for all $n \geq 2$.

so $\frac{n^2}{n^3-n} \geq \frac{n^2}{n^3} \geq 0$

$\frac{n^2}{n^3-n} \geq \frac{1}{n} \geq 0$ for all $n \geq 2$.

We know that $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges (p- series with $p=1$).

So by the DCT, $\boxed{\sum_{n=2}^{\infty} \frac{n^2}{n^3-n} \text{ diverges}}$.

Limit Comparison Test: suppose that $a_n \geq 0$, $b_n > 0$ and

let $L = \lim_{n \to \infty} \frac{a_n}{b_n}$.

$\boxed{\text{If } 0 < L < \infty, \text{ then } \sum_{n=n_0}^{\infty} a_n \text{ and } \sum_{n=n_0}^{\infty} b_n \text{ either both converge or both diverge.}}$

$\boxed{\text{If } L = 0 \text{ and } \sum_{n=n_0}^{\infty} b_n \text{ converges, then } \sum_{n=n_0}^{\infty} a_n \text{ converges.}}$

$\boxed{\text{If } L = \infty \text{ and } \sum_{n=n_0}^{\infty} b_n \text{ diverges, then } \sum_{n=n_0}^{\infty} a_n \text{ diverges.}}$

Examples: 1) Does $\sum_{n=1}^{\infty} \frac{2n}{\sqrt[5]{n^7+4}}$ converge or diverge?

LCT: we find a reference series to compare to by keeping only the dominant terms:

$$\sum_{n=1}^{\infty} \frac{n}{n^{7/5}} = \sum_{n=1}^{\infty} \frac{1}{n^{2/5}} \quad \text{diverges} \quad (\text{p-series with } p = \tfrac{2}{5} \leq 1).$$

$$L = \lim_{n \to \infty} \frac{\frac{2n}{\sqrt[5]{n^7+4}}}{\frac{1}{n^{2/5}}}$$

series we are testing for

reference series

$$= \lim_{n \to \infty} \frac{2n^{7/5}}{\sqrt[5]{n^7+4}} \cdot \frac{\frac{1}{n^{7/5}}}{\frac{1}{n^{7/5}}} = \lim_{n \to \infty} \frac{2}{\sqrt[5]{1 + 4/n^7}} = 2$$

Since $0 < L < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{n^{2/5}}$ diverges, we conclude that

$$\boxed{\sum_{n=1}^{\infty} \frac{2n}{\sqrt[5]{n^7+4}} \text{ diverges}}.$$

2) Does $\sum_{n=1}^{\infty} \frac{\sin(\frac{1}{n})}{n}$ converge or diverge ?

Intuition : " $\sin(x) \approx x$ " for x close to 0. $\left(\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \right)$

So " $\sin\left(\frac{1}{n}\right) \approx \frac{1}{n}$ " for n large

$\Rightarrow \frac{\sin(\frac{1}{n})}{n} \approx \frac{1}{n^2}$ for n large.

We use the LCT with $b_n = \frac{1}{n^2}$.

$$L = \lim_{n \to \infty} \frac{\frac{\sin(\frac{1}{n})}{n}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = 1 \quad \text{(or use L'Hôpital's).}$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p- series with $p = 2 > 1$).

So $\boxed{\sum_{n=1}^{\infty} \frac{\sin(\frac{1}{n})}{n} \text{ converges}}$.

3) Does $\sum_{n=1}^{\infty} \frac{3^n}{2^n+1}$ converge or diverge ?

We compare with $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$, using the LCT.

$$\lim_{n \to \infty} \frac{\frac{3^n}{2^n+1}}{\left(\frac{3}{2}\right)^n} = \lim_{n \to \infty} \frac{2^n}{2^n+1} = \lim_{n \to \infty} \frac{1}{1+2^{-n}} = 1 > 0.$$

$\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ diverges

So by the LCT, $\boxed{\sum_{n=1}^{\infty} \frac{3^n}{2^n+1} \text{ diverges}}$.

Practice: use a comparison test to determine if the following series converge or diverge.

1) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ 2) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$ 3) $\displaystyle\sum_{n=3}^{\infty} \frac{n^2+4\cos(n)^2}{n^3+5n^2+2}$

4) $\displaystyle\sum_{n=1}^{\infty} \frac{2\sqrt[4]{n}+3}{(n^8+n^6+1)^{1/3}}$ 5) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{3n}+n^2}{5^n-3^n}$ 6) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln(n)}$

Solutions: 1) DCT: $\dfrac{\ln(n)}{n} \geq \dfrac{\ln(2)}{n} \geq 0$ for $n \geq 2$.

it is okay if the inequality is not valid for the first few terms.

We know that $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(2)}{n}$ diverges (p-series with $p=1$).

So $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n} \text{ diverges}}$ as well.

