Section 10.6

Alternating Series and Conditional Convergence

Learning Goals

Learning Goal	*Homework Problems*
10.6.1 Determine whether a series diverges, conditionally coverges or absolutely converges	10.6: 1, 5, 6, 25, 27, 29, 31, 35, 39, 40, 41, 48, 59, 65, 66, 69, , 85, 88, 94
10.6.2 Estimate the remainder of an alternating series	Questions 14, 15, and 16 in MyLab Assignment for §10.6

Conceptual introduction: recall that $\sum_{n=n_0}^{\infty} a_n$ converges absolutely when $\sum_{n=n_0}^{\infty} |a_n|$ converges.

We have seen that: (Absolute Convergence) $\Rightarrow$ (Convergence). However, the converse is false: a series may converge, but not converge absolutely.

Definition: if $\sum_{n=n_0}^{\infty} a_n$ converges but does not converge absolutely, we say that it converges conditionally.

So $\sum_{n=n_0}^{\infty} a_n$ converges conditionally if
- $\sum_{n=n_0}^{\infty} a_n$ converges
- $\sum_{n=n_0}^{\infty} |a_n|$ diverges.

Summary of terminology:

$\sum_{n=n_0}^{\infty} a_n$	$\sum_{n=n_0}^{\infty} \|a_n\|$	Description of $\sum_{n=n_0}^{\infty} a_n$
Converges	Converges	Converges Absolutely
Converges	Diverges	Converges Conditionally
Diverges	Diverges	Diverges.

We have yet to see examples of series that converge conditionally. Common examples are some alternating series.

Definition: an alternating series is a series of the form

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

or

$$\sum_{n=1}^{\infty} (-1)^{n} a_n = -a_1 + a_2 - a_3 + a_4 - \cdots$$

with $a_n > 0$ for all n.

So it is a series where the sign of terms alternates between positive and negative.

Examples:

$$\sum_{n=1}^{\infty} (-1)^n = -1+1-1+1-1+\cdots$$

$$\sum_{n=2}^{\infty} (-1)^n n = 2-3+4-5-6+\cdots$$

$$\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

are alternating

But $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(n)}{n+1}$ is not (signs do not alternate with a period of 2)

Alternating Series Test: let $\{a_n\}_{n=n_0}^{\infty}$ be a sequence such that:

(i) $a_n \geq 0$

(ii) $\{a_n\}$ is decreasing: $a_{n+1} \leq a_n$ for all $n \geq n_0$.

(iii) $\lim_{n\to\infty} a_n = 0$

Then $\boxed{\text{the alternating series } \sum_{n=n_0}^{\infty} (-1)^n a_n, \ \sum_{n=n_0}^{\infty} (-1)^{n-1} a_n \text{ converge.}}$

Examples: 1) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$: take $a_n = \frac{1}{n}$, then

(i) $\frac{1}{n} \geq 0$

(ii) $\frac{1}{n+1} \leq \frac{1}{n}$ and (iii) $\lim_{n\to\infty} \frac{1}{n} = 0.$

So by the AST, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

However, $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p- series with $p=1$).

Therefore, $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ converges conditionally}}$.

2) For which of these series can you apply the AST ?

a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\tan^{-1}(n)}$ b) $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n\, 2^n}$ c) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3}$

a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\tan^{-1}(n)}$: $a_n = \frac{1}{\tan^{-1}(n)}$ is positive, decreasing,

but $\lim_{n\to\infty} \frac{1}{\tan^{-1}(n)} = \frac{1}{\pi/2} = \frac{2}{\pi} \neq 0,$ so AST does not apply.

Term Divergence Test : $\lim_{n\to\infty} \frac{(-1)^n}{\tan^{-1}(n)} \neq 0,$ so
$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{\tan^{-1}(n)} \text{ diverges}}$.

b) $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n\, 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n\, 2^n}$ since $\cos(\pi n) = (-1)^n$.

