Section 10.8

# Taylor and Maclaurin Series

## Learning Goals

| | |
|---|---|
| 10.8.1 Find the Nth degree Taylor polynomial for a function about a given center for a finite N | 10.8: 1,4,8  Also 10.9: 31,35 |
| 10.8.2 Find the Maclaurin series for a function | 10.8: 12,13,19,21,22 |
| 10.8.3 Find the Taylor series for a function about a given center | 10.8: 25,27,28,31,41,42 |
| 10.8.4 Find the first N terms of or the whole Maclaurin series for a function using either multiplication of another Maclaurin series  by x^n or by multiplying or dividing two other Maclaurin series | 10.8: 37,38,40.   Also 10.9:31 |

Conceptual introduction: in the previous section, we have seen that certain functions can be written as sums of power series in certain intervals:

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for} \quad -1 < x < 1}$$

$$\boxed{\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \quad \text{for} \quad -1 < x \leq 1}$$

$$\boxed{\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad \text{for} \quad -1 \leq x \leq 1}$$

How can we do this in general?
Given a function $f(x)$, we seek coefficients $c_n$ such that near $x = a$ we have:

differentiate

differentiate

differential

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

$$\xrightarrow{x=a} \boxed{c_0 = f(a)}$$

$$f'(x) = \sum_{n=1}^{\infty} c_n n (x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots$$

$$\xrightarrow{x=a} \boxed{c_1 = f'(a)}$$

$$f''(x) = \sum_{n=2}^{\infty} c_n n(n-1)(x-a)^{n-2} = 2c_2 + 6c_3(x-a) + 12c_4(x-a)^2 + \dots$$

$$\xrightarrow{x=a} f''(a) = 2c_2 \quad \text{or} \quad \boxed{c_2 = \frac{f''(a)}{2}}$$

$$f^{(3)}(x) = \sum_{n=3}^{\infty} c_n n(n-1)(n-2)(x-a)^{n-3} = 6c_3 + 24c_4(x-a) + \dots$$

$$\xrightarrow{x=a} f^{(3)}(a) = 6c_3 \quad \text{or} \quad \boxed{c_3 = \frac{f^{(3)}(a)}{6}}$$

Continuing, we find $\boxed{c_n = \dfrac{f^{(n)}(a)}{n!}}$

<u>Definition</u>: let $f$ be a function with derivatives of all orders in an open interval containing $a$.

- The <u>$N^{th}$ degree Taylor polynomial</u> of $f$ at $x = a$ is

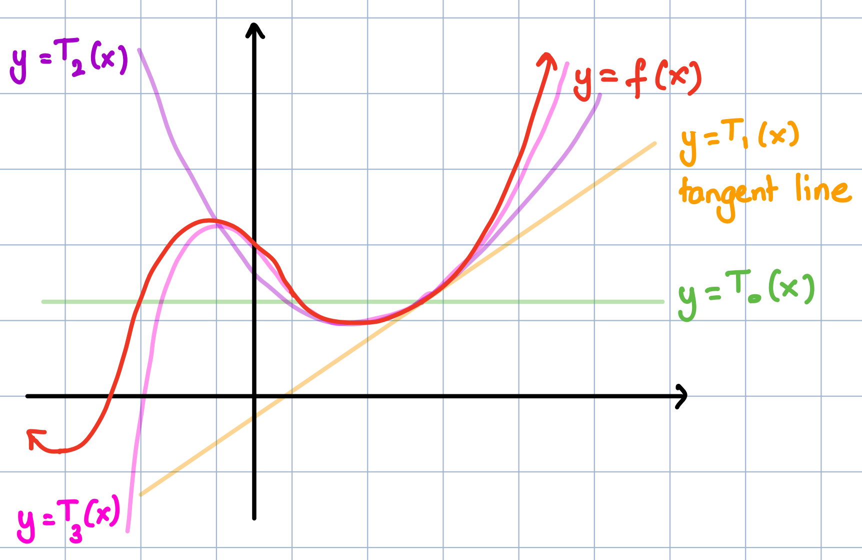
$$\boxed{\begin{aligned} T_N(x) &= \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots + \frac{f^{(N)}(a)}{N!}(x-a)^N \end{aligned}}$$

- The <u>Taylor series</u> of $f$ at $x=a$ is the power series

$$\boxed{\begin{aligned} T(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ & f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots \end{aligned}}$$

<u>Maclaurin</u> polynomial / series: case where the center $a = 0$.