2) Intuition: $\dfrac{\ln(n)}{n^2}$ will be dominated by any $\dfrac{1}{n^p}$ with $p<2$.
So if we compare with $\dfrac{1}{n^p}$ with $1<p<2$, we will be able to prove convergence.
LCT with $b_n = \dfrac{1}{n^{3/2}}$. ← works with any exponent in $(1,2)$.

- $\displaystyle\lim_{n\to\infty} \frac{\frac{\ln(n)}{n^2}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{\ln(n)}{n^{1/2}} = \lim_{x\to\infty} \frac{\ln(x)}{x^{1/2}} \overset{L'H}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (p-series with $p=\frac{3}{2}>1$)

So $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2} \text{ converges}}$.

3) Intuition: " $\dfrac{n^2+4\cos(n)^2}{n^3+5n^2+2} \approx \dfrac{n^2}{n^3} = \dfrac{1}{n}$ " when n large.

LCT with $b_n = \dfrac{1}{n}$:

- $$\lim_{n\to\infty} \frac{\frac{n^2+4\cos(n)^2}{n^3+5n^2+2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3+4n\cos(n)^2}{n^3+5n^2+2} \cdot \frac{1/n^3}{1/n^3} = \lim_{n\to\infty} \frac{1+4\cos(n)^2/n^2}{1+5/n+2/n^3}$$

$$= \frac{1+0}{1+0+0} = 1$$

(we know that $\displaystyle\lim_{n\to\infty} \frac{4\cos(n)^2}{n^2} = 0$ by the Sandwich Theorem:

$0 \leq \dfrac{4\cos(n)^2}{n^2} \leq \dfrac{4}{n^2}$

$\displaystyle\lim_{n\to\infty} \frac{4}{n^2} = 0$

$\displaystyle\Rightarrow \lim_{n\to\infty} \frac{4\cos(n)^2}{n^2} = 0$)

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

So $\boxed{\displaystyle\sum_{n=3}^{\infty} \frac{n^2+4\cos(n)^2}{n^3+5n^2+2} \text{ diverges}}$.

4) " $\dfrac{2\sqrt[4]{n}+3}{(n^8+n^6+1)^{1/3}} \approx \dfrac{2n^{1/4}}{n^{8/3}} = \dfrac{2}{n^{29/12}}$ "

We use the LCT with $b_n = \dfrac{1}{n^{29/12}}$.

- $$\lim_{n\to\infty} \frac{\frac{2\sqrt[4]{n}+3}{(n^8+n^6+1)^{1/3}}}{\frac{1}{n^{29/12}}} = \lim_{n\to\infty} \frac{n^{29/12}(2n^{1/4}+3)}{(n^8+n^6+1)^{1/3}} = \lim_{n\to\infty} \frac{\cancel{n^{29/12}n^{1/4}}(2+3n^{-1/4})}{\cancel{n^{8/3}}(1+n^{-2}+n^{-8})^{1/3}}$$

$$= \lim_{n\to\infty} \frac{2+3n^{-1/4}}{1+n^{-2}+n^{-8}} = 2.$$

- $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^{29/12}}$ converges (p-series with $p = \frac{29}{12} > 1$)

So $\boxed{\displaystyle\sum_{n=3}^{\infty} \frac{n^2+4\cos(n)^2}{n^3+5n^2+2} \text{ converges}}$.

5) DCT: we have $2^{3n} + n^2 = 8^n + n^2 \geq 8^n$

$5^n - 3^n \leq 5^n$

$$\Rightarrow \frac{2^{3n} + n^2}{5^n - 3^n} \geq \frac{8^n}{5^n} = \left(\frac{8}{5}\right)^n \geq 0$$

$\sum_{n=1}^{\infty} \left(\frac{8}{5}\right)^n$ diverges (geometric series with $r = \frac{8}{5}$, $|r| > 1$)

So $\boxed{\sum_{n=1}^{\infty} \frac{2^{3n} + n^2}{5^n - 3^n} \text{ diverges}}$.

6) LCT with $b_n = \frac{1}{n^{3/4}}$ ← any exponent in $(\frac{1}{2}, 1)$ will work

- $$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}\ln(n)}}{\frac{1}{n^{3/4}}} = \lim_{n\to\infty} \frac{n^{3/4}}{\sqrt{n}\ln(n)} = \lim_{n\to\infty} \frac{n^{1/4}}{\ln(n)} = \lim_{x\to\infty} \frac{x^{1/4}}{\ln(x)}$$

  $$\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{\frac{1}{4}x^{-3/4}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x^{1/4}}{4} = \infty$$

- $\sum_{n=2}^{\infty} \frac{1}{n^{3/4}}$ diverges (p-series with $p = \frac{3}{4} \leq 1$).

So $\boxed{\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln(n)} \text{ diverges}}$.