Rutgers University
Math 152

Section 10.6: Alternating Series & Conditional Convergence - Worksheet Solution

1. Determine if the series below converge absolutely, converge conditionally or diverge. Make sure to clearly label and justify the use of any convergence test used. **Note:** some of these problems require convergence tests from previous sections.

   (a) $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{n\log_2(n)}$

   Solution. We can prove the convergence of this series using the AST. The sequence $a_n = \frac{1}{n\log_2(n)}$ is positive when $n \geqslant 3$, decreasing (since $n\log_2(n)$ is increasing) and $\lim\limits_{n\to\infty} \frac{1}{n\log_2(n)} = 0$. So the AST applies and $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{n\log_2(n)}$ converges.

   We need to determine if the convergence is absolute or conditional, that is, we need to determine whether the series
$$\sum_{n=3}^{\infty} \left|\frac{(-1)^n}{n\log_2(n)}\right| = \sum_{n=3}^{\infty} \frac{1}{n\log_2(n)}$$
   converges or diverges. To this end, we can use the Integral Test. The function $f(x) = \dfrac{1}{x\log_2(x)}$ is continuous, positive and decreasing (because $x\log_2(x)^2$ is increasing) on $[3,\infty)$. Therefore, the Integral Test applies and we can test for convergence of the series by testing for convergence of the corresponding improper integral.

   To compute the integral, we use the substitution $u = \log_2(x)$, which gives $du = \frac{dx}{\ln(2)x}$. This gives
$$\begin{aligned}\int_3^{\infty} \frac{dx}{x\log_2(x)} &= \lim_{b\to\infty}\int_3^{b} \frac{dx}{x\log_2(x)}\\ &= \lim_{b\to\infty}\int_{\log_2(3)}^{\log_2(b)} \frac{\ln(2)du}{u}\\ &= \ln(2)\int_{\log_2(3)}^{\infty}\frac{du}{u}.\end{aligned}$$

   This last integral is a type I p-integral with $p=1$, so it diverges. Therefore, $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\log_2(n)}$ diverges.

   In conclusion, $\boxed{\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{n\log_2(n)} \text{ converges conditionally}}$.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!}$

Solution. We use the Ratio Test. We have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| && = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \\
&= \lim_{n\to\infty} \frac{2}{n+1} \\
&= 0.
\end{aligned}$$

Since $\rho < 1$, we conclude that $\boxed{\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!} \text{ converges absolutely}}$.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{n \arctan(n)}{\sqrt[3]{8n^6+1}}$

Solution. Note that this series has non-negative terms. We use the LCT with $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n}{\sqrt[3]{n^6}} = \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges as a p-series with $p = 1$. We have

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{a_n}{b_n} \\
&= \lim_{n\to\infty} \frac{\frac{n\arctan(n)}{\sqrt[3]{8n^6+1}}}{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{n^2 \arctan(n)}{\sqrt[3]{8n^6+1}} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \\
&= \lim_{n\to\infty} \frac{\arctan(n)}{\sqrt[3]{8+\frac{1}{n^6}}} \\
&= \frac{\frac{\pi}{2}}{\sqrt[3]{8}} \\
&= \frac{\pi}{4}.
\end{aligned}$$

Since $0 < L < \infty$ and $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we deduce that $\boxed{\displaystyle\sum_{n=0}^{\infty} \frac{n\arctan(n)}{\sqrt[3]{8n^6+1}} \text{ diverges}}$.

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n + \cos(n)}$

Solution. Note that this series has non-negative terms. We use the LCT with $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{1}{3^n}$, which converges as a geometric series with common ratio $r = \frac{1}{3}$, $|r| < 1$. We have

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{\frac{1}{3^n+\cos(n)}}{\frac{1}{3^n}} \\
&= \lim_{n\to\infty} \frac{3^n}{3^n+\cos(n)} \cdot \frac{\frac{1}{3^n}}{\frac{1}{3^n}} \\
&= \lim_{n\to\infty} \frac{1}{1+\frac{\cos(n)}{3^n}}.
\end{aligned}
$$

