Rutgers University
Math 152

Section 10.8: Taylor and Maclaurin Series - Worksheet Solutions

1. Find the Taylor polynomials for the following functions at the order and center indicated.

   (a) $f(x) = 2\cos\left(\frac{\pi}{3} - 5x\right)$, $T_4(x)$ at $a = 0$.

   Solution. We have

$$\begin{aligned}
f(x) &= 2\cos\left(\frac{\pi}{3} - 5x\right) &&\Rightarrow\ c_0 = f(0) = 1,\\
f'(x) &= 10\sin\left(\frac{\pi}{3} - 5x\right) &&\Rightarrow\ c_1 = f'(0) = 5\sqrt{3},\\
f''(x) &= -50\cos\left(\frac{\pi}{3} - 5x\right) &&\Rightarrow\ c_2 = \frac{f''(0)}{2!} = -\frac{25}{2},\\
f^{(3)}(x) &= -250\sin\left(\frac{\pi}{3} - 5x\right) &&\Rightarrow\ c_3 = \frac{f^{(3)}(0)}{3!} = -\frac{125\sqrt{3}}{6},\\
f^{(4)}(x) &= 1250\cos\left(\frac{\pi}{3} - 5x\right) &&\Rightarrow\ c_4 = \frac{f^{(4)}(0)}{4!} = \frac{625}{24}.
\end{aligned}$$

   Thus

$$\boxed{T_4(x) = 1 + 5\sqrt{3}x - \frac{25x^2}{2} - \frac{125\sqrt{3}x^3}{6} + \frac{625x^4}{24}}.$$

   (b) $f(x) = \sqrt[3]{4 + 2x}$, $T_3(x)$ at $a = 2$.

   Solution. We have

$$\begin{aligned}
f(x) &= \sqrt[3]{4 + 2x} &&\Rightarrow\ c_0 = f(2) = 2,\\
f'(x) &= \frac{2}{3}(4 + 2x)^{-2/3} &&\Rightarrow\ c_1 = f'(2) = \frac{1}{6},\\
f''(x) &= -\frac{8}{9}(4 + 2x)^{-5/3} &&\Rightarrow\ c_2 = \frac{f''(2)}{2!} = -\frac{1}{72},\\
f^{(3)}(x) &= \frac{80}{27}(4 + 2x)^{-8/3} &&\Rightarrow\ c_3 = \frac{f^{(3)}(2)}{3!} = \frac{5}{2592}.
\end{aligned}$$

   Thus

$$\boxed{T_3(x) = 2 + \frac{1}{6}(x - 2) - \frac{1}{72}(x - 2)^2 + \frac{5}{2592}(x - 2)^3}.$$

(c) $f(x) = 2^{3-x}$, $T_4(x)$ at $a = 1$.

Solution. We have

$$\begin{aligned}
f(x) = 2^{3-x} &\Rightarrow c_0 = f(1) = 4, \\
f'(x) = -\ln(2)2^{3-x} &\Rightarrow c_1 = f'(1) = -4\ln(2), \\
f''(x) = \ln(2)^2 2^{3-x} &\Rightarrow c_2 = \frac{f''(1)}{2!} = 2\ln(2)^2, \\
f^{(3)}(x) = -\ln(2)^3 2^{3-x} &\Rightarrow c_3 = \frac{f^{(3)}(1)}{3!} = -\frac{2\ln(2)^3}{3}, \\
f^{(4)}(x) = \ln(2)^4 2^{3-x} &\Rightarrow c_4 = \frac{f^{(4)}(1)}{4!} = \frac{\ln(2)^4}{6}.
\end{aligned}$$

Thus

$$\boxed{T_4(x) = 4 - 4\ln(2)(x-1) + 2\ln(2)^2(x-1)^2 - \frac{2\ln(2)^3}{3}(x-1)^3 + \frac{\ln(2)^4}{6}(x-1)^4}.$$

(d) $f(x) = \ln(\cos(x))$, $T_3(x)$ at $a = \frac{\pi}{4}$.

