Rutgers University
Math 152

Section 10.7: Power Series - Worksheet Solutions

1. Find the radius and interval of convergence of the power series below. Specify for which values of x in the interval of convergence the series converges absolutely and for which it converges conditionally.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{\sqrt[3]{n}5^n}$.

   Solution. We use the Ratio Test to find the radius of convergence. We have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{\sqrt[3]{n+1}\ 5^{n+1}} \cdot \frac{\sqrt[3]{n}\ 5^n}{(x-3)^n} \right| \\
&= \lim_{n\to\infty} \frac{|x-3|\sqrt[3]{n}}{5\sqrt[3]{n+1}} \cdot \frac{\frac{1}{\sqrt[3]{n}}}{\frac{1}{\sqrt[3]{n}}} \\
&= \lim_{n\to\infty} \frac{|x-3|}{5\sqrt[3]{1+\frac{1}{n}}} \\
&= \frac{|x-3|}{5}.
\end{aligned}$$

   The power series converges absolutely when $\rho < 1$, that is

$$\frac{|x-3|}{5} < 1 \ \Rightarrow\ |x-3| < 5 \ \Rightarrow\ -5 < x-3 < 5 \ \Rightarrow -2 < x < 8.$$

   Therefore the radius of convergence is $\boxed{R = 5}$.

   To find the interval of convergence, we need to determine if the power series converges at the endpoints $x = -2, 8$.

   - At $x = -2$, the power series becomes

$$\sum_{n=1}^{\infty} \frac{(-2-3)^n}{\sqrt[3]{n}5^n} = \sum_{n=1}^{\infty} \frac{(-5)^n}{\sqrt[3]{n}5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}.$$

     This series converges by the AST since $a_n = \frac{1}{\sqrt[3]{n}}$ is positive, decreasing and converges to 0. However, the series does not converge absolutely since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt[3]{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}},$$

     which is a divergent p-series with $p = \frac{1}{3} \leqslant 1$. Therefore, the power series converges conditionally at $x = -2$.

- At $x = 8$, the power series becomes

$$\sum_{n=1}^{\infty} \frac{(8-3)^n}{\sqrt[3]{n}5^n} = \sum_{n=1}^{\infty} \frac{5^n}{\sqrt[3]{n}5^n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}.$$

  As we noted above, this is a divergent p-series with $p = \frac{1}{3}$.

In conclusion, the IOC is $\boxed{[-2, 8)}$. Furthermore, the power series $\boxed{\text{converges absolutely on } (-2, 8)}$ and $\boxed{\text{converges conditionally at } x = -2}$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x-9)^{3n}}{8^n (n+1)}$.

Solution. We use the Ratio Test to find the radius of convergence. We have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-9)^{3(n+1)}}{8^{n+1}(n+2)} \cdot \frac{8^n(n+1)}{(-1)^n (x-9)^{3n}} \right| \\
&= \lim_{n\to\infty} \frac{|x-9|^3 (n+1)}{8(n+2)} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{|x-9|^3 \left(1 + \frac{1}{n}\right)}{8\left(1 + \frac{2}{n}\right)} \\
&= \frac{|x-9|^3}{8}.
\end{aligned}$$

The power series converges absolutely when $\rho < 1$, that is

$$\frac{|x-9|^3}{8} < 1 \;\Rightarrow\; |x-9| < 2 \;\Rightarrow\; -2 < x - 9 < 2 \;\Rightarrow 7 < x < 11.$$

Therefore the radius of convergence is $\boxed{R = 9}$.

To find the interval of convergence, we need to determine if the power series converges at the endpoints $x = 7, 11$.

- At $x = 7$, the power series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n (7-9)^{3n}}{8^n (n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n (-2)^{3n}}{8^n (n+1)} = \sum_{n=0}^{\infty} \frac{8^n}{8^n (n+1)} = \sum_{n=0}^{\infty} \frac{1}{n+1}.$$

  This series is a divergent p-series with $p = 1$. Therefore, the power series converges diverges at $x = 7$.

