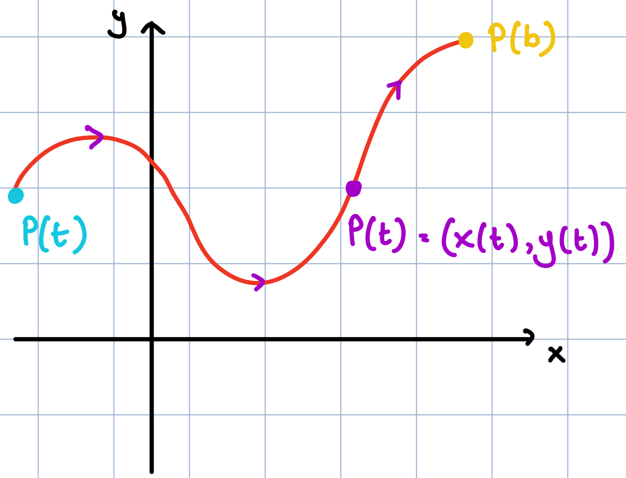
Sections 11.1, 11.2

Parametric Curves

Learning Goals

11.1.1 Express/match to a curve as y=f(x) (with direction and initial an terminal points) by eliminating the parameter	11.1: 3,5,7,8,11,19,20,21,22,23,24
11.1.2 Draw a parametric curve	11.1: 25
11.1.3 Parameterize a curve	11.1: 30,31,33,37
11.2.1 Find the tangent line to a parametric curve at a given parameter	11.2: 5,9,11,14 Also: 43
11.2.2 Find the first and the second derivative of a parametric equation	11.2: 5,11,14
11.2.3 Find the slope of an implicitly defined parametric curves at a given parameter.	11.2: 15,18,19 Also 11.2: 43
11.2.4 Find an area enclosed by parametric curves	11.2: 21,23
11.2.5 Find the arc length of a parametric curve	11.2: 25,27,29,42,47
11.2.6 Find the surface area of a surface that is generated by revolving a parametric curve about either the x or the y axis	11.2: 33,34,36,

Conceptual introduction: imagine that a particle moves in the xy-plane. As it moves, the particle traces a path on a curve.



Denote by $(x(t), y(t))$ the position of the particle at the time t.

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad a \leq t \leq b$$

are parametric equations of the curve.
or a parametrization

- t is the parameter (here time, could be another physical quantity, such as an angle)
- $a \leq t \leq b$ is the parameter interval (could be open and/or unbounded)
- $P(a)$ = initial point
  $P(b)$ = terminal point

Examples: 1) Find an equation of the curve parametrized by

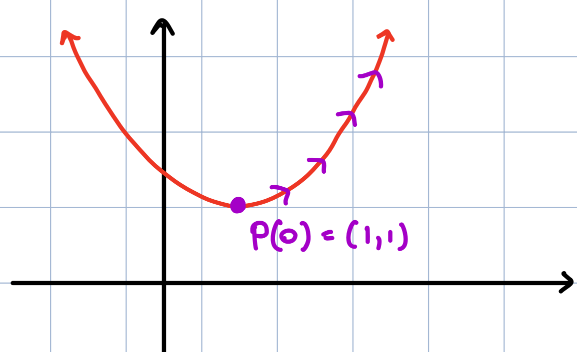
$$\begin{cases} x = 2t+1 \\ y = 1+t^2 \end{cases} \quad t \geq 0.$$

We can find an equation by solving for t in one of the equations and plugging-in the other.

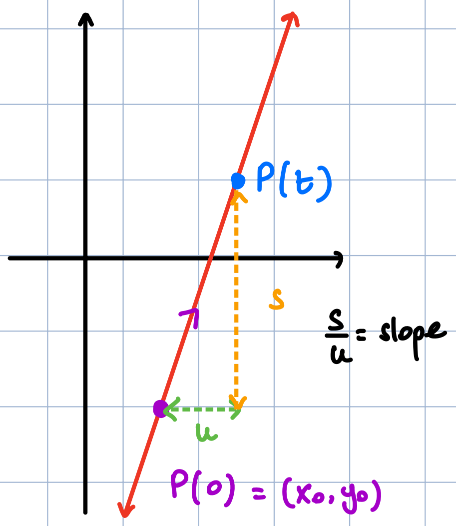
$$x = 2t+1 \Rightarrow t = \frac{x-1}{2}$$

