Rutgers University
Math 152

Sections 11.1, 11.2: Parametric Curves - Worksheet

1. Find an equation of the tangent line to the given parametric curve at the point defined by the given value of t.

(a) $\begin{cases} x = 5t^2 - 7 \\ y = t^4 - 3t \end{cases}$, $t = -1$.

Solution. We have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t-3}{10t}.$$

So the slope of the tangent line to the curve at $t = -1$ is

$$\frac{dy}{dx}_{|t=-1} = \frac{-4-3}{-10} = \frac{7}{10}.$$

Also, the tangent line to the curve at $t = -1$ passes through $(x(-1), y(-1)) = (-2, 4)$. It follows that it has equation

$$\boxed{y = \frac{7}{10}(x+2) + 4}.$$

(b) $\begin{cases} x = e^{4t} - e^t + 2 \\ y = t - 3e^{2t} \end{cases}$, $t = 0$.

Solution. We have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1-6e^{2t}}{4e^{4t} - e^t}.$$

So the slope of the tangent line to the curve at $t = 0$ is

$$\frac{dy}{dx}_{|t=0} = \frac{1-6}{4-1} = -\frac{5}{3}.$$

Also, the tangent line to the curve at $t = 0$ passes through $(x(0), y(0)) = (2, -3)$. It follows that it has equation

$$\boxed{y = -\frac{5}{3}(x-2) - 3}.$$

(c) $\begin{cases} x = \sec(3t) \\ y = \cot(2t - \pi) \end{cases}$, $t = \frac{\pi}{12}$.

Solution. We have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2\csc^2(2t-\pi)}{3\sec(3t)\tan(3t)}.$$

So the slope of the tangent line to the curve at $t = \frac{\pi}{12}$ is

$$\frac{dy}{dx}_{|t=\frac{\pi}{12}} = \frac{-2\csc^2\left(-\frac{5\pi}{6}\right)}{3\sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right)} = -\frac{8}{3\sqrt{2}}.$$

Also, the tangent line to the curve at $t = \frac{\pi}{12}$ passes through $\left(x\left(\frac{\pi}{12}\right), y\left(\frac{\pi}{12}\right)\right) = (\sqrt{2}, -\sqrt{3})$. It follows that it has equation

$$\boxed{y = \frac{8}{3\sqrt{2}}(x - \sqrt{2}) - \sqrt{3}}.$$

2. Find all points on the following parametric curves where the tangent line is (i) horizontal, and (ii) vertical.

(a) $\begin{cases} x = \sin(2t) + 1 \\ y = \cos(t) \end{cases}, 0 \leqslant t < 2\pi.$

Solution. We have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\sin(t)}{2\cos(2t)}.$$

(i) The tangent line is horizontal when $\dfrac{dy}{dx} = 0$, that is when $\dfrac{dy}{dt} = 0$ and $\dfrac{dx}{dt} \neq 0$. For $0 \leqslant t < 2\pi$, we have

$$\begin{aligned} &\frac{dy}{dt} = 0 \\ &\Rightarrow\ \sin(t) = 0 \\ &\Rightarrow\ t = 0, \pi. \end{aligned}$$

Observe that $\frac{dx}{dt} \neq 0$ for these values of t. To find the corresponding points on the curve, we plug-in these values of t in $(x(t), y(t))$ and we get the points $\boxed{(1,1), (1,-1)}$.

(ii) The tangent line is vertical when "$\dfrac{dy}{dx} = \infty$", that is when $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} \neq 0$. For $0 \leqslant t < 2\pi$, we have

$$\begin{aligned} &\frac{dx}{dt} = 0 \\ &\Rightarrow\ \cos(2t) = 0 \\ &\Rightarrow\ t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}. \end{aligned}$$

Observe that $\frac{dy}{dt} \neq 0$ for these values of t. To find the corresponding points on the curve, we plug-in these values of t in $(x(t), y(t))$ and we get the points $\boxed{\left(2, \frac{\sqrt{2}}{2}\right), \left(2, -\frac{\sqrt{2}}{2}\right), \left(0, \frac{\sqrt{2}}{2}\right), \left(0, -\frac{\sqrt{2}}{2}\right)}$.