Solution. We have

$$\begin{aligned}
f(x) = \ln(\cos(x)) &\Rightarrow c_0 = f\left(\frac{\pi}{4}\right) = -\frac{\ln(2)}{2}, \\
f'(x) = -\frac{\cos(x)}{\sin(x)} = -\tan(x) &\Rightarrow c_1 = f'\left(\frac{\pi}{4}\right) = -1, \\
f''(x) = -\sec(x)^2 &\Rightarrow c_2 = \frac{f''\left(\frac{\pi}{4}\right)}{2!} = -1, \\
f^{(3)}(x) = -2\sec(x)^2\tan(x) &\Rightarrow c_3 = \frac{f^{(3)}\left(\frac{\pi}{4}\right)}{3!} = -\frac{2}{3}.
\end{aligned}$$

Thus

$$\boxed{T_3(x) = -\frac{\ln(2)}{2} - \left(x - \frac{\pi}{4}\right) - \left(x - \frac{\pi}{4}\right)^2 - \frac{2}{3}\left(x - \frac{\pi}{4}\right)^3}.$$

(e) $f(x) = \frac{6}{5-3x}$, $T_4(x)$ at $a = 1$.

Solution. We can use the formula for the sum of a geometric series to find this Taylor polynomial. We start by finding the Taylor series, and we then keep the terms of degrees 0 through 4 to obtain the Taylor polynomial. We have

$$\begin{aligned}
\frac{6}{5-3x} &= \frac{6}{5-3(x-1)-3} \\
&= \frac{6}{2-3(x-1)} \\
&= \frac{3}{1-\frac{3(x-1)}{2}} \\
&= \sum_{n=0}^{\infty} 3\left(\frac{3(x-1)}{2}\right)^n \\
&= 3+\frac{9(x-1)}{2}+\frac{27(x-1)^2}{4}+\frac{81(x-1)^3}{8}+\frac{243(x-1)^4}{16}+\cdots.
\end{aligned}$$

Thus

$$\boxed{T_4(x) = 3+\frac{9(x-1)}{2}+\frac{27(x-1)^2}{4}+\frac{81(x-1)^3}{8}+\frac{243(x-1)^4}{16}}.$$

(f) $f(x) = \ln(5+x)$, $T_3(x)$ at $a=-4$.

Solution. We have

$$\begin{aligned}
f(x) = \ln(5+x) &\Rightarrow c_0 = f(-4) = 0, \\
f'(x) = \frac{1}{5+x} &\Rightarrow c_1 = f'(-4) = 1, \\
f''(x) = -\frac{1}{(5+x)^2} &\Rightarrow c_2 = \frac{f''(-4)}{2!} = -\frac{1}{2}, \\
f^{(3)}(x) = \frac{2}{(5+x)^3} &\Rightarrow c_3 = \frac{f^{(3)}(-4)}{3!} = \frac{1}{3}.
\end{aligned}$$

Thus

$$\boxed{T_3(x) = (x+4) - \frac{(x+4)^2}{2} - \frac{(x+4)^3}{3}}.$$

2. In 1.(b), you found the third degree Taylor polynomial of $f(x) = \sqrt[3]{4+2x}$ centered at $a=2$. Use this Taylor polynomial to estimate $\sqrt[3]{8.6}$.

Solution. We first need to find the input x to plug into $f(x)$ in order to get $\sqrt[3]{8.6}$. We want $f(x) = \sqrt[3]{4+2x} = \sqrt[3]{8.6}$, so we will need $4+2x = 8.6$, that is $x = 2.3$ Therefore, the estimate we get is

$$\sqrt[3]{8.6} = f(2.3) \simeq T_3(2.3) = 2+\frac{1}{6}(2.3-2) - \frac{1}{72}(2.3-2)^2 + \frac{5}{2592}(2.3-2)^3 \simeq \boxed{2.0488}.$$

3. Consider the function $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{9^n(n+1)}(x-4)^{2n+1}$.

(a) Find the radius and interval of convergence of f.

Solution. We use the Ratio Test. We have

$$\begin{aligned}
\rho &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+3}}{9^{n+1}(n+2)}\cdot\frac{9^n(n+1)}{(-1)^n(x-4)^{2n+1}}\right| \\
&= \lim_{n\to\infty}\frac{|x-4|^2(n+1)}{9(n+2)}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} \\
&= \lim_{n\to\infty}\frac{|x-4|^2\left(1+\frac{1}{n}\right)}{9\left(1+\frac{2}{n}\right)} \\
&= \frac{|x-4|^2}{9}.
\end{aligned}$$

The series converges absolutely when $\frac{|x-4|^2}{9}<1$, that is $1<x<7$, and diverges if $x<1$ or $x>7$. We now test the endpoints.