So $y = 1+t^2$

$$\boxed{y = 1+\left(\frac{x-1}{2}\right)^2}$$



2) Find a parametrization of the line passing through $(1,-2)$ with slope 3.



$$\begin{cases} x = x_0 + ut \\ y = y_0 + st \end{cases}, \quad -\infty < t < \infty$$

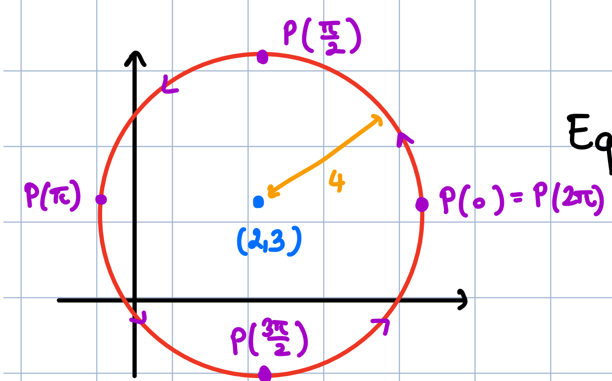
is a parametrization of the line passing through (x_0, y_0) with slope $m = \frac{s}{u}$

Here, $\boxed{\begin{cases} x = 1 + t \\ y = -2 + 3t \end{cases}}$ and $\boxed{\begin{cases} x = 1 - 2t \\ y = -2 - 6t \end{cases}}$ are two possible

parametrizations of the curve.

Remark: a curve can have many parametrizations. Each para-metrization represents a path on this curve, with a certain initial point, direction and speed.

3) Find a parametrization of the circle of center $(2,3)$ and radius 4.



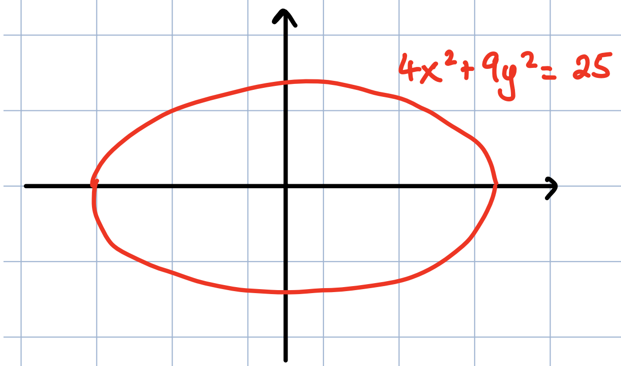
Equation : $(x-2)^2 + (y-3)^2 = 4^2$

or $\underbrace{\left(\frac{x-2}{4}\right)}_{\cos(t)}^2 + \underbrace{\left(\frac{y-3}{4}\right)}_{\sin(t)}^2 = 1$

$$\begin{cases} \frac{x-2}{4} = \cos(t) \\ \frac{y-3}{4} = \sin(t) \end{cases} \Rightarrow \boxed{\begin{cases} x = 2 + 4\cos(t) \\ y = 3 + 4\sin(t) \end{cases} \quad 0 \leq t \leq 2\pi}$$

goes around once counterclockwise

4) Parametrize the ellipse $4x^2+9y^2=25$.



$$\left(\frac{2x}{5}\right)^2+\left(\frac{3y}{5}\right)^2=1$$

$$\begin{cases}\frac{2x}{5}=\cos(t)\\ \frac{3y}{5}=\sin(t)\end{cases} \Rightarrow \boxed{\begin{cases}x=\frac{5}{2}\cos(t)\\ y=\frac{5}{3}\sin(t)\end{cases} \quad 0\leq t\leq 2\pi}$$

5) Match each parametric equation with its curve.

