Section 10.5

Absolute Convergence
Ratio and Root Tests

Learning Goals

Learning Goal	*Homework Problems*
10.5.1 Determine whether a series diverges or absolutely converges using the ratio/root tests.	10.5: 1,3,6,7,9,10,11,14,15,20,21, 22,35

Conceptual introduction: the convergence tests we have learned so far only apply to series with non-negative terms. We are now going to work with series that can have both positive and negative terms.

Definition: we say that $\sum_{n=n_0}^{\infty} a_n$ converges absolutely if the series $\sum_{n=n_0}^{\infty} |a_n|$ converges.

Absolute Convergence Test (ACT):

$$\boxed{\text{If } \sum_{n=n_0}^{\infty} |a_n| \text{ converges, then } \sum_{n=n_0}^{\infty} a_n \text{ converges.}}$$

So absolute convergence $\Rightarrow$ convergence. Said differently:

$$\boxed{\text{If } \sum_{n=n_0}^{\infty} a_n \text{ diverges, then } \sum_{n=n_0}^{\infty} |a_n| \text{ diverges.}}$$

Examples:

1) Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} \cdots$ converge or diverge?

Cannot use IT, DCT or LCT because terms are not all positive. Instead, we test for absolute convergence by looking at the "positive-term series":

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = 1 + \frac{1}{4} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$ converges (p-series with $p = 2 > 1$)

So $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \text{ converges absolutely}}$ (in particular converges by ACT).

2) Does the series $\sum_{n=1}^{\infty} \frac{\sin(n)}{3^n}$ converge or diverge ?

We test for absolute convergence:

$$\sum_{n=1}^{\infty} \left| \frac{\sin(n)}{3^n} \right| = \sum_{n=1}^{\infty} \frac{|\sin(n)|}{3^n}$$

Observe that
- $0 \leq \frac{|\sin(n)|}{3^n} \leq \frac{1}{3^n}$.
- $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges (geometric series with common ratio $r = \frac{1}{3}$, $|r| < 1$)

So $\sum_{n=1}^{\infty} \frac{|\sin(n)|}{3^n}$ converges by the DCT.

Therefore, $\boxed{\sum_{n=1}^{\infty} \frac{\sin(n)}{3^n} \text{ converges absolutely}}$.

Ratio Test: consider a series $\sum_{n=n_0}^{\infty} a_n$ and let $\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$.

> If $\rho < 1$, then $\sum_{n=n_0}^{\infty} a_n$ converges absolutely.
>
> If $\rho > 1$, then $\sum_{n=n_0}^{\infty} a_n$ diverges.
>
> If $\rho = 1$, then the test is inconclusive and does not tell us anything about $\sum_{n=n_0}^{\infty} a_n$.

Examples: 1) Does the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converge or diverge ?

Ratio Test: $\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \overbrace{\frac{(n+1)^2}{2^{n+1}}}^{a_{n+1}} \cdot \overbrace{\frac{2^n}{n^2}}^{\frac{1}{a_n}} \right| = \lim_{n \to \infty} \frac{2^n}{2^{n+1}} \cdot \left(\frac{n+1}{n}\right)^2$

$$= \lim_{n \to \infty} \frac{1}{2}\left(1 + \frac{1}{n}\right)^2 = \frac{1}{2}$$

Since $\rho < 1$, $\boxed{\sum_{n=1}^{\infty} \frac{n^2}{2^n} \text{ converges absolutely}}$.

2) Does the series $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$ converge or diverge?

Ratio Test: $\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \overbrace{\frac{(-3)^{n+1}}{(n+1)!}}^{a_{n+1}} \cdot \overbrace{\frac{n!}{(-3)^n}}^{\frac{1}{a_n}} \right|$

$$= \lim_{n\to\infty} \left| \frac{(-3)^{n+1}}{(-3)^n} \cdot \frac{n!}{(n+1)!} \right| \qquad \frac{n!}{(n+1)!} = \frac{\cancel{n}\cancel{(n-1)}\cdots\cancel{2}\cdot\cancel{1}}{(n+1)\cancel{n}\cancel{(n-1)}\cdots\cancel{2}\cdot\cancel{1}} = \frac{1}{n+1}$$

$$= \lim_{n\to\infty} \left| \frac{-3}{n+1} \right| = 0$$

Since $\rho < 1$, $\boxed{\sum_{n=1}^{\infty} \frac{(-3)^n}{n!} \text{ converges absolutely}}$.