<u>Interpretation</u>: $T_N(x)$ is the best $N^{th}$ degree polynomial approximation of $f$ near $x = a$.




$P_0(x) = f(a) =$ constant approximation

$P_1(x) = f(a) + f'(a)(x-a)$
linear approximation (tangent line).

**Examples**: 1) Find the $3^{rd}$ degree Taylor polynomial of $f(x) = \sqrt{x}$ at $x=1$ and use it to approximate $\sqrt{1.4}$.

We calculate $c_0, c_1, c_2, c_3$.

| $n$ | $f^{(n)}(x)$ | $c_n = \frac{f^{(n)}(1)}{n!}$ |
|---|---|---|
| 0 | $x^{1/2}$ | $\frac{1}{0!} = 1$ |
| 1 | $\frac{1}{2}x^{-1/2}$ | $\frac{\frac{1}{2}}{1!} = \frac{1}{2}$ |
| 2 | $-\frac{1}{4}x^{-3/2}$ | $\frac{-\frac{1}{4}}{2!} = -\frac{1}{8}$ |
| 3 | $\frac{3}{8}x^{-5/2}$ | $\frac{-\frac{3}{8}}{6} = \frac{1}{16}$ |

So $\boxed{T_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3}$.

Approximation of $\sqrt{1.4}$:

$\sqrt{1.4} = f(1.4)$ and $f(x) \approx T_3(x)$ if $x$ close to 1.

So $\sqrt{1.4} \approx T_3(1.4) = 1 + \frac{1}{2}(1.4-1) - \frac{1}{8}(1.4-1)^2 + \frac{1}{16}(1.4-1)^3$
$= \boxed{1.184}$

Actual value: $\sqrt{1.4} = 1.18321\cdots$

We will investigate issues of accuracy of approximation in the next section.

2) Find the Maclaurin series of $f(x) = e^x$ and its radius and interval of convergence.

We have $f^{(n)}(x) = e^x$

So $c_n = \dfrac{f^{(n)}(0)}{n!} = \dfrac{e^0}{n!} = \dfrac{1}{n!}$

Therefore $\boxed{T(x) = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}}$

Ratio Test : $\lim_{n\to\infty} \left| \dfrac{x^{n+1}}{(n+1)!} \cdot \dfrac{n!}{x^n} \right| = \lim_{n\to\infty} \dfrac{|x|}{n+1} = 0$ so convergence for all $x$.

So $\boxed{R = \infty}$, interval of convergence is $\boxed{(-\infty, \infty)}$

3) Find the $4^{th}$ degree Taylor polynomial and the Taylor series of $f(x) = \dfrac{1}{x}$ centered at $x = 2$.

We calculate $c_0, c_1, c_2, c_3, c_4$

| $n$ | $f^{(n)}(x)$ | $c_n = \dfrac{f^{(n)}(2)}{n!}$ |
|---|---|---|
| 0 | $x^{-1}$ | $\frac{1}{2}$ |
| 1 | $-x^{-2}$ | $-\frac{1}{4}$ |
| 2 | $2x^{-3}$ | $\frac{1}{8}$ |
| 3 | $-6x^{-4}$ | $-\frac{1}{16}$ |
| 4 | $24x^{-5}$ | $\frac{1}{32}$ |

$\Rightarrow \boxed{T_4(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3 + \frac{1}{32}(x-2)^4}$

We notice the pattern $f^{(n)}(x) = \dfrac{(-1)^n n!}{x^{n+1}}$ so $c_n = \dfrac{f^{(n)}(2)}{n!} = \dfrac{(-1)^n}{2^{n+1}}$

Therefore $\boxed{T(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n (x-2)^n}{2^{n+1}}}$

Can we check that $T(x) = f(x)$ in this case ?