- At $x = 8$, the power series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n (11-9)^{3n}}{8^n (n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{3n}}{8^n (n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n 8^n}{8^n (n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}.$$

  This series converges by the AST since $a_n = \frac{1}{n+1}$ is positive, decreasing and converges to 0. However, the series does not converge absolutely since

$$\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{n+1} \right| = \sum_{n=0}^{\infty} \frac{1}{n+1},$$

which is a divergent p-series with $p = 1$ as noted above. Therefore, the power series converges conditionally at $x = 11$.

In conclusion, the IOC is $\boxed{(9, 11]}$. Furthermore, the power series $\boxed{\text{converges absolutely on } (9, 11)}$ and $\boxed{\text{converges conditionally at } x = 11}$.

(c) $\displaystyle\sum_{n=0}^{\infty} n3^n(2x+1)^n$.

Solution. We use the Ratio Test to find the radius of convergence. We have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)3^{n+1}(2x+1)^{n+1}}{n3^n(2x+1)^n} \right| \\
&= \lim_{n\to\infty} 3\left(1 + \frac{1}{n}\right)|2x+1| \\
&= 3|2x+1|.
\end{aligned}$$

The power series converges absolutely when $\rho < 1$, that is

$$3|2x+1| < 1 \;\Rightarrow\; \left|x + \frac{1}{2}\right| < \frac{1}{6} \;\Rightarrow\; -\frac{1}{6} < x + \frac{1}{2} < \frac{1}{6} \;\Rightarrow -\frac{2}{3} < x < -\frac{1}{3}.$$

Therefore the radius of convergence is $\boxed{R = \frac{1}{6}}$.

To find the interval of convergence, we need to determine if the power series converges at the endpoints $x = -\frac{2}{3}, -\frac{1}{3}$.

- At $x = -\frac{2}{3}$, the power series becomes

$$\sum_{n=0}^{\infty} n3^n\left(2\left(-\frac{2}{3}\right)+1\right)^n = \sum_{n=0}^{\infty}\sum_{n=0}^{\infty} n3^n\left(-\frac{1}{3}\right)^n = \sum_{n=0}^{\infty}(-1)^n n.$$

  Since $\lim_{n\to\infty}(-1)^n n$ does not exist (and in particular is not equal to 0), this series diverges by the Term Divergence Test. Therefore, the power series diverges at $x = -\frac{2}{3}$.

- At $x = -\frac{1}{3}$, the power series becomes

$$\sum_{n=0}^{\infty} n3^n\left(2\left(-\frac{1}{3}\right)+1\right)^n = \sum_{n=0}^{\infty}\sum_{n=0}^{\infty} n3^n\left(\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} n.$$

  Since $\lim_{n\to\infty} n = \infty$ (and in particular is not equal to 0), this series diverges by the Term Divergence Test. Therefore, the power series diverges at $x = -\frac{1}{3}$.

In conclusion, the IOC is $\boxed{\left(-\frac{2}{3}, -\frac{1}{3}\right)}$. Furthermore, the power series $\boxed{\text{converges absolutely on } \left(-\frac{2}{3}, -\frac{1}{3}\right)}$ and $\boxed{\text{never converges conditionally}}$.

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{n^n(x+2)^n}{6^n}$.

Solution. We use the Root Test to find the radius of convergence. We have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} |a_n|^{1/n} \\
&= \lim_{n\to\infty} \left|\frac{n^n(x+2)^n}{6^n}\right|^{1/n} \\
&= \lim_{n\to\infty} \frac{n|x+2|}{6} \\
&= \begin{cases} 0 & \text{if } x=-2, \\ \infty & \text{if } x \neq -2. \end{cases}
\end{aligned}$$

Therefore, the series converges conditionally for $x=-2$ and diverges otherwise. So the radius of convergence is $\boxed{R=0}$ and the IOC is $\{-2\}$. The power series $\boxed{\text{converges absolutely at } x=-2}$ and $\boxed{\text{never converges conditionally}}$.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(x-4)^{2n}}{36^n\sqrt{n}}$.