(b) $\begin{cases} x = 3t - t^3 \\ y = t^2 + 4t + 3 \end{cases}$

Solution. We have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t + 4}{3 - 3t^2}.$$

(i) The tangent line is horizontal when $\frac{dy}{dx} = 0$, that is when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$. We have

$$\begin{aligned} &\frac{dy}{dt} = 0 \\ &\Rightarrow\ 2t + 4 = 0 \\ &\Rightarrow\ t = -2. \end{aligned}$$

Observe that $\frac{dx}{dt} \neq 0$ for this values of t. To find the corresponding point on the curve, we plug-in $t = -2$ in $(x(t), y(t))$ and we get the point $\boxed{(2, -1)}$.

(ii) The tangent line is vertical when "$\frac{dy}{dx} = \infty$", that is when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$. We have

$$\begin{aligned} &\frac{dx}{dt} = 0 \\ &\Rightarrow\ 3 - 3t^2 = 0 \\ &\Rightarrow\ t = -1, 1. \end{aligned}$$

Observe that $\frac{dy}{dt} \neq 0$ for these values of t. To find the corresponding points on the curve, we plug-in these values of t in $(x(t), y(t))$ and we get the points $\boxed{(-2, 0), (2, 8)}$.

(c) $\begin{cases} x = 4t - e^{2t} \\ y = t^2 - 18\ln|t| \end{cases}$

Solution. We have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t - \frac{18}{t}}{4 - 2e^{2t}}.$$

(i) The tangent line is horizontal when $\frac{dy}{dx} = 0$, that is when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$. We have

$$\begin{aligned} &\frac{dy}{dt} = 0 \\ &\Rightarrow\ 2t - \frac{18}{t} = 0 \\ &\Rightarrow\ t^2 = 9 \\ &\Rightarrow\ t = -3, 3. \end{aligned}$$

Observe that $\frac{dx}{dt} \neq 0$ for these values of t. To find the corresponding point on the curve, we plug-in these values of t in $(x(t), y(t))$ and we get the points $\boxed{\left(-12 - e^{-6}, 9 - 18\ln(3)\right), \left(12 - e^{6}, 9 - 18\ln(3)\right)}$.

(ii) The tangent line is vertical when "$\frac{dy}{dx} = \infty$", that is when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$. We have

$$\begin{aligned} &\frac{dx}{dt} = 0 \\ &\Rightarrow\ 4 - 2e^{2t} = 0 \\ &\Rightarrow\ t = \frac{1}{2}\ln(2). \end{aligned}$$

Observe that $\frac{dy}{dt} \neq 0$ for this value of t. To find the corresponding points on the curve, we plug-in this value of t in $(x(t), y(t))$ and we get the point $\boxed{\left(2\ln(2) - 2, \frac{\ln(2)^2}{4} - 18\ln\left(\frac{\ln(2)}{2}\right)\right)}$.

3. Consider the ellipse of equation $x^2 + 4y^2 = 4$.

(a) Find a parametrization of the ellipse.

Solution. The equation of the ellipse can be written as

$$\left(\frac{x}{2}\right)^2 + y^2 = 1,$$

from which we see that a possible parametrization is

$$\boxed{\begin{cases} x = 2\cos(t) \\ y = \sin(t) \end{cases}, 0 \leqslant t < 2\pi},$$

where we have chosen the parameter interval to ensure that the ellipse is traced exactly one.

(b) Find the area enclosed by the ellipse.

Solution. By symmetry, it suffices to find the area of the region inside the ellipse in the first quadrant and multiply it by 4. This region is bounded by the parametric curve $x = 2\cos(t), y = \sin(t)$ from $x = 0$ (which corresponds to $t = \frac{\pi}{2}$) to $x = 2$ (which corresponds to $t = 0$). Therefore, the area is given by

$$\begin{aligned}
A &= 4\int_0^2 y dx \\
&= 4\int_{\pi/2}^0 y(t)x'(t)dt \\
&= 4\int_{\pi/2}^0 \sin(t)(-2\sin(t))dt \\
&= 8\int_0^{\pi/2} \sin(t)^2 dt \\
&= 8\int_0^{\pi/2} \frac{1-\cos(2t)}{2} dt \\
&= 4\left[t - \frac{\sin(2t)}{2}\right]_0^{\pi/2} \\
&= \boxed{2\pi \text{ square units}}.
\end{aligned}$$

(c) Find the area of the surface obtained by revolving the top-half of the ellipse about the x-axis.