At $x=1$, we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{9^n(n+1)}(1-4)^{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{9^n(n+1)}(-3)^{2n+1}=-3\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}.$$

At $x=7$, we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{9^n(n+1)}(7-4)^{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{9^n(n+1)}3^{2n+1}=3\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}.$$

Both series converge by the AST since $a_n=\frac{1}{n+1}$ is positive, decreasing and converges to 0.

In conclusion, the radius of convergence is $\boxed{R=3}$ and the interval of convergence is $\boxed{[2,7]}$.

(b) Find $f^{(7)}(4)$, $f^{(8)}(4)$ and $f^{(9)}(4)$.

Solution. Since the given series must be the Taylor series of f at $a=4$, the coefficient of $(x-4)^7$ in the series is $\dfrac{f^{(7)}(4)}{7!}$. The term in $(x-4)^7$ is obtained in the series when $2n+1=7$, that is for $n=3$. So looking at the resulting coefficient gives

$$\frac{f^{(7)}(4)}{7!}=\frac{(-1)^3}{9^3(3+1)}.$$

So

$$\boxed{f^{(7)}(4)=-\frac{7!}{4\cdot 9^3}}.$$

Similarly, the coefficient of $(x-4)^8$ in the series is $\dfrac{f^{(8)}(4)}{8!}$. The term in $(x-4)^8$ is obtained in the series when $2n+1=8$. Since this equation has no solution with n being an integer, we deduce that there is no term in $(x-4)^8$ appearing in the series. Therefore

$$\boxed{f^{(8)}(4)=0}.$$

The coefficient of $(x-4)^9$ in the series is $\dfrac{f^{(9)}(4)}{9!}$. The term in $(x-4)^9$ is obtained in the series when $2n+1=9$, that is for $n=4$. So looking at the resulting coefficient gives

$$\frac{f^{(9)}(4)}{9!} = \frac{(-1)^4}{9^4(4+1)}.$$

So

$$\boxed{f^{(9)}(4) = \frac{9!}{5 \cdot 9^4}}.$$

4. Use the reference Maclaurin series to calculate the Maclaurin series of the following functions.

(a) $f(x) = x^7 \cos(4x^5)$.

Solution. We know

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

So we get

$$\begin{aligned} x^7 \cos(4x^5) &= x^7 \sum_{n=0}^{\infty} \frac{(-1)^n (4x^5)^{2n}}{(2n)!} \\ &= x^7 \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{10n}}{(2n)!} \\ &= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{10n+7}}{(2n)!}}. \end{aligned}$$

(b) $f(x) = e^{-x^3} - 1 + x^3$.

Solution. From

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

we get

$$e^{-x^3} = \sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!} = 1 - x^3 + \frac{x^6}{2} - \frac{x^9}{6} + \cdots.$$

We then see that adding $-1 + x^3$ to e^{-x^3} will cancel out the first two terms of this Maclaurin series, giving

$$\begin{aligned} e^{-x^3} - 1 + x^3 &= \left(1 - x^3 + \frac{x^6}{2} - \frac{x^9}{6} + \cdots\right) - 1 + x^3 \\ &= \frac{x^6}{2} - \frac{x^9}{6} + \cdots \\ &= \boxed{\sum_{n=2}^{\infty} \frac{(-1)^n x^{3n}}{n!}}. \end{aligned}$$

(c) $f(x) = \sin(2x) - 2\tan^{-1}(x)$

Solution. We know that

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots,$$

so

$$\sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!} = 2x - \frac{8x^3}{6} + \frac{32x^5}{120} + \cdots.$$

On the other hand, we have

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots,$$

so

$$2\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{2(-1)^n x^{2n+1}}{2n+1} = 2x - \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots.$$

When we subtract these two Maclaurin series, the terms $2x$ will cancel out. We can group together the remaining terms of same degree to obtain

$$\begin{aligned} \sin(2x) - 2\tan^{-1}(x) &= \left(2x - \frac{8x^3}{6} + \frac{32x^5}{120} + \cdots\right) - \left(2x - \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots\right) \\ &= -\left(\frac{8}{6} - \frac{2}{3}\right)x^3 + \left(\frac{32}{120} - \frac{2}{5}\right)x^5 + \cdots \\ &= \boxed{\sum_{n=1}^{\infty} (-1)^n \left(\frac{2^{2n+1}}{(2n+1)!} - \frac{2}{2n+1}\right) x^{2n+1}}. \end{aligned}$$