Solution. We use the Ratio Test to find the radius of convergence. We have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \left|\frac{(-1)^{n+1}(x-4)^{2(n+1)}}{36^{n+1}\sqrt{n+1}} \cdot \frac{36^n\sqrt{n}}{(-1)^n(x-4)^{2n}}\right| \\
&= \lim_{n\to\infty} \frac{|x-4|^2\sqrt{n}}{36\sqrt{n+1}} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} \\
&= \lim_{n\to\infty} \frac{|x-4|^2}{36\sqrt{1+\frac{1}{n}}} \\
&= \frac{|x-4|^2}{36}.
\end{aligned}$$

The power series converges absolutely when $\rho < 1$, that is

$$\frac{|x-4|^2}{36} < 1 \;\Rightarrow\; |x-4| < 6 \;\Rightarrow\; -6 < x-4 < 6 \;\Rightarrow -2 < x < 10.$$

Therefore the radius of convergence is $\boxed{R=6}$.

To find the interval of convergence, we need to determine if the power series converges at the endpoints $x=-2, 10$.

- At $x=-2$, the power series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n(-2-4)^{2n}}{36^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n(-6)^{2n}}{36^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n 36^n}{36^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$

This series converges by the AST since $a_n = \frac{1}{\sqrt{n}}$ is positive, decreasing and converges to 0. However, the series does not converge absolutely since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}},$$

which is a divergent p-series with $p = \frac{1}{2} \leqslant 1$. Therefore, the power series converges conditionally at $x = -2$.

- At $x = 10$, the power series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n (10-4)^{2n}}{36^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n 6^{2n}}{36^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n 36^n}{36^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$

As noted above, this series converges conditionally. Therefore, the power series converges conditionally at $x = 10$.

In conclusion, the IOC is $\boxed{[-2, 10]}$. Furthermore, the power series $\boxed{\text{converges absolutely on } (-2, 10)}$ and $\boxed{\text{converges conditionally at } x = -2, 10}$.

(f) $\displaystyle\sum_{n=0}^{\infty} \frac{(3x+2)^n}{n^2+4}$.

Solution. We use the Ratio Test to find the radius of convergence. We have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n\to\infty} \left| \frac{(3x+2)^{n+1}}{(n+1)^2+4} \cdot \frac{n^2+4}{(3x+2)^n} \right| \\
&= \lim_{n\to\infty} \frac{|3x+2|(n^2+4)}{n^2+2n+5} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \\
&= \lim_{n\to\infty} \frac{|3x+2| \left(1 + \frac{4}{n^2}\right)}{1 + \frac{2}{n} + \frac{5}{n^2}} \\
&= |3x+2|.
\end{aligned}$$

The power series converges absolutely when $\rho < 1$, that is

$$|3x+2| < 1 \;\Rightarrow\; \left| x + \frac{2}{3} \right| < \frac{1}{3} \;\Rightarrow\; -\frac{1}{3} < x + \frac{2}{3} < \frac{1}{3} \;\Rightarrow -1 < x < -\frac{1}{3}.$$

Therefore the radius of convergence is $\boxed{R = \frac{1}{3}}$.

To find the interval of convergence, we need to determine if the power series converges at the endpoints $x = -1, -\dfrac{1}{3}$.

- At $x = -1$, the power series becomes

$$\sum_{n=0}^{\infty} \frac{(3(-1)+2)^n}{n^2+4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+4}.$$

We can show that this series converges absolutely using the DCT. We have

$$0 \leqslant \left|\frac{(-1)^n}{n^2+4}\right| \leqslant \frac{1}{n^2},$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges as a p-series with $p = 2 > 1$. Therefore, the power series converges absolutely at $x = -1$.