Solution. The top-half of the ellipse is parametrized by $x = 2\cos(t), y = \sin(t)$ for $0 \leqslant t \leqslant \pi$. Note that by symmetry, it suffices to find the area obtained by revolving the curve for $0 \leqslant t \leqslant \frac{\pi}{2}$ and multiply by 2. Therefore, the surface area is

$$\begin{aligned}
A &= 2\int_0^{\pi/2} 2\pi y(t)\sqrt{x'(t)^2 + y'(t)^2} dt \\
&= 4\pi\int_0^{\pi/2} \sin(t)\sqrt{4\sin(t)^2 + \cos(t)^2} dt.
\end{aligned}$$

This integral can be computed by expressing the inside of the square root in terms of $\cos(t)$ only (with the help of the Pythagorean identity) and then using a substitution. This gives

$$\begin{aligned} A &= 4\pi\int_0^{\pi/2} \sin(t)\sqrt{4(1-\cos(t)^2)+\cos(t)^2}dt \\ &= 4\pi\int_0^{\pi/2} \sin(t)\sqrt{4-3\cos(t)^2}dt \\ &= 8\pi\int_0^{\pi/2} \sin(t)\sqrt{1-\left(\frac{\sqrt{3}\cos(t)}{2}\right)^2}dt \\ &= \frac{16\pi}{\sqrt{3}}\int_0^{\sqrt{3}/2} \sqrt{1-u^2}du \quad \left(u = dfrac\sqrt{3}\cos(t)2\right). \end{aligned}$$

This integral can be computed with a trigonometric substitution $u = \sin(\theta)$, so that $du = \cos(\theta)d\theta$ and $\sqrt{1-u^2} = \cos(\theta)$. The bounds become

$$\begin{aligned} u = 0 &\Rightarrow \theta = \sin^{-1}(0) = 0, \\ u = \frac{\sqrt{3}}{2} &\Rightarrow \theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}. \end{aligned}$$

So

$$\begin{aligned} A &= \frac{16\pi}{\sqrt{3}}\int_0^{\pi/3} \cos(\theta)\cos(\theta)d\theta \\ &= \frac{16\pi}{\sqrt{3}}\int_0^{\pi/3} \cos(\theta)^2 d\theta \\ &= \frac{16\pi}{\sqrt{3}}\int_0^{\pi/3} \frac{1+\cos(2\theta)}{2}d\theta \\ &= \frac{8\pi}{\sqrt{3}}\left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{\pi/3} \\ &= \boxed{\frac{8\pi}{\sqrt{3}}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right) \text{ square units}}. \end{aligned}$$

4. For each of the following parametric curves: (i) find the arc length, (ii) set-up (but do not evaluate) an integral that computes the area of the surface obtained by revolving the curve about the x-axis and (iii) set-up (but do not evaluate) an integral that computes the area of the surface obtained by revolving the curve about the y-axis.

   (a) $\begin{cases} x = e^{4t} \\ y = e^{5t} \end{cases}, \ 0 \leqslant t \leqslant 1.$

   Solution. (i) The arc length is given by

$$L = \int_0^1 \sqrt{x'(t)^2 + y'(t)^2}dt$$

$$= \int_0^1 \sqrt{16e^{8t} + 25e^{10t}} dt$$
$$= \int_0^1 e^{4t}\sqrt{16 + 25e^{2t}} dt$$

We can calculate that integral by substituting $u = 16 + 25e^{2t}$, which gives $du = 50e^{2t}dt$. The extraneous factor e^{2t} in the integrand can be expressed in terms of u as $e^{2t} = \frac{u-16}{25}$. Finally, the bounds become

$$x = 0 \ \Rightarrow \ u = 16 + 25 = 41,$$
$$x = 1 \ \Rightarrow \ u = 16 + 25e.$$

We obtain

$$L = \frac{1}{50}\int_{41}^{16+25e} \frac{u-16}{25}\sqrt{u} du$$
$$= \frac{1}{175}\int_{41}^{16+25e} \left(u^{3/2} - 16u^{1/2}\right) du$$
$$= \frac{1}{175}\left[\frac{2}{5}u^{5/2} - \frac{32}{3}u^{3/2}\right]_{41}^{16+25e}$$
$$= \boxed{\frac{1}{175}\left(\frac{2}{5}(16+25e)^{5/2} - \frac{32}{3}(16+25e)^{3/2} - \frac{2}{5}41^{5/2} - \frac{32}{3}41^{3/2}\right) \text{ units}}.$$

(ii) Revolution about the x-axis: $\boxed{A = \int_0^1 2\pi e^{5t}\sqrt{16e^{8t} + 25e^{10t}} dt}$.