$T(x) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{2-x}{2}\right)^n$ geometric series with common ratio $r = \frac{2-x}{2}$, so converges if $-1 < \frac{2-x}{2} < 1 \Rightarrow \underbrace{0 < x < 4}_{\text{IOC}}$

$$= \frac{1}{2}\cdot\frac{1}{1-\frac{2-x}{2}} = \frac{1}{2-(2-x)} = \frac{1}{x} = f(x).$$

It is not always the case that $T(x) = f(x)$ in the interval of convergence of $T(x)$ - see example at the end of notes.

But if we have a power series representation of $f$ centered at $x=a$ $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$, then this power series must be the Taylor series of $f$ at $x=a$ so we must have $c_n = \frac{f^{(n)}(a)}{n!}$.

Examples: 1) Suppose that $f(x) = \sum_{n=0}^{\infty}\frac{2n}{n^2+1}(x-5)^n$.
Find $f^{(7)}(5)$.

We know that $\sum_{n=0}^{\infty}\frac{2n}{n^2+1}(x-5)^n$ is the Taylor series of $f$.

take $n=7$

So $\frac{f^{(7)}(5)}{7!}$ = coefficient of $(x-5)^7$

$$\frac{f^{(7)}(5)}{7!} = \frac{2\cdot 7}{7^2+1} \Rightarrow \boxed{f^{(7)}(5) = 7!\,\frac{14}{50} = \frac{7056}{5}}$$

2) Suppose that $f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{3^n n!}$. Find $f^{(4)}(0)$, $f^{(5)}(0)$.

$2n = 4 \Rightarrow$ take $n=2$

$\frac{f^{(4)}(0)}{4!}$ = coefficient of $x^4$

$$\frac{f^{(4)}(0)}{4!} = \frac{(-1)^2}{3^2 2!} \Rightarrow \boxed{f^{(4)}(0) = \frac{4!}{3^2 2!} = \frac{4}{3}}$$

$$\frac{f^{(5)}(0)}{5!} = \text{coefficient of } x^5$$

$2n=5 \Rightarrow$ impossible

$x^5$ does not appear in the power series

$$= 0$$

So $\boxed{f^{(5)}(0) = 0}$.

## Important Maclaurin Series to memorize.

| $f(x)$ | $T(x)$ | Interval where $f=T$ |
|---|---|---|
| $\frac{1}{1-x}$ | $\sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+\cdots$ | $(-1,1)$ |
| $\ln(1+x)$ | $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$ | $(-1,1]$ |
| $e^x$ | $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$ | $(-\infty,\infty)$ |
| $\cos(x)$ | $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots$ | $(-\infty,\infty)$ |
| $\sin(x)$ | $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$ | $(-\infty,\infty)$ |
| $\tan^{-1}(x)$ | $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots$ | $[-1,1]$ |

For a computation of the Maclaurin series of $\cos(x)$, $\sin(x)$, see end of notes.

We can use these reference Maclaurin series to compute other Maclaurin / Taylor series.

Examples: 1) Find the Maclaurin series of $f(x) = xe^{3x^2}$.

Start with $\quad e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Substitute $3x^2$: $\quad e^{3x^2} = \sum_{n=0}^{\infty} \frac{(3x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n x^{2n}}{n!}$

Multiply by $x$: $\quad xe^{3x^2} = x \sum_{n=0}^{\infty} \frac{3^n x^{2n}}{n!}$

$$= \boxed{\sum_{n=0}^{\infty} \frac{3^n x^{2n+1}}{n!}}$$

2) Find the Maclaurin series of $f(x) = \cos(2x) + \frac{1}{1+x^2}$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n}}{(2n)!}$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Combine the two:

$$\cos(x) + \frac{1}{1+x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$= \sum_{n=0}^{\infty} \left( \frac{(-1)^n 4^n x^{2n}}{(2n)!} + (-1)^n x^{2n} \right)$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \left( \frac{4^n}{(2n)!} + 1 \right) x^{2n}}$$

3) Find the first three non-zero terms of the Maclaurin series of $f(x) = \ln(1+x)\sin(x)$.