- At $x = -\frac{1}{3}$, the power series becomes

$$\sum_{n=0}^{\infty} \frac{\left(3\left(-\frac{1}{3}\right)+2\right)^n}{n^2+4} = \sum_{n=0}^{\infty} \frac{1}{n^2+4}.$$

As noted above, this series converges absolutely. Therefore, the power series converges absolutely at $x = -\frac{1}{3}$.

In conclusion, the IOC is $\boxed{\left[-1, -\frac{1}{3}\right]}$. Furthermore, the power series $\boxed{\text{converges absolutely on } \left[-1, -\frac{1}{3}\right]}$ and $\boxed{\text{never converges conditionally}}$.

2. Find the radius of convergence of the following power series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} x^{2n}$.

Solution. We use the Ratio Test. We have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \left|\frac{((n+1)!)^2 x^{2(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{(n!)^2 x^{2n}}\right| \\
&= \lim_{n\to\infty} |x|^2 \left(\frac{(n+1)!}{n!}\right)^2 \frac{(2n)!}{(2n+2)!} \\
&= \lim_{n\to\infty} |x|^2 \frac{(n+1)^2}{(2n+1)(2n+2)} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \\
&= \lim_{n\to\infty} |x|^2 \frac{\left(1+\frac{1}{n}\right)^2}{\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)} \\
&= \frac{|x|^2}{4}.
\end{aligned}$$

The power series converges absolutely when $\rho < 1$, that is $|x|^2 < 4$, or $|x| < 2$. Therefore, the radius of convergence is $\boxed{R = 2}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \left(1 - \frac{3}{n}\right)^{n^2} (x+5)^n$.

Solution. We use the Root Test. We have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} |a_n|^{1/n} \\
&= \lim_{n\to\infty} \left| \left(1 - \frac{3}{n}\right)^{n^2} (x+5)^n \right|^{1/n} \\
&= \lim_{n\to\infty} |x+5| \left(1 - \frac{3}{n}\right)^{n} \\
&= |x+5| \lim_{n\to\infty} \left(1 - \frac{3}{n}\right)^{n}.
\end{aligned}$$

This last limit is an indeterminate exponent 1^∞. To compute it, we can write the expression in base e

$$\lim_{n\to\infty} \left(1 - \frac{3}{n}\right)^{n} = \lim_{n\to\infty} e^{n \ln\left(1 - \frac{3}{n}\right)}$$

and compute the limit of the exponent using L'Hôpital's Rule. This gives

$$\begin{aligned}
\lim_{n\to\infty} n \ln\left(1 - \frac{3}{n}\right) &= \lim_{x\to\infty} \frac{\ln\left(1 - \frac{3}{x}\right)}{\frac{1}{x}} \\
&\overset{\mathrm{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty} \frac{-\frac{3}{x^2} \cdot \frac{1}{1+\frac{3}{x}}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} -\frac{3}{1 - \frac{3}{x}} \\
&= -3.
\end{aligned}$$

Therefore

$$\rho = |x+5| \lim_{n\to\infty} e^{n \ln\left(1 - \frac{3}{n}\right)} = |x+5|e^{-3}.$$

The power series converges absolutely when $\rho < 1$, that is $|x+5|e^{-3} < 1$, or $|x+5| < e^3$. Therefore, the radius of convergence is $\boxed{R = e^3}$.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{n^n} x^n.$

Solution. We use the Ratio Test. We have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)!x^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{n!x^n} \right| \\
&= \lim_{n\to\infty} |x| \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} \\
&= \lim_{n\to\infty} |x|(n+1) \frac{n^n}{(n+1)^{n+1}} \\
&= \lim_{n\to\infty} |x| \frac{n^n}{(n+1)^{n}} \\
&= |x| \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^{n}.
\end{aligned}$$

This last limit is an indeterminate exponent 1^∞. To compute it, we can write the expression in base e