$a_n = \frac{1}{n 2^n}$ is positive, decreasing (since reciprocal $n2^n$ is increasing) and $\lim_{n\to\infty} \frac{1}{n2^n} = 0.$ So the AST applies and
$\boxed{\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n\, 2^n} \text{ converges}}$.

Does it converge absolutely or conditionally ?

$\sum_{n=1}^{\infty} \left| \frac{\cos(\pi n)}{n 2^n} \right| = \sum_{n=1}^{\infty} \frac{1}{n 2^n}$ DCT : $0 \leqslant \frac{1}{n2^n} \leqslant \frac{1}{2^n}$ and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges (geometric series with common ratio $r = \frac{1}{2}$, $|r| < 1$).

So $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n 2^n}$ converges absolutely.

<u>Remark:</u> we could have tested for absolute converge directly with the DCT without doing the AST first. We could have also used the Ratio or Root Tests.

c) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3}$ is not alternating, so AST does not apply.

We can use the DCT to test for absolute convergence:

$$0 \leqslant \left| \frac{\sin(n)}{n^3} \right| \leqslant \frac{1}{n^3}$$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (p-series with $p = 3 > 1$).

So $\boxed{\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3} \text{ converges absolutely}}$.

<u>Important remarks:</u>

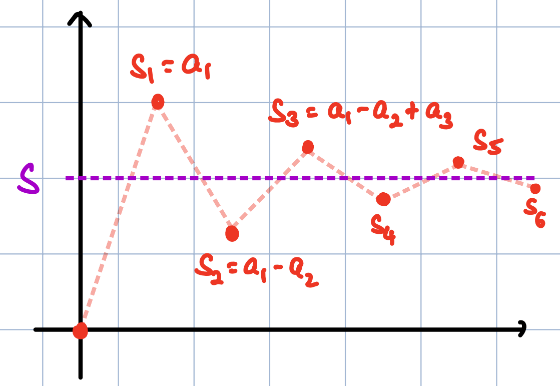
- The AST can never be used to show that a series diverges.
- Just because the AST does not apply does not mean that the series diverges.
- When the AST applies, we can conclude that the series converges, but we do not know if the convergence is absolute or conditional without further analysis.

Ex: $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ AST applies to both.

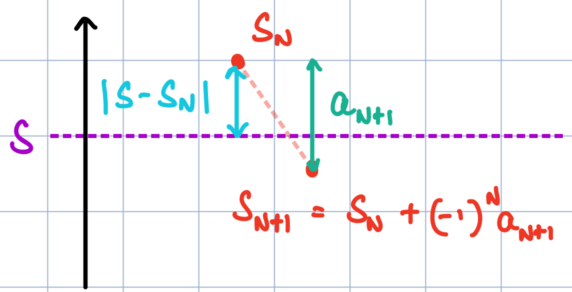
converges conditionally.   converges absolutely.

Approximating the sum of an alternating series.

Consider an alternating series $S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$ with $\{a_n\}_n$ positive, decreasing and $\lim_{n \to \infty} a_n = 0$.



The sequence of partial sums $\{S_N\}_N$ oscillates back and forth across the sum S with decreasing amplitude.



The error made when estimating S using S_N is at most:

$$\boxed{\underbrace{|S - S_N|}_{\text{error}} \leq a_{N+1}}$$

Alternating Series Estimation Theorem

$S - S_N = \sum_{n=N+1}^{\infty} (-1)^{n-1} a_n$ is called the remainder R_N.

Upshot: if an alternating series meets the conditions of the AST, then a_{N+1} is the best estimate of the error $|S - S_N|$.

Examples: 1) The alternating series $\sum_{n=4}^{\infty} \frac{(-1)^n}{\sqrt{3n}}$ converges (conditionally) by the AST + p-test.

How many terms must be summed for the remainder to be less than 0.1 in magnitude?