3) Does the series $\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$ converge or diverge?

Ratio Test: $\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \overbrace{\frac{(2(n+1))!}{(n+1)!(n+1)!}}^{a_{n+1}} \cdot \overbrace{\frac{n!\,n!}{(2n)!}}^{\frac{1}{a_n}}$

$$= \lim_{n\to\infty} \frac{(2n+2)!}{(2n)!} \cdot \frac{n!}{(n+1)!} \cdot \frac{n!}{(n+1)!} = \lim_{n\to\infty} (2n+2)(2n+1) \cdot \frac{1}{n+1} \cdot \frac{1}{n+1}$$

$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2} = 4 > 1$ so $\boxed{\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!} \text{ diverges}}$.

4) Let a_n be the sequence defined inductively by

$$a_1 = 8, \qquad a_{n+1} = \frac{3n}{2n+1} a_n.$$

Does $\sum_{n=1}^{\infty} a_n$ converge or diverge?

Because this recursive relation gives us information about $\frac{a_{n+1}}{a_n}$, it is a good idea to try the Ratio Test.

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3n}{2n+1}\cancel{a_n}\cdot\frac{1}{\cancel{a_n}}\right| = \lim_{n\to\infty}\frac{3n}{2n+1} = \frac{3}{2}.$$

Since $\rho > 1$, we conclude that $\boxed{\sum_{n=1}^{\infty} a_n \text{ diverges}}$.

The ratio test is very helpful for series whose terms involve factorials / exponentials. It has the advantage of not needing to find a reference series to compare to. But it has the disadvantage of being often inconclusive (ex: for any series with algebraic terms, such as $\sum_{n=1}^{\infty}\frac{1}{n^p}$ or $\sum_{n=1}^{\infty}\frac{2n+1}{\sqrt{n^2+1}}$).

Root Test: consider a series $\sum_{n=n_0}^{\infty} a_n$ and let $\rho = \lim_{n\to\infty}\sqrt[n]{|a_n|}$

> If $\rho < 1$, then $\sum_{n=n_0}^{\infty} a_n$ converges absolutely.
>
> If $\rho > 1$, then $\sum_{n=n_0}^{\infty} a_n$ diverges.
>
> If $\rho = 1$, then the test is inconclusive and does not tell us anything about $\sum_{n=n_0}^{\infty} a_n$.

The following limits we saw in 10.1 will be helpful when using the Root Test:

$$\boxed{\lim_{n\to\infty} n^{1/n} = 1} \qquad \boxed{\lim_{n\to\infty}\left(1+\frac{c}{n}\right)^n = e^c \ (\text{any } c)}$$

(Show full details with L'Hôpital's Rule when using them.)

Examples: 1) Does the series $\sum_{n=1}^{\infty}\left(1-\frac{3}{n}\right)^{n^2}$ converge or diverge?

Root Test: $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty}\left(\left(1-\frac{3}{n}\right)^{n^2}\right)^{1/n} = \lim_{n\to\infty}\left(1-\frac{3}{n}\right)^{n}$

$= \lim_{n\to\infty} e^{n\ln\left(1-\frac{3}{n}\right)}$

We compute the limit of the exponent using L'Hôpital's Rule:

$$\lim_{n\to\infty} n\ln\left(1-\frac{3}{n}\right) = \lim_{x\to\infty}\frac{\ln\left(1-\frac{3}{x}\right)}{\frac{1}{x}} \underset{\frac{0}{0}}{\overset{L'H}{=}} \lim_{x\to\infty}\frac{\frac{1}{1-3/x}\cdot\frac{3}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{-3}{1-\frac{3}{x}} = -3.$$

So $\rho = \lim_{n\to\infty} e^{n\ln\left(1-\frac{3}{n}\right)} = e^{-3}.$

Since $\rho<1$, $\boxed{\sum_{n=1}^{\infty}\left(1-\frac{3}{n}\right)^{n^2} \text{ converges absolutely}}$.

2) Does the series $\sum_{n=1}^{\infty}\frac{5n^n}{2^n n^3}$ converge or diverge?