① $\begin{cases}x=t-\sin(t)\\ y=1-\cos(t)\end{cases}$ $t\geq 0$

② $\begin{cases}x=\cos(t)\\ y=2\sin(t)\end{cases}$ $0\leq t\leq \pi$

③ $\begin{cases}x=\cos(3t)\\ y=\sin(t)\end{cases}$

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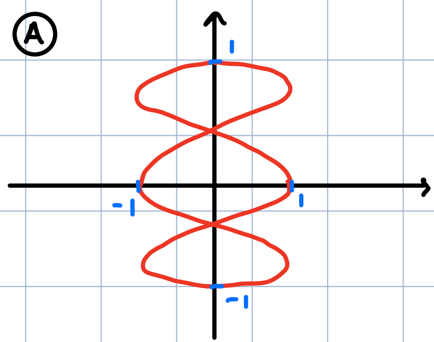

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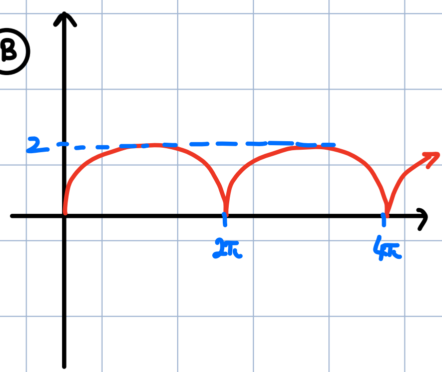

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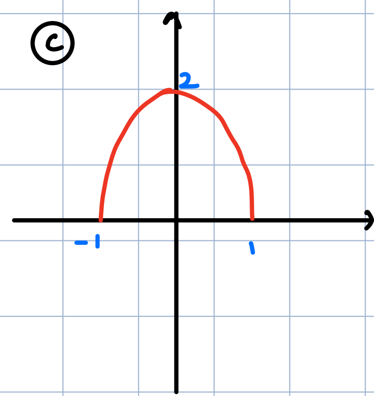


Solution : 1: B, 2: C, 3: A.

Calculus with parametric curves:

- Tangent line: the slope of the tangent line to a parametric curve $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ is given by:

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}}$$

Example: find the equation of the tangent line to $\begin{cases} x = \sec(t) \\ y = \tan(t) \end{cases}$ at the point corresponding to $t = \frac{\pi}{3}$.

We have $\frac{dy}{dt} = \sec(t)^2$, $\frac{dx}{dt} = \sec(t)\tan(t)$

so $\frac{dy}{dx} = \frac{\sec(t)^2}{\sec(t)\tan(t)} = \frac{\sec(t)}{\tan(t)} = \csc(t)$.

At $t = \frac{\pi}{3}$: $\begin{cases} x = \sec\left(\frac{\pi}{3}\right) = 2 \\ y = \tan\left(\frac{\pi}{3}\right) = \sqrt{3} \\ \frac{dy}{dx} = \csc\left(\frac{\pi}{3}\right) = \frac{2}{\sqrt{3}} \end{cases}$

So the tangent line has equation: $\boxed{y = \frac{2}{\sqrt{3}}(x-2) + \sqrt{3}}$

- Second derivative: if we apply $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ to $y' = \frac{dy}{dx}$, we get:

$$\boxed{\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}}$$

y' needs to be expressed in terms of t to use this formula.

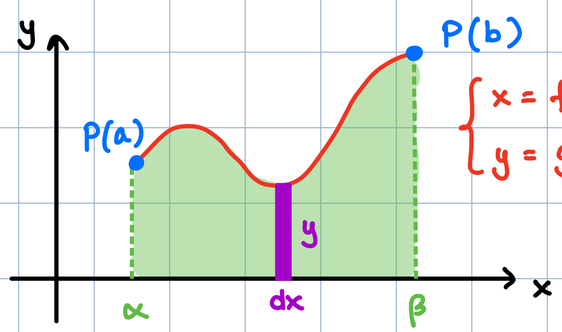
Example: find $\frac{d^2y}{dx^2}$ for $\begin{cases} x = t - t^2 \\ y = t - t^3 \end{cases}$

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t}$$

So $$\frac{dy'}{dt} = \frac{-6t(1-2t)-(1-3t^2)(-2)}{(1-2t)^2} = \frac{6t^2-6t+2}{(1-2t)^2}$$

and $$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \boxed{\frac{6t^2-6t+2}{(1-2t)^3}}$$