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} e^{n\ln\left(\frac{n}{n+1}\right)}$$

and compute the limit of the exponent using L'Hôpital's Rule. This gives

$$\begin{aligned}\lim_{n\to\infty} n\ln\left(\frac{n}{n+1}\right) &= \lim_{x\to\infty}\frac{\ln(x)-\ln(x+1)}{\frac{1}{x}}\\ &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty}\frac{\frac{1}{x}-\frac{1}{x+1}}{-\frac{1}{x^2}}\\ &= \lim_{x\to\infty} -\frac{x^2}{x(x+1)}\\ &= \lim_{x\to\infty} -\frac{1}{1+1/x}\\ &= -1.\end{aligned}$$

Therefore

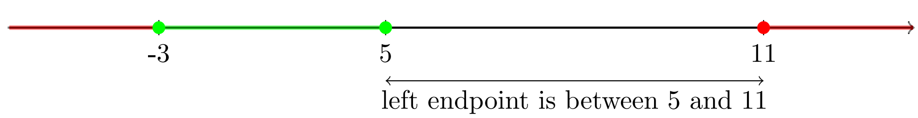
$$\rho = |x|\lim_{n\to\infty} e^{n\ln\left(\frac{n}{n+1}\right)} = |x|e^{-1}.$$

The power series converges absolutely when $\rho < 1$, that is $|x|e^{-1} < 1$, or $|x| < e$. Therefore, the radius of convergence is $\boxed{R = e}$.

3. Suppose that a power series converges absolutely at $x = 5$, converges conditionally at $x = -3$ and diverges at $x = 11$. What can you say, if anything, about the convergence or divergence of the power series at the following values of x?

(a) $x = -4$. (b) $x = 2$. (c) $x = 15$. (d) $x = 7$.

Solution. We know that a power series converges absolutely in the interior of its interval of convergence. Therefore, we know that $x = -3$ is an endpoint of the interval of convergence. Since $x = 5$ is in the interval of convergence, we know that $x = -3$ is the left endpoint of the interval of convergence and the right endpoint of the interval of convergence is at least 5. Since the power series diverges at $x = 11$, we know that the right endpoint of the interval of convergence must be less than 11. These observations are summarized on the figure below, where red indicates divergence and green indicates convergence.



We conclude that

(a) $\boxed{\text{the power series diverges at } x = -4}$.

(b) $\boxed{\text{the power series converges at } x = 2}$.

(c) $\boxed{\text{the power series diverges at } x = 15}$.

(d) $\boxed{\text{We cannot conclude anything about the behavior of the power series at } x = 7}$.

4. Let $f(x) = \dfrac{3}{2+7x}$. Use the power series representation of $\dfrac{1}{1-x}$ and power series operations to find a power series representation of $f(x)$ centered at $a = 0$. What are the radius and interval of convergence of the resulting power series?

Solution. A bit of algebra will help us make $\frac{3}{2+7x}$ look like $\frac{1}{1-x}$ up to a suitable substitution. Namely, if we factor 3 from the numerator and 2 from the denominator, we get

$$\frac{3}{2+7x} = \frac{3}{2} \cdot \frac{1}{1+\frac{7x}{2}} = \frac{3}{2} \cdot \frac{1}{1-\left(-\frac{7x}{2}\right)} = \frac{3}{2} \cdot \frac{1}{1-u},$$

with $u = -\frac{7x}{2}$. For $|u| < 1$, we know that $\frac{1}{1-u}$ is the sum of a geometric series of common ratio u and first term 1, that is:

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n.$$

So

$$\frac{3}{2+7x} = \frac{3}{2} \cdot \frac{1}{1-\left(-\frac{7x}{2}\right)} = \frac{3}{2}\sum_{n=0}^{\infty} \left(-\frac{7x}{2}\right)^n = \boxed{\sum_{n=0}^{\infty} \frac{3(-7)^n x^n}{2^{n+1}}}.$$

This geometric series converges when the common ratio $r = -\frac{7x}{2}$ satisfies $|r| < 1$. This gives

$$\left|-\frac{7x}{2}\right| < 1 \;\Rightarrow\; |x| < \frac{2}{7} \;\Rightarrow\; -\frac{2}{7} < x < \frac{2}{7}.$$

Therefore, the radius of convergence is $\boxed{R = \frac{2}{7}}$ and the interval of convergence is $\boxed{\left(-\frac{2}{7}, \frac{2}{7}\right)}$.