(ii) Revolution about the y-axis: $\boxed{A = \int_0^1 2\pi e^{4t}\sqrt{16e^{8t} + 25e^{10t}} dt}$.

(b) $\begin{cases} x = \ln(t) \\ y = \sin^{-1}(t) \end{cases}$, $\frac{1}{2} \leqslant t \leqslant \frac{1}{\sqrt{2}}$.

Solution. (i) The arc length is given by

$$L = \int_{1/2}^{1/\sqrt{2}} \sqrt{x'(t)^2 + y'(t)^2} dt$$
$$= \int_{1/2}^{1/\sqrt{2}} \sqrt{\frac{1}{t^2} + \frac{1}{1-t^2}} dt$$
$$= \int_{1/2}^{1/\sqrt{2}} \sqrt{\frac{1-t^2+t^2}{t^2(1-t^2)}} dt$$
$$= \int_{1/2}^{1/\sqrt{2}} \sqrt{\frac{1}{t^2(1-t^2)}} dt$$
$$= \int_{1/2}^{1/\sqrt{2}} \frac{1}{t\sqrt{1-t^2}} dt.$$

This integral can be computed with the trigonometric substitution $t = \sin(\theta)$, which gives $dt = \cos(\theta)d\theta$ and $\sqrt{1-t^2} = \sqrt{1-\sin(\theta)^2} = \cos(\theta)$. The bounds become

$$x = \frac{1}{2} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6},$$
$$x = \frac{1}{\sqrt{2}} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}.$$

So

$$\begin{aligned}
L &= \int_{\pi/6}^{\pi/4} \frac{1}{\sin(\theta)\cos(\theta)} \cos(\theta)d\theta \\
&= \int_{\pi/6}^{\pi/4} \csc(\theta)d\theta \\
&= [-\ln|\csc(\theta)+\cot(\theta)]_{\pi/6}^{\pi/4} \\
&= \boxed{\ln\left(2+\sqrt{3}\right) - \ln(\sqrt{2}+1) \text{ units}}.
\end{aligned}$$

(ii) Revolution about the x-axis: $\boxed{A = \int_{1/2}^{1/\sqrt{2}} 2\pi \sin^{-1}(t)\sqrt{\frac{1}{t^2}+\frac{1}{1-t^2}}dt}$.

(ii) Revolution about the y-axis: $\boxed{A = \int_{1/2}^{1/\sqrt{2}} 2\pi \ln(t)\sqrt{\frac{1}{t^2}+\frac{1}{1-t^2}}dt}$.

(c) $\begin{cases} x = t^3 - t \\ y = \sqrt{3}t^2 \end{cases}$, $0 \leqslant t \leqslant 1$.

Solution. (i) The arc length is given by

$$\begin{aligned}
L &= \int_0^1 \sqrt{x'(t)^2 + y'(t)^2}dt \\
&= \int_0^1 \sqrt{(3t^2-1)^2 + \left(2\sqrt{3}t\right)^2}dt \\
&= \int_0^1 \sqrt{9t^4 - 6t^2 + 1 + 12t^2}dt \\
&= \int_0^1 \sqrt{9t^4 + 6t^2 + 1}dt \\
&= \int_0^1 \sqrt{(3t+1)^2}dt \\
&= \int_0^1 (3t+1)dt \\
&= \left[\frac{3t^2}{2} + t\right]_0^1 \\
&= \boxed{\frac{5}{2} \text{ units}}.
\end{aligned}$$

(ii) Revolution about the x-axis: $\boxed{A = \int_0^1 2\pi\sqrt{3}t^2\sqrt{(3t^2-1)^2 + \left(2\sqrt{3}t\right)^2}\,dt}$.

(ii) Revolution about the y-axis: $\boxed{A = \int_0^1 2\pi\left(t^3 - t\right)\sqrt{(3t^2-1)^2 + \left(2\sqrt{3}t\right)^2}\,dt}$.