- **Area under curves:**

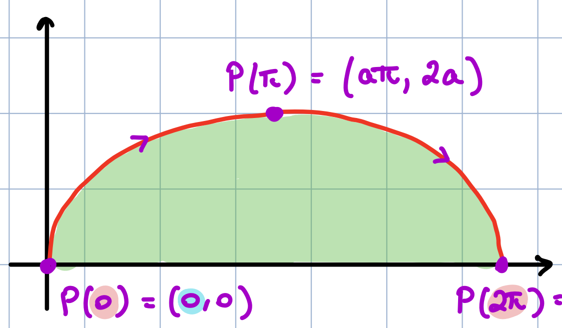


$$A = \int_\alpha^\beta y\,dx$$

$$\boxed{A = \int_a^b y(t)\frac{dx}{dt}\,dt = \int_a^b g(t)f'(t)\,dt}$$

Example: find the area under one arch of the cycloid

$$\begin{cases} x = a(t-\sin(t)) \\ y = a(1-\cos(t)) \end{cases} \quad (a>0)$$



$$A = \int_0^{2\pi a} y\,dx \qquad (x\text{- integral})$$

$$= \int_0^{2\pi} y\frac{dx}{dt}\,dt \qquad (t\text{- integral})$$

$$= \int_0^{2\pi} a(1-\cos(t))\,a(1-\cos(t))\,dt$$

$$= a^2\int_0^{2\pi}(1-\cos(t))^2\,dt$$

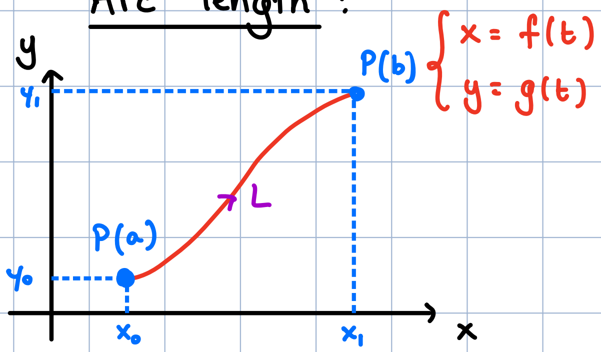
$$= a^2\int_0^{2\pi}(1-2\cos(t)+\cos(t)^2)\,dt$$

$$= a^2 \int_0^{2\pi} \left(1 - 2\cos(t) + \frac{1+\cos(2t)}{2} \right) dt$$

$$= a^2 \left[t - 2\sin(t) + \frac{t}{2} + \frac{\sin(2t)}{4} \right]_0^{2\pi}$$

$$= \boxed{3\pi a^2}$$

- <u>Arc length</u> :



We can compute L with :

- an x - integral
- a y - integral

$\Big\}$ section 6.3.

- a t - integral

$$L = \int_{curve} ds \quad \text{with} \quad ds = \sqrt{dx^2 + dy^2}$$

- x- integral : $L = \displaystyle\int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ (provided $\frac{dy}{dx}$ continuous)

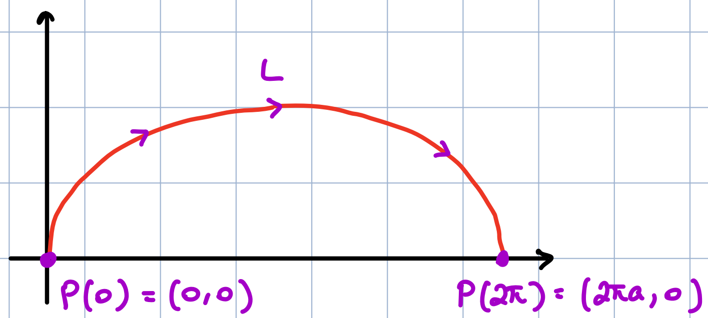
- y - integral : $L = \displaystyle\int_{y_0}^{y_1} \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$ (provided $\frac{dx}{dy}$ continuous)

- t - integral : $\boxed{L = \displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt}$

provided $\frac{dx}{dt}$, $\frac{dy}{dt}$ do not equal zero simultaneously and the path is traced once for $a \leq t \leq b$.

