Rutgers University
Math 152

Section 10.9: Convergence of Taylor Series - Worksheet Solutions

1. Use the Remainder Estimation Theorem to estimate the error made when approximating $f(x) = \sqrt{1+3x}$ by its 2^{nd} degree Maclaurin polynomial $T_2(x)$ on the interval $[0, 0.1]$.

   Solution. The Remainder Estimation Theorem gives us the error bound

   $$|R_2(x)| \leqslant \frac{M\,|x|^3}{3!} \leqslant \frac{M(0.1)^3}{6}.$$

   In this estimate, M is any number such that $M \geqslant \left|f^{(3)}(x)\right|$ for x in the interval $[0, 0.1]$. We have

   $$f'(x) = \frac{3}{2\sqrt{1+3x}},$$
   $$f''(x) = -\frac{9}{4(1+3x)^{3/2}},$$
   $$f^{(3)}(x) = \frac{81}{8(1+3x)^{5/2}}.$$

   Since the smallest value of $1+3x$ is 1 on the interval $[0, 0.1]$, we have $\left|f^{(3)}(x)\right| \leqslant \frac{81}{8}$. So we can choose $M = \frac{81}{8}$ for our error estimation. We obtain the error

   $$\frac{81(0.1)^3}{8 \cdot 6} = \boxed{\frac{27}{16000}}.$$

2. Consider the function $f(x) = \dfrac{1}{1+2x}$.

   (a) Find the Maclaurin series of f using geometric series. What are its radius and interval of convergence?

   Solution. We recognize that $f(x)$ is the sum of a geometric series with first term 1 and common ratio $-2x$, so

   $$f(x) = \sum_{n=0}^{\infty} (-2x)^n = \boxed{\sum_{n=0}^{\infty} (-2)^n x^n}.$$

   We know that a geometric series converges when its common ratio r satisfies $-1 < r < 1$. So the Maclaurin series of f converges when $-1 < -2x < 1$, which gives $-\frac{1}{2} < x < \frac{1}{2}$. Therefore, the ROC is $\boxed{R = \frac{1}{2}}$ and the IOC is $\boxed{\left(-\frac{1}{2}, \frac{1}{2}\right)}$.

(b) Find the Maclaurin series of f using its definition. (Hint: compute the first few derivatives of f and identify a pattern to find a formula for $f^{(n)}(x)$.)

Solution. We have

$$\begin{aligned} f'(x) &= -\frac{2}{(1+2x)^2}, \\ f''(x) &= \frac{2^2 \cdot 2}{(1+2x)^3}, \\ f^{(3)}(x) &= -\frac{2^3 \cdot 2 \cdot 3}{(1+2x)^3}, \\ f^{(4)}(x) &= \frac{2^4 \cdot 2 \cdot 3 \cdot 4}{(1+2x)^4}, \\ f^{(5)}(x) &= -\frac{2^5 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(1+2x)^5}. \end{aligned}$$

From this, we can see the following pattern for the higher derivatives of f:

$$f^{(n)}(x) = (-1)^n \frac{2^n n!}{(1+2x)^n}.$$

Therefore, the coefficients of the Maclaurin series of f are given by

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^n 2^n n!}{n!} = (-2)^n.$$

We deduce that the Maclaurin series of f is given by

$$\boxed{T(x) = \sum_{n=0}^{\infty} (-2)^n x^n}.$$

(c) Find is the smallest integer N for which the Maclaurin polynomial $T_N(x)$ of $f(x)$ approximates $f(x)$ with an error of at most 10^{-4} on the interval $[-0.1, 0.1]$.