5. Consider the power series $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(x+1)^n}{3^n(n+1)}$.

(a) Find the radius and interval of convergence of f.

Solution. We use the Ratio Test to find the radius of convergence. We have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \left|\frac{(x+1)^{n+1}}{3^{n+1}(n+2)} \cdot \frac{3^n(n+1)}{(x+1)^n}\right| \\
&= \lim_{n\to\infty} \frac{|x+1|(n+1)}{3(n+2)} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{|x+1|\left(1+\frac{1}{n}\right)}{3\left(1+\frac{2}{n}\right)} \\
&= \frac{|x+1|}{3}.
\end{aligned}$$

The power series converges absolutely when $\rho < 1$, that is

$$\frac{|x+1|}{3} < 1 \;\Rightarrow\; |x+1| < 3 \;\Rightarrow\; -3 < x+1 < 3 \;\Rightarrow -4 < x < 2.$$

Therefore the radius of convergence is $\boxed{R=3}$.

To find the interval of convergence, we need to determine if the power series converges at the endpoints $x=-4,2$.

- At $x=-4$, the power series becomes

$$\sum_{n=0}^{\infty} \frac{(-4+1)^n}{3^n(n+1)} = \sum_{n=0}^{\infty} \frac{(-3)^n}{3^n(n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}.$$

  This series converges by the AST since $a_n = \frac{1}{n+1}$ is positive, decreasing and converges to 0. Therefore, the power series converges at $x=-4$.

- At $x=2$, the power series becomes

$$\sum_{n=0}^{\infty} \frac{(2+1)^n}{3^n(n+1)} = \sum_{n=0}^{\infty} \frac{3^n}{3^n(n+1)} = \sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n}.$$

  This is a divergent p-series with $p=1$. Therefore, the power series converges diverges at $x=2$.

In conclusion, the IOC is $\boxed{[-4,2)}$.

(b) Find a power series representation of $f'(x)$ centered at $a=-1$. What are its radius and interval of convergence?

Solution. Differentiating term-by-term gives

$$\begin{aligned} f'(x) &= \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{(x+1)^n}{3^n(n+1)}\right) \\ &= \sum_{n=0}^{\infty} \frac{n(x+1)^{n-1}}{3^n(n+1)} \\ &= \boxed{\sum_{n=1}^{\infty} \frac{n(x+1)^{n-1}}{3^n(n+1)}}. \end{aligned}$$

We know that the radius of convergence does not change when differentiating term-by-term, so $\boxed{R=3}$. The interval, however, may change and we need to test the endpoints to determine it.

- At $x=-4$, the power series becomes

$$\sum_{n=1}^{\infty} \frac{n(-4+1)^{n-1}}{3^n(n+1)} = \sum_{n=1}^{\infty} \frac{n(-3)^{n-1}}{3^n(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{3(n+1)}.$$

  This series diverges by the Term Divergence Test since

$$\lim_{n\to\infty} \frac{n}{3(n+1)} = \lim_{n\to\infty} \frac{1}{3(1+1/n)} = \frac{1}{3},$$

  and therefore $\lim\limits_{n\to\infty} \frac{(-1)^{n-1}n}{3(n+1)}$ does not exist. Thus, the power series diverges at $x=-4$.