<u>Examples</u>: 1) find the length of one arch of the cycloid

$$\begin{cases} x = a(t - \sin(t)) \\ y = a(1 - \cos(t)) \end{cases} \quad (a > 0)$$



$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{a^2(1-\cos(t))^2 + a^2\sin(t)^2}\, dt$$

$$= a\int_0^{2\pi} \sqrt{1 - 2\cos(t) + \cos(t)^2 + \sin(t)^2}\, dt$$

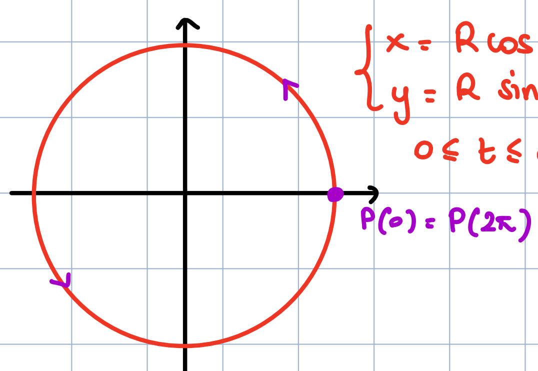
$$= a\int_0^{2\pi} \sqrt{2 - 2\cos(t)}\, dt$$

$$\boxed{1-\cos(t) = 2\sin\left(\frac{t}{2}\right)^2}$$

$$= a\int_0^{2\pi} \sqrt{4\sin\left(\frac{t}{2}\right)^2}\, dt$$

$$= 2a\int_0^{2\pi} \sin\left(\frac{t}{2}\right) dt = 2a\left[-2\cos\left(\frac{t}{2}\right)\right]_0^{2\pi} = \boxed{8a}$$

2) Find the circumference of a circle of radius R using a parametric integral.



$$\begin{cases} x = R\cos(t) \\ y = R\sin(t) \end{cases}$$

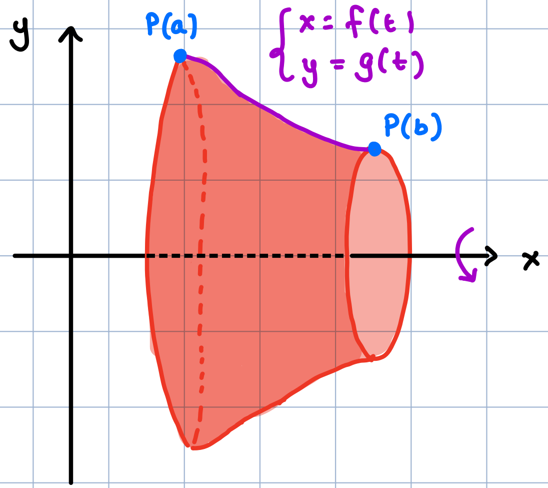
$$0 \leq t \leq 2\pi$$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{R^2\sin(t)^2 + R^2\cos(t)^2}\, dt$$

$$= \int_0^{2\pi} R\,dt = \boxed{2\pi R}.$$

- **Areas of surfaces of revolution:**



We revolve a curve $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ $a \le t \le b$ around the x-axis.

We can compute the resulting surface area using

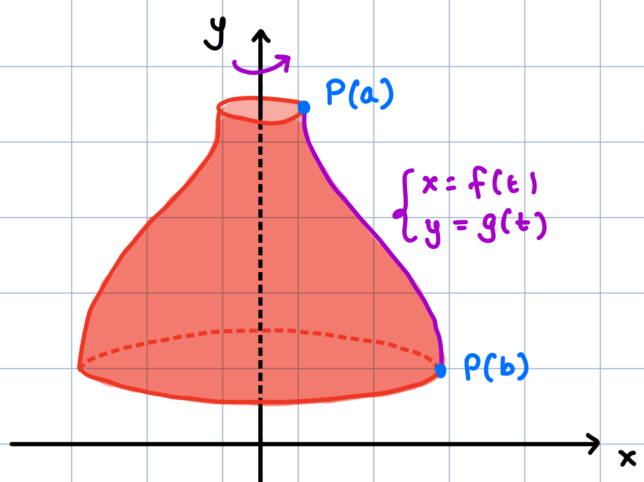
- an x-integral
- a y-integral
- a t-integral

