Rutgers University
Math 152

Section 10.5: Absolute Convergence, Ratio & Root Tests - Worksheet Solutions

1. Determine if the series below converge or diverge. Make sure to clearly label and justify the use of any convergence test used. **Note:** some of these problems require convergence tests from previous sections.

   (a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{3^{2n}}$

   Solution. We use the Root Test. We have

$$\begin{aligned}\rho &= \lim_{n\to\infty} \sqrt[n]{\left|(-1)^n \frac{n^n}{3^{2n}}\right|}\\ &= \lim_{n\to\infty} \frac{n}{9}\\ &= \infty.\end{aligned}$$

   Since $\rho > 1$, we conclude that $\boxed{\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{3^{2n}} \text{ diverges}}$.

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{8n^6+7n+11}}{3n^7+8n^5-1}$

   Solution. We use the LCT with $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n^{6/3}}{n^7} = \sum_{n=1}^{\infty} \frac{1}{n^5}$, which converges as a p-series with $p = 5 > 1$. We have

$$\begin{aligned}L &= \lim_{n\to\infty} \frac{a_n}{b_n}\\ &= \lim_{n\to\infty} \frac{\frac{\sqrt[3]{8n^6+7n+11}}{3n^7+8n^5-1}}{\frac{n^{6/3}}{n^7}}\\ &= \lim_{n\to\infty} \frac{\sqrt[3]{8n^6+7n+11}}{3n^7+8n^5-1} \cdot \frac{n^7}{n^{6/3}}\\ &= \lim_{n\to\infty} \frac{\sqrt[3]{8+\frac{7}{n^5}+\frac{11}{n^6}}}{3+\frac{8}{n^2}-\frac{1}{n^7}}\\ &= \frac{\sqrt[3]{8+0+0}}{3+0-0}\\ &= \frac{2}{3}.\end{aligned}$$

   Since $0 < L < \infty$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$ converges, we conclude that $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{8n^6+7n+11}}{3n^7+8n^5-1} \text{ converges}}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)!}{e^n n!(n+1)!}$

Solution. We use the Ratio Test. We have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \frac{(2(n+1)+1)!}{e^{n+1}(n+1)!(n+2)!} \cdot \frac{e^n n!(n+1)!}{(2n+1)!} \\
&= \lim_{n\to\infty} \frac{e^n}{e^{n+1}} \cdot \frac{(2n+3)!}{(2n+1)!} \cdot \frac{n!(n+1)!}{(n+1)!(n+2)!} \\
&= \lim_{n\to\infty} \frac{(2n+2)(2n+3)}{e(n+1)(n+2)} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \\
&= \lim_{n\to\infty} \frac{(2+\frac{2}{n})(2+\frac{3}{n})}{e(1+\frac{1}{n})(1+\frac{2}{n})} \\
&= \frac{4}{e}.
\end{aligned}$$

Since $e < 4$, we have $\rho > 1$. So $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)!}{e^n n!(n+1)!} \text{ diverges}}$.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(8n)+3}{4^n}$

Solution. We use the DCT. Observe that $-1 \leqslant \cos(8n) \leqslant 1$, so $2 \leqslant \cos(8n)+3 \leqslant 4$. It follows that

$$0 < \frac{\cos(8n)+3}{4^n} < \frac{4}{4^n}.$$

Furthermore, $\displaystyle\sum_{n=1}^{\infty} \frac{4}{4^n}$ converges as a geometric series with common ratio $r = \frac{1}{4}$ satisfying $|r| < 1$.
Therefore, $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{\cos(8n)+3}{4^n} \text{ converges}}$.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{\ln(\ln(n))}$

Solution. Observe that

$$\begin{aligned}
\lim_{n\to\infty} \frac{\ln(n)}{\ln(\ln(n))} &= \lim_{x\to\infty} \frac{\ln(x)}{\ln(\ln(x))} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x\ln(x)}} \\
&= \lim_{x\to\infty} \ln(x) \\
&= \infty.
\end{aligned}$$

Since the general term of the series does not approach 0, the Term Divergence Test tells us that $\boxed{\sum_{n=1}^{\infty} \frac{\ln(n)}{\ln(\ln(n))} \text{ diverges}}$.

(f) $\sum_{n=1}^{\infty} 4^n \left(\frac{n-2}{n}\right)^{n^2}$

Solution. We use the Root Test. We have

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left(4^n \left(\frac{n-2}{n}\right)^{n^2}\right)^{1/n} \\ &= \lim_{n\to\infty} 4 \left(\frac{n-2}{n}\right)^{n} \\ &= 4 \lim_{n\to\infty} \left(1 - \frac{2}{n}\right)^{n}.\end{aligned}$$

This limit is an indeterminate power 1^{∞}. We can start by writing the power in exponential form

$$\lim_{n\to\infty} \left(1 - \frac{2}{n}\right)^{n} = \lim_{n\to\infty} e^{n \ln\left(1-\frac{2}{n}\right)}.$$

We now compute the limit of the exponent using L'Hôpital's Rule:

$$\begin{aligned}\lim_{n\to\infty} n \ln\left(1 - \frac{2}{n}\right) &= \lim_{x\to\infty} \frac{\ln\left(1-\frac{2}{x}\right)}{\frac{1}{x}} \\ &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \frac{\frac{2}{x^2} \cdot \frac{1}{1-\frac{2}{x}}}{-\frac{1}{x^2}} \\ &= \lim_{x\to\infty} -\frac{2}{1-\frac{2}{x}} \\ &= -2.\end{aligned}$$

Therefore

$$\rho = 4 \lim_{n\to\infty} e^{n\left(1-\frac{2}{n}\right)} = 4e^{-2}.$$

Since $e > 2$, we have $e^2 > 4$ and therefore $\rho < 1$. Hence, $\boxed{\sum_{n=1}^{\infty} 4^n \left(\frac{n-2}{n}\right)^{n^2} \text{ converges absolutely}}$.