- At $x=2$, the power series becomes

$$\sum_{n=1}^{\infty} \frac{n(2+1)^{n-1}}{3^n(n+1)} = \sum_{n=1}^{\infty} \frac{n3^{n-1}}{3^n(n+1)} = \sum_{n=1}^{\infty} \frac{n}{3(n+1)}.$$

This series diverges by the Term Divergence Test since

$$\lim_{n\to\infty} \frac{n}{3(n+1)} = \lim_{n\to\infty} \frac{1}{3(1+1/n)} = \frac{1}{3}.$$

Thus, the power series diverges at $x = 2$.

In conclusion, the IOC is $\boxed{(-4, 2)}$.

(c) Let $g(x)$ be the antiderivative of $f(x)$ such that $g(-1) = -8$. Find a power series representation of $g(x)$ centered at $a = -1$. What are its radius and interval of convergence?

Solution. Integrating term-by-term gives

$$\begin{aligned} g(x) &= \sum_{n=0}^{\infty} \int \frac{(x+1)^n}{3^n(n+1)} dx \\ &= \sum_{n=0}^{\infty} \frac{(x+1)^{n+1}}{3^n(n+1)^2} + C. \end{aligned}$$

To find C, we use $g(-1) = -8$, which gives

$$\sum_{n=0}^{\infty} \frac{(-1+1)^{n+1}}{3^n(n+1)^2} + C = -8 \ \Rightarrow \ C = -8.$$

Therefore,

$$\boxed{g(x) = -8 + \sum_{n=0}^{\infty} \frac{(x+1)^{n+1}}{3^n(n+1)^2}}.$$

We know that the radius of convergence does not change when differentiating term-by-term, so $\boxed{R = 3}$. The interval, however, may change and we need to test the endpoints to determine it.

- At $x = -4$, the power series becomes

$$\sum_{n=0}^{\infty} \frac{(-4+1)^{n+1}}{3^n(n+1)^2} = \sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{3^n(n+1)^2} = \sum_{n=0}^{\infty} \frac{3(-1)^n}{(n+1)^2}.$$

  This series converges absolutely since

$$\sum_{n=0}^{\infty} \left| \frac{3(-1)^n}{(n+1)^2} \right| = 3\sum_{n=1}^{\infty} \frac{1}{n^2},$$

  which is a convergent p-series with $p = 2 > 1$. So the power series converges at $x = -4$.

- At $x = 2$, the power series becomes

$$\sum_{n=0}^{\infty} \frac{(2+1)^{n+1}}{3^n(n+1)^2} = \sum_{n=0}^{\infty} \frac{3^{n+1}}{3^n(n+1)^2} = \sum_{n=0}^{\infty} \frac{3}{(n+1)^2} = 3\sum_{n=1}^{\infty} \frac{1}{n^2},$$

  which is a convergent p-series with $p = 2 > 1$. So the power series converges at $x = 2$.

In conclusion, the IOC is $\boxed{[-4, 2]}$.

6. (a) Use term-by-term differentiation to find a power series representation of $\frac{1}{(1-x)^2}$. What is its raidus of convergence?

Solution. We have the power series representation

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

with radius of convergence $R = 1$. Differentiating term-by-term gives

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{1-x}\right) &= \sum_{n=0}^{\infty} \frac{d}{dx} x^n \\ \Rightarrow \boxed{\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}}. \end{aligned}$$

The radius of convergence does not change when differentiating term-by-term, so $\boxed{R = 1}$.

(b) Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{5^n}$.

Solution. Observe that

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{5^n} = \sum_{n=1}^{\infty} n\left(-\frac{1}{5}\right)^n = -\frac{1}{5}\sum_{n=1}^{\infty} n\left(-\frac{1}{5}\right)^{n-1}.$$

Using the power series representation from the previous part with $x = -\frac{1}{5}$, we get

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^n n}{5^n} &= -\frac{1}{5} \cdot \frac{1}{\left(1 - \left(-\frac{1}{5}\right)\right)^2} \\ &= \boxed{-\frac{5}{36}}. \end{aligned}$$