Because $-1 \leqslant \cos(n) \leqslant 1$, we have

$$-\frac{1}{3^n} \leqslant \frac{\cos(n)}{3^n} \leqslant \frac{1}{3^n}$$

and $\lim_{n\to\infty} -\frac{1}{3^n} = \lim_{n\to\infty} \frac{1}{3^n} = 0$. By the Sandwich Theorem, we obtain $\lim_{n\to\infty} \frac{\cos(n)}{3^n} = 0$ and

$$L = \lim_{n\to\infty} \frac{1}{1+\frac{\cos(n)}{3^n}} = \frac{1}{1+0} = 1.$$

Since $0 < L < \infty$ and $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{1}{3^n}$ converges, we deduce that $\boxed{\sum_{n=0}^{\infty} \frac{1}{3^n+\cos(n)} \text{ converges absolutely}}$.

(e) $\sum_{n=2}^{\infty} \frac{\sec(\pi n)}{\sqrt{n}}$

Solution. Note that $\sec(\pi n) = (-1)^n$ for any integer n. So

$$\sum_{n=2}^{\infty} \frac{\sec(\pi n)}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}},$$

and the series is alternating. Let us use the AST. The sequence $a_n = \frac{1}{\sqrt{n}}$ is positive, decreasing (since $\sqrt{n}$ is increasing) and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. Therefore, the AST applies and $\sum_{n=2}^{\infty} \frac{\sec(\pi n)}{\sqrt{n}}$ converges.

We need to determine if the convergence is absolute or conditional, so we consider the series

$$\sum_{n=2}^{\infty} \left|\frac{\sec(\pi n)}{\sqrt{n}}\right| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}.$$

This series is a p-series with $p = \frac{1}{2} \leqslant 1$, so it diverges. In conclusion, $\boxed{\sum_{n=2}^{\infty} \frac{\sec(\pi n)}{\sqrt{n}} \text{ converges conditionally}}$.

(f) $\sum_{n=2}^{\infty} (-1)^n \ln\left(\frac{n+1}{n}\right)$

Solution. Let us use the AST with $a_n = \ln\left(\frac{n+1}{n}\right)$. The sequence a_n has positive terms since $\frac{n+1}{n} > 1$, so $\ln\left(\frac{n+1}{n}\right) > \ln(1) = 0$. The sequence a_n is decreasing since

$$\frac{d}{dx} \ln\left(\frac{x+1}{x}\right) = \frac{d}{dx}\left(\ln(x+1) - \ln(x)\right) = \frac{1}{x+1} - \frac{1}{x} = -\frac{1}{x(x+1)} < 0 \quad \text{for} \quad x > 0.$$

Finally, observe that

$$\lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right) = \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right) = \ln(1+0) = \ln(1) = 0.$$

Therefore, the AST applies and $\displaystyle\sum_{n=2}^{\infty} (-1)^n \ln\left(\frac{n+1}{n}\right)$ converges.

We need to determine if the convergence is absolute or conditional, so we consider the series

$$\sum_{n=2}^{\infty} \left|(-1)^n \ln\left(\frac{n+1}{n}\right)\right| = \sum_{n=2}^{\infty} \ln\left(\frac{n+1}{n}\right) = \sum_{n=2}^{\infty} \left(\ln(n+1) - \ln(n)\right).$$

This series looks telescoping, and inspecting the partial sums, we see that

$$\begin{aligned} S_N &= \sum_{n=2}^{N} \left(\ln(n+1) - \ln(n)\right) \\ &= (\ln(3) - \ln(2)) + (\ln(4) - \ln(3)) + \cdots + (\ln(N) - \ln(N-1)) + (\ln(N+1) - \ln(N)) \\ &= \ln(N+1) - \ln(2). \end{aligned}$$

Therefore,

$$\sum_{n=2}^{\infty} (\ln(n+1) - \ln(n)) = \lim_{N\to\infty} S_N = \lim_{N\to\infty} (\ln(N+1) - \ln(2)) = \infty.$$

So $\displaystyle\sum_{n=2}^{\infty} (\ln(n+1) - \ln(n))$ diverges, and $\boxed{\displaystyle\sum_{n=2}^{\infty} (-1)^n \ln\left(\frac{n+1}{n}\right) \text{ converges conditionally}}$.