(g) $\sum_{n=1}^{\infty} (-1)^n \frac{((2n)!)^2}{(4n)!}$

Solution. We use the Ratio Test. We have

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty} \frac{((2n+2)!)^2}{(4n+4)!} \cdot \frac{(4n)!}{((2n)!)^2}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \left(\frac{(2n+2)!}{(2n)!}\right)^2 \cdot \frac{(4n)!}{(4n+4)!} \\
&= \lim_{n\to\infty} \frac{(2n+2)^2(2n+1)^2}{(4n+1)(4n+2)(4n+3)(4n+4)} \cdot \frac{\frac{1}{n^4}}{\frac{1}{n^4}} \\
&= \lim_{n\to\infty} \frac{(2+\frac{2}{n})^2(2+\frac{1}{n})^2}{(4+\frac{1}{n})(4+\frac{2}{n})(4+\frac{3}{n})(4+\frac{4}{n})} \\
&= \frac{2^2 2^2}{4^4} \\
&= \frac{1}{16}.
\end{aligned}
$$

Since $\rho < 1$, we conclude that $\boxed{\sum_{n=1}^{\infty} (-1)^n \frac{((2n)!)^2}{(4n)!} \text{ converges absolutely}}$.

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^n(n+2)!}$

Solution. We use the Ratio Test. We have

$$
\begin{aligned}
\rho &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{3^{n+1}(n+3)!} \cdot \frac{3^n(n+2)!}{n^n} \\
&= \lim_{n\to\infty} \frac{(n+1)(n+1)^n}{3(n+3)n^n} \\
&= \lim_{n\to\infty} \frac{n+1}{3(n+3)} \left(1+\frac{1}{n}\right)^n \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{3(1+\frac{3}{n})} e^{n\ln\left(1+\frac{1}{n}\right)}.
\end{aligned}
$$

We can compute the limit of the exponent using L'Hôpital's Rule:

$$
\begin{aligned}
\lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) &= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \frac{-\frac{1}{x^2}\cdot\frac{1}{1+\frac{1}{x}}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} \\
&= 1.
\end{aligned}
$$

Therefore

$$
\rho = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{3(1+\frac{3}{n})} e^{n\ln\left(1+\frac{1}{n}\right)} = \frac{1+0}{3(1+0)} e^1 = \frac{e}{3}.
$$

Since $e < 3$, we have $\rho < 1$ so $\boxed{\sum_{n=1}^{\infty} \frac{n^n}{3^n(n+2)!} \text{ converges absolutely}}$.

(i) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n + 5\sin(n)}{3n}\right)^n$

Solution. We use the Root Test. We have

$$\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{2n + 5\sin(n)}{3n} = \lim_{n\to\infty} \left(\frac{2}{3} + \frac{5\sin(n)}{3n}\right).$$

We have $-1 \leqslant \sin(n) \leqslant 1$, so

$$-\frac{5}{3n} \leqslant \frac{5\sin(n)}{3n} \leqslant \frac{5}{3n}.$$

Furthermore, $\lim_{n\to\infty} -\frac{5}{3n} = \lim_{n\to\infty} \frac{5}{3n} = 0$. So by the Sandwich Theorem, we have $\lim_{n\to\infty} \frac{5\sin(n)}{3n} = 0$ and

$$\rho = \lim_{n\to\infty} \left(\frac{2}{3} + \frac{5\sin(n)}{3n}\right) = \frac{2}{3}.$$

Since $\rho < 1$, we conclude that $\boxed{\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n + 5\sin(n)}{3n}\right)^n \text{ converges absolutely}}$.

2. Let a_n be the sequence defined recursively by

$$a_1 = 7, \quad a_{n+1} = a_n \left(\frac{n}{n+3}\right)^n \text{ for } n \geqslant 1.$$

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges or diverges. Make sure to clearly label and justify the use of any convergence test used.

Solution. The recursive relation gives us information about $\dfrac{a_{n+1}}{a_n}$, so the Ratio Test seems like a good option here. We have

$$\begin{aligned} \rho &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty} \left|\frac{a_n \left(\frac{n}{n+3}\right)^n}{a_n}\right| \\ &= \lim_{n\to\infty} \left(\frac{n}{n+3}\right)^n. \end{aligned}$$

This limit is an indeterminate power 1^∞. We can start by writing the power in exponential form

$$\rho = \lim_{n\to\infty} \left(\frac{n}{n+3}\right)^n = \lim_{n\to\infty} e^{n\ln\left(\frac{n}{n+3}\right)}.$$

We now compute the limit of the exponent using L'Hôpital's Rule:

$$\lim_{n\to\infty} n\ln\left(\frac{n}{n+3}\right) = \lim_{x\to\infty} \frac{\ln(x) - \ln(x+3)}{\frac{1}{x}}$$

$$
\begin{aligned}
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \frac{\frac{1}{x} - \frac{1}{x+3}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} -x^2 \frac{(x+3)-x}{x(x+3)} \\
&= \lim_{x\to\infty} -\frac{3x}{x+3} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x\to\infty} -\frac{3}{1+3/x} \\
&= -3.
\end{aligned}
$$

So

$$\rho = \lim_{n\to\infty} e^{n\ln\left(\frac{n}{n+3}\right)} = e^{-3}.$$

Since $\rho < 1$, we conclude that $\boxed{\sum_{n=1}^{\infty} a_n \text{ converges absolutely}}$.