Section 10.7

Power Series

Learning Goals

10.7.1 Find the interval and radius of convergence for a power series	10.7: 2,4,5,8,14,15,22,23,26,30,37,39, 40
10.7.2 Find the interval of convergence and using Theorem 20	10.7: 41,44,45,48
10.7.3 Find the power series representation of a rational function using the power series representation of 1/(1+x)	10.7: 50
10.7.4 Find the power series representation of a new function by using the power series representation of other functions by either term by term differentiation or integration	10.7: 53,54,56,60; Also: 10.10: 61
10.7.5 Using some information on convergence at some values to find the convergence on other values.	10.7:62

Conceptual introduction :

A <u>polynomial</u> is a function of x of the form
$$c_0 + c_1 x + c_2 x^2 + \cdots + c_d x^d = \sum_{n=0}^{d} c_n x^n .$$

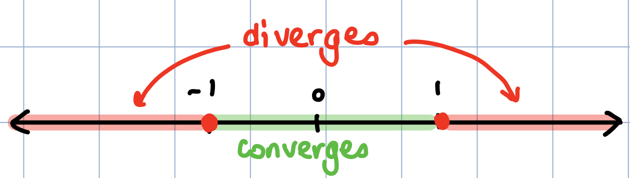
A <u>power series (about $x=0$)</u> is an infinite series of the form
$$c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots = \sum_{n=0}^{\infty} c_n x^n$$

A <u>power series about $x=a$</u> is an infinite series of the form
$$c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \cdots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

The <u>interval of convergence</u> (IOC) of a power series is the interval consisting of all values of x for which the series converges. It is an interval <u>centered at a</u>, whose radius R is called the <u>radius of convergence</u> (ROC).

<u>Examples</u> : 1) $f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$.
Find the radius and interval of convergence of f.

$f(x) = \sum_{n=0}^{\infty} x^n$ is a geometric series with common ratio x.
So f converges when $|x| < 1 \Rightarrow -1 < x < 1$.



So $\boxed{\begin{array}{c} IOC = (-1,1) \\ R = 1 \end{array}}$

Sum of the series : $\boxed{f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}$ for $-1 < x < 1$.

2) $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{n 3^n} = \frac{-(x-1)}{3} + \frac{(x-1)^2}{2 \cdot 9} - \frac{(x-1)^3}{3 \cdot 27} + \cdots$

Find the radius and interval of convergence of f.

This time, f is not a geometric series. We can try use the Ratio Test.

$$\rho = \lim_{n \to \infty} \left| \frac{(-1)^{n+1} (x-1)^{n+1}}{(n+1)\, 3^{n+1}} \cdot \frac{n 3^n}{(-1)^n (x-1)^n} \right| = \lim_{n\to\infty} \frac{n|x-1|}{3(n+1)} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{|x-1|}{3(1+\frac{1}{n})} = \frac{|x-1|}{3}.$$

Series converges absolutely when

$$\rho = \frac{|x-1|}{3} < 1 \Rightarrow |x-1| < 3 \Rightarrow -3 < x-1 < 3 \Rightarrow -2 < x < 4.$$

Series diverges when $\rho = \frac{|x-1|}{3} > 1$, i.e. $x < -2$ or $x > 4$.

Test inconclusive when $\rho = \frac{|x-1|}{3} = 1$, i.e $x = -2, 4$.

So we must test $x = -2, 4$ separately.

- At $x = -2$: $\sum_{n=1}^{\infty} \frac{(-1)^n (-2-1)^n}{n 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n (-3)^n}{n 3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p- series with $p = 1$)

- At $x = 4$: $\sum_{n=1}^{\infty} \frac{(-1)^n (4-1)^n}{n 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the AST since $\left\{\frac{1}{n}\right\}$ is decreasing, positive and $\lim_{n\to\infty} \frac{1}{n} = 0$.

Conclusion:

$$\boxed{\begin{array}{l} IOC = (-2, 4] \\ R = 3 \end{array}}$$

Remark: what if we change f to:

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{n^2 3^n} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{(-1)^n n (x-1)^n}{3^n} \; ?$$

Radius stays the same but behavior at endpoints changes.

- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{n^2 3^n} \xrightarrow{x=-2} \sum_{n=1}^{\infty} \frac{1}{n^2}$, $\quad \xrightarrow{x=4} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ $\Bigg\}$ both converge (p-series, $p=2>1$), so $\boxed{IOC = [-2,4]}$.

- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n (x-1)^n}{3^n} \xrightarrow{x=-2} \sum_{n=1}^{\infty} n$, $\quad \xrightarrow{x=4} \sum_{n=1}^{\infty} (-1)^n n$ $\Bigg\}$ both diverge (Term Divergence Test) so $\boxed{IOC = (-2,4)}$.

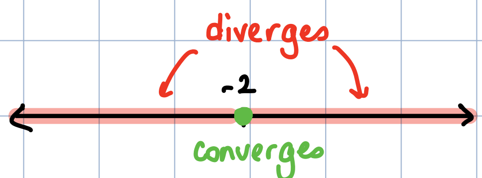
3) $f(x) = \displaystyle\sum_{n=0}^{\infty} n!(x+2)^n = 1 + (x+2) + 2(x+2)^2 + 6(x+2)^3 + 24(x+2)^4 + \cdots$

Find the radius and interval of convergence of f.

We use the Ratio Test.

$$\rho = \lim_{n\to\infty} \left| \frac{(n+1)!\,(x+2)^{n+1}}{n!\,(x+2)^n} \right| = \lim_{n\to\infty} (n+1)|x+2| = \begin{cases} 0 & \text{if } x=-2 \\ \infty & \text{if } x \neq -2 \end{cases}$$

So f converges for $x=-2$ only.



$$\boxed{\begin{aligned} IOC &= \{2\} \\ R &= 0 \end{aligned}}$$

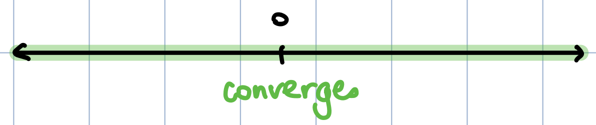
4) $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^n} = x + \frac{x^2}{4} + \frac{x^3}{27} + \cdots$

Find the radius and interval of convergence of f.

We use the Root Test.

$$\rho = \lim_{n\to\infty} \sqrt[n]{\left| \frac{x^n}{n^n} \right|} = \lim_{n\to\infty} \frac{|x|}{n} = 0$$

So f converges for any value of x.



$$\boxed{\begin{array}{l} IOC = (-\infty, \infty) \\ R = \infty \end{array}}$$

Important remarks:

- A power series always converges absolutely in the interior of the IOC.
- If $0 < R < \infty$, anything can happen at the endpoints. Series may converge absolutely, converge conditionally or diverge $\Rightarrow$ test case by case.
- Ratio / Root Tests are helpful to find RoC, but always inclusive at endpoints.

Example: if $f(x) = \sum_{n=0}^{\infty} c_n x^n$ converges at $x = 2$ and diverges at $x = -5$, what can we say about convergence or divergence at $x = -1, 7, -2$?

We know the radius R is at least 2, at most 5.

$x = -1$: absolute convergence since $|x| < R$.

$x = 7$: diverges since $|x| > R$.

$x = -2$: cannot conclude anything.

Operations on power series :

- **Substitution** : if $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely for $|x|<R$

  then $\sum_{n=0}^{\infty} c_n f(x)^n$ converges absolutely for $|f(x)|<R$.

Example: Find a power series representation of $g(x) = \frac{3}{4+x^2}$.

We can write g as a power series using the power series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ converges absolutely if $|x|<1$

$$g(x) = \frac{3}{4+x^2} = \frac{3}{4} \cdot \frac{1}{1-\left(-\frac{x^2}{4}\right)} = \frac{3}{4} \sum_{n=0}^{\infty} \left(\frac{-x^2}{4}\right)^n = \frac{3}{4} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n}$$, and this

converges absolutely if $\left|-\frac{x^2}{4}\right| < 1$, i.e. $|x^2| < 4$

i.e $\boxed{|x| < 2.}$

- **Term-by-term differentiation and integration:**

If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R>0$,

then

$$\boxed{f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}}$$

can omit $n=0$ since the term would be 0

$$\boxed{\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{c_n (x-a)^{n+1}}{n+1} + C}$$

and both series have same ROC as f R.

⚠ IOC / behavior at endpoints may change.