(g) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{e^{\sqrt{n}}}$

Solution. This series is a bit tricky because it is not geometric. Indeed, the exponent of e is $\sqrt{n}$, and not n. The Root Test is also inconclusive since

$$\lim_{n\to\infty} \left(\frac{1}{e^{\sqrt{n}}}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{e^{\sqrt{n}/n}} = \lim_{n\to\infty} \frac{1}{e^{1/\sqrt{n}}} = \frac{1}{e^0} = 1.$$

Still, we expect the series to converge because $e^{\sqrt{n}}$ grows faster than any power of n (even though it grows slower than the exponential e^n). This hints that we might be able to prove convergence by comparing with a convergent p-series.

So let us use the LCT with $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which converges as a p-series with $p = \frac{3}{2} > 1$. We have

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{a_n}{b_n} \\ &= \lim_{n\to\infty} \frac{\frac{1}{e^{\sqrt{n}}}}{\frac{1}{n^{3/2}}} \\ &= \lim_{x\to\infty} \frac{x^{3/2}}{e^{\sqrt{x}}} \end{aligned}$$

$$
\begin{aligned}
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \frac{\frac{3}{2}x^{1/2}}{\frac{1}{2}x^{-1/2}e^{\sqrt{x}}} \\
&= \lim_{x\to\infty} \frac{3x}{e^{\sqrt{x}}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \frac{3}{\frac{1}{2}x^{-1/2}e^{\sqrt{x}}} \\
&= \lim_{x\to\infty} \frac{6\sqrt{x}}{e^{\sqrt{x}}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \frac{3x^{-1/2}}{\frac{1}{2}x^{-1/2}e^{\sqrt{x}}} \\
&= \lim_{x\to\infty} \frac{6}{e^{\sqrt{x}}} \\
&= 0.
\end{aligned}
$$

Since $L = 0$ and $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, we conclude that $\boxed{\displaystyle\sum_{n=0}^{\infty} \frac{1}{e^{\sqrt{n}}} \text{ converges absolutely}}$.

(h) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n}{2n+1}$

Solution. This series is alternating, but the AST does not apply. Indeed, we have

$$\lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2}.$$

It follows that $\lim_{n\to\infty} (-1)^n \frac{n}{2n+1}$ does not exists. Therefore, the Term Divergence Test tells us that $\boxed{\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n}{2n+1} \text{ diverges}}$.

(i) $\displaystyle\sum_{n=3}^{\infty} \cos\left(\frac{\pi}{n}\right)^{n^2}$

Solution. Given the exponent n^2, the Root Test is tempting here, but it would turn out to be inconclusive (try it). Let us try to directly compute the limit of the general term to see if the Term Divergence Test would apply. The limit $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right)^{n^2}$ is an indeterminate power 1^∞. Let us write it as

$$\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right)^{n^2} = \lim_{n\to\infty} e^{n^2 \ln\left(\cos\left(\frac{\pi}{n}\right)\right)}$$

and compute the limit of the exponent using L'Hôpital's Rule. We have

$$
\begin{aligned}
\lim_{n\to\infty} n^2 \ln\left(\cos\left(\frac{\pi}{n}\right)\right) &= \lim_{x\to\infty} \frac{\ln\left(\cos\left(\frac{\pi}{x}\right)\right)}{\frac{1}{x^2}} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty} \frac{\frac{\pi}{x^2}\tan\left(\frac{\pi}{x}\right)}{-\frac{2}{x^3}} \\
&= \lim_{x\to\infty} -\frac{\pi\tan\left(\frac{\pi}{x}\right)}{\frac{2}{x}}
\end{aligned}
$$

$$\begin{aligned}
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty} -\frac{-\frac{\pi^2}{x^2}\sec\left(\frac{\pi}{x}\right)^2}{-\frac{2}{x^2}} \\
&= \lim_{x\to\infty} -\frac{\pi^2}{2}\sec\left(\frac{\pi}{x}\right)^2 \\
&= -\frac{\pi^2}{2}\sec(0)^2 \\
&= -\frac{\pi^2}{2}.
\end{aligned}$$

So

$$\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right)^{n^2} = \lim_{n\to\infty} e^{n^2 \ln\left(\cos\left(\frac{\pi}{n}\right)\right)} = e^{-\pi^2/2}.$$

Since this limit is not equal to zero, the Term Divergence Test tells us that $\boxed{\sum_{n=3}^{\infty} \cos\left(\frac{\pi}{n}\right)^{n^2} \text{ diverges}}$.

2. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{7n+4}}$.

   (a) Show that this series meets the conditions of the Alternating Series Estimation Theorem.

   Solution. The sequence $a_n = \frac{1}{\sqrt[3]{7n+4}}$ satisfies the following conditions.

   - $\frac{1}{\sqrt[3]{7n+4}} > 0$ for any $n \geqslant 1$.
   - The sequence $a_n = \frac{1}{\sqrt[3]{7n+4}}$ is decreasing since $\sqrt[3]{7n+4}$ is increasing.
   - $\lim_{n\to\infty} \frac{1}{\sqrt[3]{7n+4}} = 0$.

   Therefore, $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{7n+4}} \text{ meets the conditions of the Alternating Series Estimation Theorem}}$.

   (b) Find the smallest integer N for which the partial sum $S_N = \displaystyle\sum_{n=1}^{N} \frac{(-1)^n}{\sqrt[3]{7n+4}}$ approximates the sum of the series with an error of at most 0.1.

   Solution. The Alternating Series Estimation Theorem tells us that the best estimate for the error is $|S - S_N| \leqslant a_{N+1}$. Therefore, we will want $a_{N+1} \leqslant 0.1$. This gives

$$\begin{aligned}
&\frac{1}{\sqrt[3]{7(N+1)+4}} \leqslant 0.1 \\
&\Rightarrow \sqrt[3]{7N+11} \geqslant 10 \\
&\Rightarrow 7N + 11 \geqslant 1000 \\
&\Rightarrow 7N \geqslant 989 \\
&\Rightarrow N \geqslant \frac{989}{7} \simeq 141.3.
\end{aligned}$$

   Therefore, the smallest value of N giving us the desired error is $\boxed{N = 142}$.

3. Consider the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{3n-7}+9}$.

(a) Show that this series meets the conditions of the Alternating Series Estimation Theorem.

Solution. The sequence $a_n = \frac{1}{2^{3n-7}+9}$ satisfies the following conditions.

- $\frac{1}{2^{3n-7}+9} > 0$ for any $n \geqslant 0$.
- The sequence $a_n = \frac{1}{2^{3n-7}+9}$ is decreasing since $2^{3n-7} + 11$ is increasing.
- $\lim\limits_{n\to\infty} \frac{1}{2^{3n-7}+9} = 0$.

Therefore, $\boxed{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{3n-7}+9} \text{ meets the conditions of the Alternating Series Estimation Theorem}}$.

(b) Find the smallest integer N for which the partial sum $S_N = \displaystyle\sum_{n=0}^{N} \frac{(-1)^{n+1}}{2^{3n-7}+9}$ approximates the sum of the series with an error of at most 10^{-3}.

Solution. The Alternating Series Estimation Theorem tells us that the best estimate for the error is $|S - S_N| \leqslant a_{N+1}$. Therefore, we will want $a_{N+1} \leqslant 10^{-3}$. This gives

$$\begin{aligned}
&\frac{1}{2^{3(N+1)-7}+9} \leqslant 10^{-3} \\
&\Rightarrow\ 2^{3N-4} + 9 \geqslant 1000 \\
&\Rightarrow 2^{3N-4} \geqslant 991 \\
&\Rightarrow 3N - 4 \geqslant 10 \quad (2^9 = 512, 2^{10} = 1024) \\
&\Rightarrow N \geqslant \frac{14}{3} \simeq 4.7.
\end{aligned}$$

Therefore, the smallest value of N giving us the desired error is $\boxed{N = 5}$.