$\Big\}$ section 6.4. (for the x-integral and y-integral)

$$A = \int_{curve} 2\pi y \, ds = \underbrace{\int_{x_0}^{x_1} 2\pi y(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx}_{(x\text{-integral})} = \underbrace{\int_{y_0}^{y_1} 2\pi y \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} \, dy}_{(y\text{-integral})}$$

With a t-integral:

$$\boxed{\begin{aligned} A &= \int_a^b 2\pi y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \\ &= \int_a^b 2\pi g(t) \sqrt{f'(t)^2 + g'(t)^2} \, dt \end{aligned}}$$



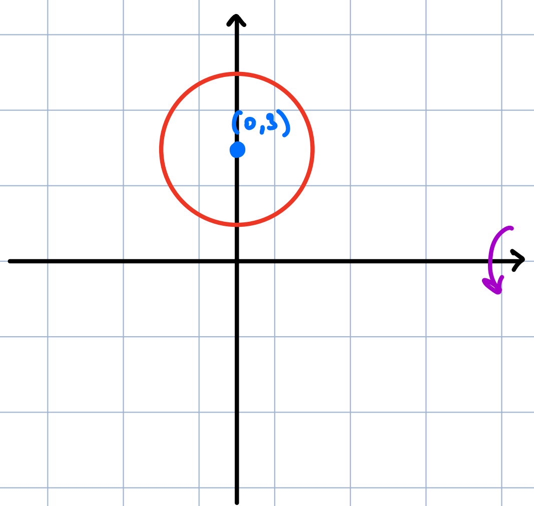
We can do the same for a curve revolved around the y-axis.

$$A = \int_{curve} 2\pi x \, ds = \underbrace{\int_{y_0}^{y_1} 2\pi x(y) \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} \, dy}_{(y\text{-integral})} = \underbrace{\int_{x_0}^{x_1} 2\pi \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx}_{(x\text{-integral})}$$

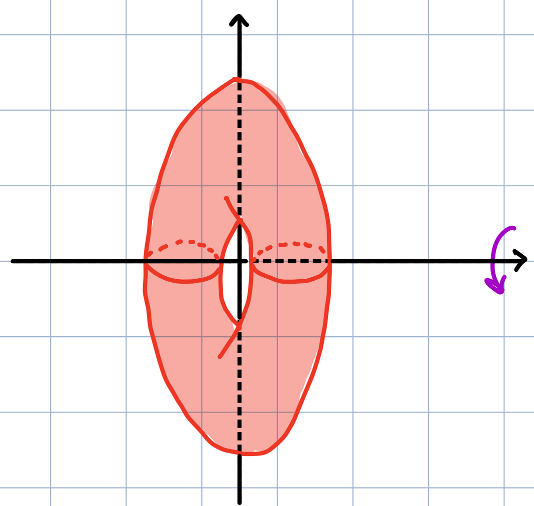
With a t-integral:

$$\boxed{\begin{aligned} A &= \int_a^b 2\pi x(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \\ &= \int_a^b 2\pi f(t)\sqrt{f'(t)^2+g'(t)^2}\,dt \end{aligned}}$$

Example: the circle of radius 2 centered at (0,3) is revolved around the x-axis. Find the resulting surface area.



$\Rightarrow$



Parametrization of the circle : $\begin{cases} x = 2\cos(t) \\ y = 3+2\sin(t) \end{cases}$ $\quad 0 \leq t \leq 2\pi$

$$A = \int_0^{2\pi} 2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^{2\pi} 2\pi\,(3+2\sin(t))\sqrt{4\sin(t)^2+4\cos(t)^2}\,dt$$

$$= 4\pi\int_0^{2\pi}(3+2\sin(t))\,dt = 4\pi\Big[3t-2\cos(t)\Big]_0^{2\pi} = \boxed{24\pi^2}$$