Solution. Let us start by finding an expression for the error made when approximating $f(x)$ with $T_N(x)$ on the interval $[-0.1, 0.1]$. The Remainder Estimation Theorem gives us the error bound

$$|R_N(x)| \leqslant \frac{M\,|x|^{N+1}}{(N+1)!} \leqslant \frac{M(0.1)^{N+1}}{(N+1)!}.$$

In this estimate, M is any number such that $M \geqslant \left|f^{(N+1)}(x)\right|$ for x in the interval $[-0.1, 0.1]$. Using the formula for the higher derivatives of f that we found in the previous question, we have

$$\left|f^{(N+1)}(x)\right| = \left|(-1)^{N+1} \frac{2^{N+1}(N+1)!}{(1+2x)^{N+1}}\right| = \frac{2^{N+1}(N+1)!}{|1+2x|^{N+1}}.$$

This expression is largest when the denominator is smallest. In the interval $[-0.1, 0.1]$, this occurs when $x = -0.1$. So we can take for M the quantity

$$M = \frac{2^{N+1}(N+1)!}{(1+2(-0.1))^{N+1}} = \frac{2^{N+1}(N+1)!}{0.8^{N+1}} = 2.5^{N+1}(N+1)!.$$

So the error becomes

$$\frac{M(0.1)^{N+1}}{(N+1)!} = \frac{2.5^{N+1}(N+1)!(0.1)^{N+1}}{(N+1)!} = 0.25^{N+1} = \frac{1}{4^{N+1}}.$$

Now we want this error to be less than 10^{-4}, that is

$$\frac{1}{4^{N+1}} < 10^{-4}.$$

Taking reciprocals, this is equivalent to $4^{N+1} > 10000$. Computing the first few powers of 4, we see that this condition is met when the exponent of 4 is at least 7. So the smallest value of N giving the desired error is when $N+1 = 7$, that is $\boxed{N = 6}$.

3. Find the Maclaurin series of the following functions.

(a) $f(x) = x^4 \ln(1 - 5x^3)$.

Solution. From

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

we get

$$\begin{aligned} x^4 \ln(1-5x^3) &= x^4 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-5x^3)^n}{n} \\ &= x^4 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-5)^n x^{3n}}{n} \\ &= \boxed{\sum_{n=1}^{\infty} -\frac{5^n x^{3n+4}}{n}}. \end{aligned}$$

(b) $f(x) = e^{-x^2/2} - \cos(x)$.

Solution. We know

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots,$$

so

$$\begin{aligned} e^{-x^2/2} &= \sum_{n=0}^{\infty} \frac{\left(-\frac{x^2}{2}\right)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!} \\ &= 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \cdots \end{aligned}$$

We also know that

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots .$$

When we subtract these two Maclaurin series, the first two terms will cancel out. Combining the remaining terms gives

$$\begin{aligned} e^{-x^2/2} - \cos(x) &= \left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \cdots\right) - \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right) \\ &= \left(\frac{1}{8} - \frac{1}{24}\right) x^4 - \left(\frac{1}{48} - \frac{1}{720}\right) x^6 + \cdots \\ &= \boxed{\sum_{n=2}^{\infty} (-1)^n \left(\frac{1}{2^n n!} - \frac{1}{(2n)!}\right) x^{2n}}. \end{aligned}$$

4. Find the first three non-zero terms of the Maclaurin series of $\sin(2x)e^{3x}$.

Solution. We have

$$\sin(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} = 2x - \frac{4x^3}{3} + \cdots ,$$

and

$$e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = 1 + 3x + \frac{9x^2}{2} + \cdots .$$

So

$$\sin(2x)e^{3x} = \left(2x - \frac{4x^3}{3} + \cdots\right)\left(1 + 3x + \frac{9x^2}{2} + \cdots\right).$$

We can now distribute the terms and collect terms of same degree. Let us detail degree by degree what we obtain.

- We do not get any terms in degree 0.
- We get one term in degree 1, from multiplying the $2x$ in the left factor with the 1 in the right factor. This gives us $2x$ as our first non-zero term.
- We get one term in degree 2, from multiplying the $2x$ in the left factor with the $3x$ in the right factor. This gives us $6x^2$ as our second non-zero term.
- We get two terms in degree 3, the first one from multiplying the $2x$ in the left factor with the $\frac{9x^2}{2}$ in the right factor, and the second one from multiplying the $-\dfrac{4x^3}{3}$ in the left factor with the 1 in the right factor. This gives us $9x^3 - \frac{4x^3}{3} = \frac{23x^3}{3}$ as our third non-zero term.

Therefore, the first three non-zero terms of the Maclaurin of $\sin(2x)e^{3x}$ are

$$\boxed{2x + 6x^2 + \frac{23x^3}{3}}.$$