Examples: 1) Find a power series representation of $\frac{1}{(1-x)^2}$ and give its RoC.

We know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad R=1$

So $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{n=0}^{\infty} \frac{d}{dx} x^n$ (term-by-term differentiation)

$$\boxed{\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}, \quad R=1}.$$

2) Find a power series representation of $\ln(1+x)$ and find its RoC and IoC.

We know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x|<1$

$\Rightarrow \frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$ for $|-x|<1 \Rightarrow R=1.$

$\Rightarrow \int \frac{dx}{1+x} = \sum_{n=0}^{\infty} (-1)^n \int x^n dx$ (term-by-term integration)

$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$ with $R=1.$

To find C, plug-in $x=0$: $\ln(1) = C$ so $C=0$.

Conclusion:

$$\boxed{\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

We already know $\boxed{R=1}$ since term-by-term integration does not change the RoC. To find the IoC, we must test the endpoints $x = -1, 1$

- At $x = 1$: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges by AST

- At $x = -1$: $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{-1}{n}$ diverges (p-series, $p=1$)

So $\boxed{\text{IOC} = (-1, 1]}$.

3) Find a power series representation of $\tan^{-1}(x)$ and find its ROC and IOC.

We know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$

$$\Rightarrow \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad \text{for } |-x^2| < 1 \Rightarrow R = 1.$$

$$\Rightarrow \int \frac{dx}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n \int x^{2n}$$

(term-by-term integration)

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C \quad \text{with } R = 1.$$

To find C, plug-in $x = 0$: $\tan^{-1}(0) = C$ so $C = 0$.

Conclusion :

$$\boxed{\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

We already know $\boxed{R = 1}$. Test endpoints $x = -1, 1$:

- At $x = 1$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$
- At $x = -1$: $\sum_{n=0}^{\infty} \frac{-(-1)^n}{2n+1}$

$\left.\right\}$ both converge by AST $\Rightarrow \boxed{\text{IOC} = [-1, 1]}$.

Practice :

1) Find the radius and interval of convergence of the following power series.

a) $\sum_{n=0}^{\infty} \frac{(x+3)^{2n}}{4^n \sqrt{n}}$ b) $\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$ c) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n(n^2+1)}$

2) Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{(n!)^2}{2^n (2n)!} x^n$

3) a) Find a power series representation of $\frac{1}{2-3x}$ centered at $a=-1$ and give its interval of convergence.

b) Find a power series representation of $\ln(2-3x)$ centered at $a=-1$ and give its interval of convergence.

4) Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = 1 + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \cdots$

Solutions :

1) a) Root Test:

$$\rho = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(x+3)^{2n}}{4^n\sqrt{n}}\right|} = \lim_{n\to\infty} \frac{|x+3|^2}{4\, n^{1/2n}} = \frac{|x+3|^2}{4}$$

The series converges when $\frac{|x+3|^2}{4} < 1 \Rightarrow |x+3|^2 < 4$

$\Rightarrow |x+3| < 2 \Rightarrow -5 < x < -1$

So the radius of convergence is $\boxed{R=2}$

We need to test for convergence at the endpoints $x=-5,-1$.

At $x=-5$: $\sum_{n=0}^{\infty} \frac{(-5+3)^{2n}}{4^n\sqrt{n}} = \sum_{n=0}^{\infty} \frac{(-2)^{2n}}{4^n\sqrt{n}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series $p=1/2$)

At $x=-1$: $\sum_{n=0}^{\infty} \frac{(-1+3)^{2n}}{4^n\sqrt{n}} = \sum_{n=0}^{\infty} \frac{2^{2n}}{4^n\sqrt{n}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series, $p=1/2$)

So the IOC is $\boxed{(-5,-1)}$.

b) Ratio Test:

$$\rho = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(2(n+1))!} \cdot \frac{(2n)!}{x^n} \right| = \lim_{n\to\infty} |x| \frac{(2n)!}{(2n+2)!} = \lim_{n\to\infty} \frac{|x|}{(2n+1)(2n+2)} = 0$$

So the series converges absolutely for any x.

