Section 10.9

# Convergence of Taylor Series

## Learning Goals

| *Learning Goal* | *Homework Problems* |
|---|---|
| 10.9.1 Find the Maclaurin series for a function by substituting for x and other simple algebraic manipulation of other Maclaurin series | 10.9: 1,7,9,11,15,16,22,23,26 |
| 10.9.2 Find the Maclaurin series for a function using a combination of two Maclaurin series | 10.9: 27 |
| 10.9.3 Estimate error using either the Remainder estimation Theorem or the alternating Series Estimation Theorem and find the number of term to attain a given error. | 10.9:45,46 Also 10.10: 45,46 |

Conceptual introduction: in the previous section, we learned that the $N^{th}$ degree Taylor polynomial of a function $f$ at $x=a$ gives the best $N^{th}$ degree polynomial approximation of $f$ near $x=a$. In this section, we investigate how accurate this approximation is.

Recall that $T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$

The remainder is $\boxed{R_N(x) = f(x) - T_N(x)}$.

$|R_N(x)|$ is the error made when approximating $f(x)$ with $T_N(x)$.

Remark: in most cases, $f(x) = T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

So:

$$f(x) = \underbrace{f(a) + f'(a)(x-a) + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N}_{T_N(x)} + \underbrace{\frac{f^{(N+1)}(a)}{(N+1)!}(x-a)^{N+1} + \cdots}_{R_N(x)}$$

So $\boxed{R_N(x) = \sum_{n=N+1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$

Example: for $f(x) = e^x$, find $T_2(x)$ and $R_2(x)$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \underbrace{1 + x + \frac{x^2}{2}}_{T_2(x)} + \underbrace{\frac{x^3}{6} + \frac{x^4}{24} + \cdots}_{R_2(x)}$$

So $T_2(x) = 1 + x + \frac{x^2}{2}$ and $R_2(x) = \sum_{n=3}^{\infty} \frac{x^n}{n!} = \frac{x^3}{6} + \frac{x^4}{24} + \cdots$

**Taylor's Theorem:** if $f$ has derivatives of all orders on an open interval $I$ containing $a$, then for all $x$ in $I$, there exists $c$ between $a$ and $x$ such that:

$$\boxed{R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}}$$

**Remark:** the case $N=0$ is simply the Mean Value Theorem from Calculus I: $f(x) - f(a) = f'(c)(x-a)$ for some $c$ between $x$ and $a$.

**Remainder Estimation Theorem:**

$$\boxed{|R_N(x)| \leqslant \frac{M|x-a|^{N+1}}{(N+1)!}}$$

where $M > 0$ is any number such that $|f^{(N+1)}(t)| \leqslant M$ for all $t$ between $x$ and $a$.

**Examples:** 1) Estimate the error made when using the approximation $e^x \approx 1 + x + \frac{x^2}{2}$ for $\underbrace{|x| < 0.5}_{\text{interval } (-0.5,\, 0.5)}$.

To use the Remainder Estimation Theorem, we must find a possible value for $M$. We need $|f^{(3)}(t)| \leqslant M$ in $(-0.5, 0.5)$.

We have $f^{(3)}(t) = e^t$.

So $|f^{(3)}(t)| = e^t \leqslant \underbrace{e^{0.5}}_{\text{we use this as } M}$ on $(-0.5, 0.5)$

So $|R_2(x)| \leqslant \frac{M|x|^3}{3!} \underset{|x| \leqslant 0.5}{\leqslant} \frac{e^{0.5}(0.5)^3}{6} \simeq 0.034$.

Therefore, the error made is at most $\boxed{0.034}$.

2) $f(x) = \sqrt{x}$

Estimate the error made when using the $2^{nd}$ degree Taylor polynomial $T_2(x)$ at $x=1$ to approximate $f(x)$ for $\underbrace{|x-1| < 0.1}_{\text{interval } (0.9,\, 1.1)}$.

To use the Remainder Estimation Theorem, need a possible value of M. Want $|f^{(3)}(t)| \leq M$ for $t$ in $(0.9, 1.1)$.

$$f(x) = x^{1/2}$$
$$f'(x) = \frac{1}{2}x^{-1/2}$$
$$f''(x) = -\frac{1}{4}x^{-3/2}$$
$$f^{(3)}(x) = \frac{3}{8}x^{-5/2}$$

largest when $|t|$ is smallest

$$|f^{(3)}(t)| = \frac{3}{8|t|^{5/2}} \leq \underbrace{\frac{3}{8(0.9)^{5/2}}}_{\text{this is the M we can use}}$$

$$\text{So} \quad |R_2(x)| \leq \frac{M|x-1|^3}{3!} \leq \frac{3}{8(0.9)^{5/2}} \cdot \frac{(0.1)^3}{6} \simeq 0.00008.$$

$|x-1|$ is at most $0.1$

Therefore, the error made is at most $\boxed{0.00008}$.