Root Test: $\rho = \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left(\frac{5n^n}{2^n n^3}\right)^{1/n} = \lim_{n\to\infty}\frac{5^{1/n}\cdot n}{2n^{3/n}} = \lim_{n\to\infty}\frac{5^{1/n}\cdot n}{2e^{\frac{3\ln(n)}{n}}}.$

We compute the limit of the exponent using L'Hôpital's Rule:

$$\lim_{x\to\infty}\frac{\ln(x)}{x} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = 0.$$

So $\rho = \lim_{n\to\infty}\frac{5^{1/n}\cdot n}{2e^{\frac{3\ln(n)}{n}}} = \text{“}\frac{5^0\cdot\infty}{2\cdot e^0}\text{”} = \infty.$

Since $\rho>1$, $\boxed{\sum_{n=1}^{\infty}\frac{n^n}{2^n n^3} \text{ diverges}}$.

3) Does the series $\sum_{n=1}^{\infty} \left(\frac{n+3}{2n+1}\right)^n$ converge or diverge?

Root Test: $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\left(\frac{n+3}{2n+1}\right)^n\right)^{1/n} = \lim_{n\to\infty} \frac{n+3}{2n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$

$= \lim_{n\to\infty} \frac{1+\frac{3}{n}}{2+\frac{1}{n}} = \frac{1}{2}.$

Since $\rho < 1$, $\boxed{\sum_{n=1}^{\infty} \left(\frac{n+3}{2n+1}\right)^n \text{ converges absolutely}}$.

What to choose between ratio and root test? If the general term involves factorials, ratio will be better. Otherwise, root test is stronger than ratio test.

Practice: determine if the following series converge or diverge.

1) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ 2) $\sum_{n=1}^{\infty} \frac{5^{n^2}}{n^{2n}}$ 3) $\sum_{n=1}^{\infty} \frac{(-1)^n n^2 (n+2)!}{n!\, 3^{2n}}$

4) $\sum_{n=1}^{\infty} a_n$ where $a_1 = 2$, $a_{n+1} = \left(1+\frac{1}{n}\right)^n$.

Solutions:

1) Ratio Test: $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n\to\infty} \frac{(n+1)(n+1)^n}{n^n} \cdot \frac{n!}{(n+1)!}$

$= \lim_{n\to\infty} \cancel{(n+1)}\left(\frac{n+1}{n}\right)^n \frac{1}{\cancel{n+1}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} e^{n\ln\left(1+\frac{1}{n}\right)}$

Exponent: $\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \overset{L'H}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot \cancel{\frac{-1}{x^2}}}{\cancel{-\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1.$

So $\rho = \lim_{n\to\infty} e^{n\ln\left(1+\frac{1}{n}\right)} = e^1 = e.$

Since $\rho > 1$, $\boxed{\sum_{n=1}^{\infty} \frac{n^n}{n!} \text{ diverges}}$.

2) Root Test: $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{5^{n^2}}{n^{2n}}\right)^{1/n} = \lim_{n\to\infty} \frac{5^n}{n^2} = \lim_{x\to\infty} \frac{5^x}{x^2}$

$\underset{\frac{\infty}{\infty}}{\overset{L'H}{=}} \lim_{x\to\infty} \frac{\ln(5)\cdot 5^x}{2x} \underset{\frac{\infty}{\infty}}{\overset{L'H}{=}} \lim_{x\to\infty} \frac{\ln(5)^2\cdot 5^x}{2} = \infty$

Since $\rho > 1$, $\boxed{\sum_{n=1}^{\infty} \frac{5^{n^2}}{n^{2n}} \text{ diverges}}$.

3) Ratio Test: $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2 (n+3)!}{(n+1)!\, 3^{2n+2}} \cdot \frac{n!\, 3^{2n}}{n^2 (n+2)!}$

$= \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot \frac{3^{2n}}{3^{2n+2}} \cdot \frac{n!}{(n+1)!} \cdot \frac{(n+3)!}{(n+2)!} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{9} \cdot \frac{1}{n+1}(n+3) = \frac{1}{9} < 1$

Since $\rho < 1$, $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n n^2 (n+2)!}{n!\, 3^{2n}} \text{ converges absolutely}}$.

4) Ratio Test: $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\cancel{a_n}\left(1+\frac{1}{n}\right)^n}{\cancel{a_n}}\right|$

$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} e^{n\ln\left(1+\frac{1}{n}\right)}$.

Exponent: $\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \underset{\frac{0}{0}}{\overset{L'H}{=}} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot \cancel{\frac{-1}{x^2}}}{\cancel{-\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1.$

So $\rho = \lim_{n\to\infty} e^{n\ln\left(1+\frac{1}{n}\right)} = e^1 = e.$

Since $\rho > 1$, $\boxed{\sum_{n=1}^{\infty} a_n \text{ diverges}}$.