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$
$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \dots$$

$$\ln(1+x)\sin(x) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} \dots\right)$$
$$= x^2 - \frac{x^3}{4} + \left(\frac{1}{3} - \frac{1}{6}\right)x^4 + \dots$$
$$= \boxed{x^2 - \frac{x^3}{4} + \frac{x^4}{6} + \dots}$$

distribute and collect terms by degree.

4) Find the first four terms of the Maclaurin series of $\cos(x)^2$ two ways:

a) multiplying Maclaurin series.

b) using a double angle formula.

a) $\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots$

$$\cos(x)^2 = \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots\right)$$
$$= 1 + \left(-\frac{1}{2} - \frac{1}{2}\right)x^2 + \left(\frac{1}{24} + \frac{1}{24} + \frac{1}{4}\right)x^4 + \left(-\frac{1}{720} - \frac{1}{720} - \frac{1}{48} - \frac{1}{48}\right)x^6 + \dots$$
$$= \boxed{1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45} + \dots}$$

b) $$\cos(x)^2 = \frac{1 + \cos(2x)}{2} = \frac{1}{2} + \frac{1}{2}\left(1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{24} - \frac{(2x)^6}{720} + \dots\right)$$

$$= \frac{1}{2} + \frac{1}{2} - \frac{4x^2}{4} + \frac{16x^4}{48} - \frac{64x^6}{1440} + \cdots$$

$$= \boxed{1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45} + \cdots}$$

5) Find the Maclaurin series of $x^2 \tan^{-1}(x) - x^3$.

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$\Rightarrow \quad x^2 \tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{2n+1} = x^3 - \frac{x^5}{3} + \frac{x^7}{5} - \frac{x^9}{7} + \cdots$$

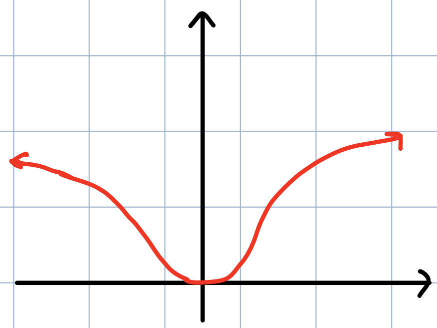
$$x^2 \tan^{-1}(x) - x^3 = -\frac{x^5}{3} + \frac{x^7}{5} - \frac{x^9}{7} + \cdots$$

$$= \boxed{\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+3}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n-1} x^{2n+1}}{2n-1}}$$

## Complements:

1) Example of a function for which the Maclaurin series does not converge to $f(x)$.

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



We have $f^{(n)}(0) = 0$ for all $n$.

So $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0.$

$T(x)$ converges on $(-\infty, \infty)$, but $T(x) \neq f(x)$.

2) Maclaurin series of $\cos(x)$

| $n$ | $f^{(n)}(x)$ | $c_n = \frac{f^{(n)}(0)}{n!}$ |
|---|---|---|
| 0 | $\cos(x)$ | 1 |
| 1 | $-\sin(x)$ | 0 |
| 2 | $-\cos(x)$ | $-\frac{1}{2}$ |
| 3 | $\sin(x)$ | 0 |
| 4 | $\cos(x)$ | $\frac{1}{24}$ |
| 5 | $-\sin(x)$ | 0 |

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ odd} \\ (-1)^{n/2} & \text{if } n \text{ even} \end{cases}$$

$$\text{so} \quad c_n = \begin{cases} 0 & \text{if } n \text{ odd} \\ \frac{(-1)^{n/2}}{n!} & \text{if } n \text{ even} \end{cases}$$

$$\Rightarrow c_{2n} = \frac{(-1)^n}{(2n)!}$$

$$T(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

For $\sin(x)$, we can integrate term-by-term :

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\Rightarrow \sin(x) = \int \cos(x)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{x^{2n+1}}{2n+1} + C$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} + C.$$

Plugging-in $x = 0$ gives $C = 0$.

So $$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$