3) $f(x) = \cos(3x)$, $T(x) =$ Maclaurin series of $f$. How many terms of $T(x)$ must be summed to ensure that for $|x| < \frac{1}{9}$, $T(x)$ approximates $f(x)$ with an error of at most $10^{-4}$?

We are looking for $N$ such that $|R_N(x)| \leq 10^{-4}$ in the interval $(-\frac{1}{9}, \frac{1}{9})$.

Let us compute the error $|R_N(x)| \leq \frac{M|x|^{N+1}}{(N+1)!}$.

$$f(x) = \cos(3x) \quad \text{so} \quad f^{(n)}(x) = \pm 3^n \cdot \begin{cases} \cos(3x) & n \text{ even} \\ \sin(3x) & n \text{ odd} \end{cases}$$

In particular, $|f^{(N+1)}(t)| \leqslant \underbrace{3^{N+1}}_{\text{take this for } M}$ in $\left(-\frac{1}{9}, \frac{1}{9}\right)$.

So $$|R_N(x)| \leqslant \frac{M|x|^{N+1}}{(N+1)!} \underset{|x| \leqslant \frac{1}{9}}{\leqslant} \frac{3^{N+1}}{(N+1)!} \cdot \frac{1}{9^{N+1}} = \underbrace{\frac{1}{(N+1)!\,3^{N+1}}}_{\text{error}}$$

We want error $\leqslant 10^{-4}$

$$\frac{1}{(N+1)!\,3^{N+1}} \leqslant 10^{-4}$$

$$(N+1)!\,3^{N+1} \geqslant 10^4$$

This holds if $N \geqslant 4$

| $N$ | $(N+1)!\,3^{N+1}$ |
|---|---|
| 0 | 3 ✗ |
| 1 | 18 ✗ |
| 2 | 162 ✗ |
| 3 | 2016 ✗ |
| 4 | 29160 ✓ |

So the Maclaurin polynomial giving an approximation with error at most $10^{-4}$ for $|x| < \frac{1}{9}$ is

$T_4(x) = 1 - \frac{(3x)^2}{2} + \frac{(3x)^4}{24}$ : we sum $\boxed{\text{3 terms}}$.

4) $f(x) = \frac{1}{1+2x}$, $T(x)$ = Maclaurin series of $f$.

How many terms of $T(x)$ must be summed to ensure that for $|x| < 0.05$, $T(x)$ approximates $f(x)$ with an error of at most $0.001$ ?

We are looking for the smallest value of $N$ for which $|R_N(x)| \leqslant 0.001$ if $x$ is in the interval $(-0.05, 0.05)$.

we start by estimating the error.

Remainder Estimation Theorem : $|R_N(x)| \leq \frac{M|x|^{N+1}}{(N+1)!}$

Let us find M.

$$f(x) = (1+2x)^{-1}$$
$$f'(x) = -(1+2x)^{-2}\cdot 2$$
$$f''(x) = 2(1+2x)^{-3}\cdot 2^2$$
$$f'''(x) = -6(1+2x)^{-4}\cdot 2^3$$
$$\vdots$$
$$f^{(n)}(x) = (-1)^n n!\,(1+2x)^{-n-1}\,2^n$$
$$f^{(N+1)}(x) = \frac{(-1)^{N+1}(N+1)!\,2^{N+1}}{(1+2x)^{N+2}}$$

So on the interval $(-0.05, 0.05)$ :

$$|f^{(N+1)}(t)| \leq \frac{(N+1)!\,2^{N+1}}{(1+2(-0.05))^{N+2}} = \underbrace{\frac{(N+1)!\,2^{N+1}}{(0.9)^{N+2}}}_{\text{we can take this for } M}.$$

$|x| \leq 0.05$

$$|R_N(x)| \leq \frac{M|x|^{N+1}}{(N+1)!} \leq \frac{\cancel{(N+1)!}\,2^{N+1}}{(0.9)^{N+2}}\cdot\frac{(0.05)^{N+1}}{\cancel{(N+1)!}}$$
$$= \frac{(2\cdot 0.05)^{N+1}}{(0.9)^{N+2}} = \frac{(0.1)^{N+1}}{(0.9)^{N+2}} = \underbrace{\frac{10}{9^{N+2}}}_{\text{error}}$$

Now we want $\frac{10}{9^{N+2}} \leq 0.001 \Rightarrow 9^{N+2} \geq \frac{10}{0.001} = 10\,000$

So $N+2 \geq \log_9(10000)$

$N \geq \log_9(10000) - 2 \simeq 4.2$

So the smallest value of N giving the desired error is $N=5$, which means we must sum $\boxed{6 \text{ terms}}$ $(N=0 \text{ to } 5)$.