We are looking for N such that $|S - S_N| < 0.1$. We know that $|S - S_N| \leq a_{N+1}$, so it suffices to have $a_{N+1} < 0.1$

$$\Rightarrow \frac{1}{\sqrt{3(N+1)}} < 10^{-1}$$

$\Rightarrow \sqrt{3N+3} > 10 \Rightarrow 3N+3 > 100 \Rightarrow 3N > 97 \Rightarrow N > \frac{97}{3} \approx 32.33\ldots$

So the smallest value of N for which the remainder is less than 0.1 is $N = 33$.

$S_{33} = \sum_{n=4}^{33} \frac{(-1)^n}{\sqrt{3n}}$ has $33 - 4 + 1 = \boxed{30 \text{ terms}}$.

2) How many terms of $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ must be summed for the partial sum to approximate S with an error of less than 10^{-3}?

We want $a_{N+1} < 10^{-3} \Rightarrow \frac{1}{(N+1)!} < 10^{-3} \Rightarrow (N+1)! > 1000$

We cannot solve this algebraically, so we solve numerically with a calculator / table of values.

N	$(N+1)!$	
0	1	×
1	2	×
2	6	×
3	24	×
4	120	×
5	720	×
6	5040	✓

The smallest value of N that works is $N = 6$. So the partial sum $S_6 = \sum_{n=0}^{6} \frac{(-1)^n}{n!}$ has $6 - 0 + 1 = \boxed{7 \text{ terms}}$.

Practice: determine if the following series converge absolutely, converge conditionally or diverge.

1) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ 2) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{2n+1}$ 3) $\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2 3^n}{(2n+1)!}$

4) $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ 5) $\sum_{n=2}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$

1) AST : $a_n = \frac{1}{\ln(n)}$ is positive, decreasing (since ln increasing) and $\lim_{n\to\infty} \frac{1}{\ln(n)} = 0$.

So $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ converges.

Observe that $\ln(n) \leq n$ if $n \geq 2$

so $\left|\frac{(-1)^n}{\ln(n)}\right| \geq \frac{1}{n} \geq 0$, and $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges (p- series with $p=1$).

Therefore, $\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{\ln(n)}\right|$ diverges.

So $\boxed{\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)} \text{ converges conditionally}}$.

2) $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$, so $\lim_{n\to\infty} \frac{(-1)^n n}{2n+1}$ DNE and by the

Term Divergence Test, $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n n}{2n+1} \text{ diverges}}$.

3) Ratio Test :

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{((n+1)!)^2 3^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{(n!)^2 3^n} = \lim_{n\to\infty} \frac{3(n+1)^2}{(2n+2)(2n+3)} = \frac{3}{4}$$

Since $\rho < 1$, $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2 3^n}{(2n+1)!} \text{ converges absolutely}}$.

4) DCT : $0 \leq \left|\frac{\cos(n)}{n^2}\right| \leq \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series with $p=2>1$).

So $\boxed{\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2} \text{ converges absolutely}}$.

5) AST : $a_n = \sin\left(\frac{1}{n}\right)$ is positive, decreasing (because sin is increasing on $\left(0, \frac{\pi}{2}\right)$) and $\lim\limits_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin(0) = 1$.

So $\sum\limits_{n=2}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$ converges.

To test for absolute convergence, we use the LCT with $b_n = \frac{1}{n}$.

- $\lim\limits_{n\to\infty} \dfrac{\left|(-1)^n \sin\left(\frac{1}{n}\right)\right|}{\frac{1}{n}} = \lim\limits_{n\to\infty} \dfrac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim\limits_{u\to 0} \dfrac{\sin(u)}{u} = 1.$
- $\sum\limits_{n=2}^{\infty} \frac{1}{n}$ diverges (p-series with $p=1$).

So $\sum\limits_{n=2}^{\infty} \left|(-1)^n \sin\left(\frac{1}{n}\right)\right|$ diverges.

Therefore, $\boxed{\sum\limits_{n=2}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right) \text{ converges conditionally}}$.