Therefore, $\boxed{R = \infty}$, interval of convergence = $\boxed{(-\infty, \infty)}$

c) Root Test:

$$\rho = \lim_{n\to\infty} \sqrt[n]{\left| \frac{(x-2)^n}{3^n (n^2+1)} \right|} = \lim_{n\to\infty} \frac{|x+2|}{3(n^2+1)^{1/n}} = \lim_{n\to\infty} \frac{|x+2|}{3\, n^{2/n} (1+1/n^2)^{1/n}} = \frac{|x+2|}{3}$$

So the series converges absolutely when $\frac{|x+2|}{3} < 1$

$|x+2| < 3$

$-5 < x < 1. \Rightarrow \boxed{R = 3}$.

Test endpoints:

At $x = -5$: $\sum_{n=0}^{\infty} \frac{(-5+2)^n}{3^n (n^2+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$ converges absolutely

by the DCT since $0 \leq \left| \frac{(-1)^n}{n^2+1} \right| \leq \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p- series with $p = 2 > 1$)

At $x = 1$: $\sum_{n=0}^{\infty} \frac{(1+2)^n}{3^n (n^2+1)} = \sum_{n=0}^{\infty} \frac{1}{n^2+1}$ converges (same argument as before).

So the interval of convergence is $\boxed{[-5, 1]}$.

2) Ratio Test:

$$\rho = \lim_{n\to\infty} \left| \frac{((n+1)!)^2 x^{n+1}}{2^{n+1} (2n+2)!} \cdot \frac{2^n (2n)!}{(n!)^2 x^n} \right| = \lim_{n\to\infty} \frac{(n+1)^2 |x|}{2(2n+1)(2n+2)} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{(1+\frac{1}{n})^2 |x|}{2(2+\frac{1}{n})(2+\frac{2}{n})} = \frac{|x|}{8}$$

So $\sum_{n=1}^{\infty} \frac{(n!)^2}{2^n (2n)!} x^n$ converges absolutely when $\frac{|x|}{8} < 1$, i.e. $|x| < 8$

diverges if $\frac{|x|}{8} > 1$, i.e. $|x| > 8$.

So $\boxed{R = 8}$.

3) a) $$\frac{1}{2-3x} = \frac{1}{5-3(x+1)} = \frac{1}{5} \cdot \frac{1}{1-\frac{3(x+1)}{5}} = \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{3(x+1)}{5}\right)^n = \boxed{\sum_{n=0}^{\infty} \frac{3^n (x+1)^n}{5^{n+1}}}$$

Interval of convergence: $\left|\frac{3(x+1)}{5}\right| < 1 \Rightarrow -\frac{5}{3} < x+1 < \frac{5}{3}$

$-\frac{8}{3} < x < \frac{2}{3}$

$\Rightarrow \boxed{\left(-\frac{8}{3}, \frac{2}{3}\right)}$

b) $$\frac{1}{2-3x} = \sum_{n=0}^{\infty} \frac{3^n (x+1)^n}{5^{n+1}}$$

integrate term-by-term

$$\int \frac{dx}{2-3x} = \sum_{n=0}^{\infty} \frac{3^n}{5^{n+1}} \int (x+1)^n dx$$

$$-\frac{\ln(2-3x)}{3} = \sum_{n=0}^{\infty} \frac{3^n (x+1)^{n+1}}{5^{n+1}(n+1)} +$$

$$\ln(2-3x) = -\sum_{n=0}^{\infty} \frac{3^{n+1}(x+1)^{n+1}}{5^{n+1}(n+1)} + C$$

Plug in $x = -1 \Rightarrow C = \ln(5)$. So

$$\boxed{\begin{array}{c} \ln(2-3x) = \ln(5) - \sum_{n=1}^{\infty} \frac{3^n (x+1)^n}{5^n n} \\ \text{Interval of convergence: } \left[-\frac{8}{3}, \frac{2}{3}\right) \end{array}}$$

4) $$\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = \sum_{n=1}^{\infty} n x^{n-1} \quad \text{with} \quad x = \frac{1}{2}.$$

Let $f(x) = \sum_{n=1}^{\infty} n x^{n-1}$. Then $\int f(x) dx = \sum_{n=1}^{\infty} n \int x^{n-1} dx$

$$= \sum_{n=1}^{\infty} x^n + C$$

$$= \frac{1}{1-x} + C$$

So $f(x) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$

Therefore $\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = f\left(\frac{1}{2}\right) = \frac{1}{\left(1-\frac{1}{2}\right)^2} = \boxed{4}$.