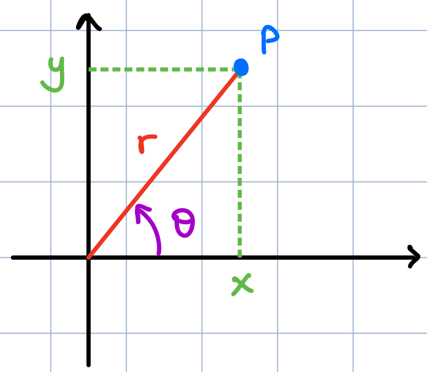
Sections 11.3, 11.4

Polar Coordinates

Learning Goals

11.3.1 Find different labels to the same point.	11.3: 1,2
11.3.2 Convert polar coordinates to rectangular coordinates and vice versa	11.3: 6,7,8
11.3.3 Sketch a polar curve	11.3: 11,13,14,15,19 Also 10.5: 6,19,29
11.3.4 Convert polar curves to Cartesian curves and vice versa	11.3: 28,31,35,40,41,43,48.49,56,60,65,68
11.4.1 Identify symmetries and sketch a polar curve in the xy-plane.	11.4:1,6
11.4.2 Find a slope and a tangent line to a polar curve at a given theta	11.4: 19,20

Conceptual introduction: we commonly use Cartesian / rectangular coordinates (x, y) to locate a point in the xy-plane. **Polar coordinates** are a different way of locating a point.

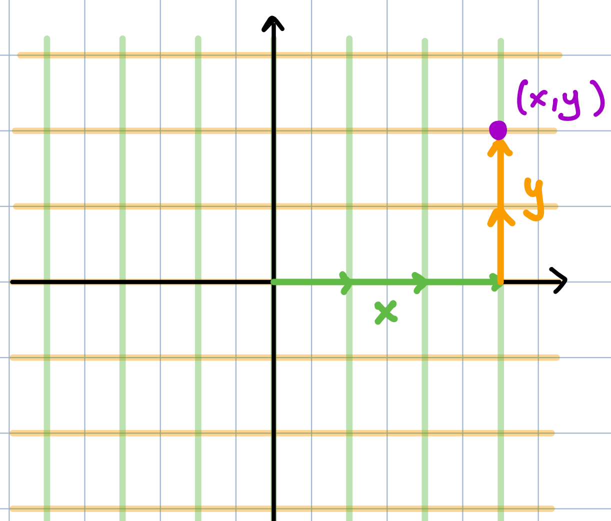


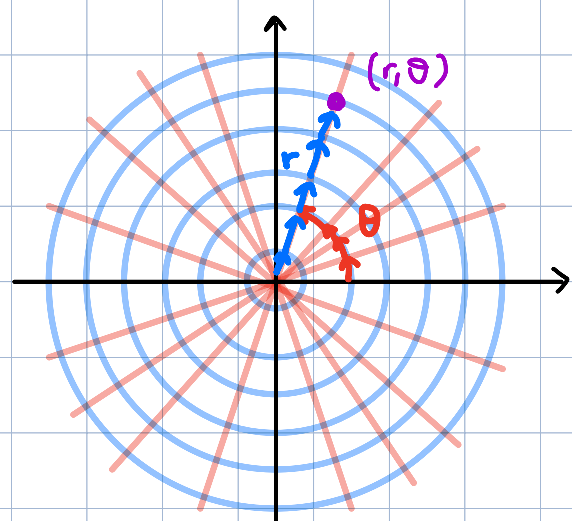
r = distance from the origin
θ = angle from positive x-axis
(r, θ) = polar coordinates.

Conversion formulas:

Cartesian to Polar	Polar to Cartesian
$r = \sqrt{x^2+y^2}$	$x = r\cos(\theta)$
$\tan(\theta) = \frac{y}{x}$	$y = r\sin(\theta)$

Grids:





Cartesian grid:
vertical lines are x = constant
horizontal lines are y = constant

Polar grid:
circles are r = constant
rays are θ = constant

Examples: 1) Find the cartesian coordinates of the points having polar coordinates a) $\left(3, \frac{5\pi}{6}\right)$ and b) $\left(2, -\frac{\pi}{4}\right)$.

a) $\left(3, \frac{5\pi}{6}\right)$: $x = r\cos(\theta) = 3\cos\left(\frac{5\pi}{6}\right) = -\frac{3\sqrt{3}}{2}$ $\Rightarrow$ $\boxed{\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)}$

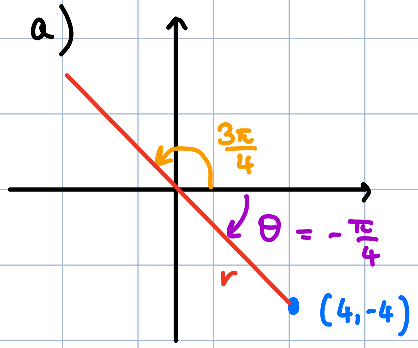
$y = r\sin(\theta) = 3\sin\left(\frac{5\pi}{6}\right) = \frac{3}{2}$

b) $\left(2, -\frac{\pi}{4}\right)$: $x = r\cos(\theta) = 2\cos\left(-\frac{\pi}{4}\right) = \sqrt{2}$ $\Rightarrow$ $\boxed{\left(\sqrt{2}, -\sqrt{2}\right)}$

$y = r\sin(\theta) = 2\sin\left(-\frac{\pi}{4}\right) = -\sqrt{2}$

2) Find a possible polar coordinate pair for the points having cartesian coordinates a) $(4,-4)$ and b) $(-1, \sqrt{3})$

a)



$r = \sqrt{4^2 + (-4)^2} = \sqrt{32} = 4\sqrt{2}$.

$\tan(\theta) = \frac{-4}{4} = -1 \Rightarrow \theta = -\frac{\pi}{4}$ or $\frac{3\pi}{4}$

Point in QIII, so $\theta = -\frac{\pi}{4}$.

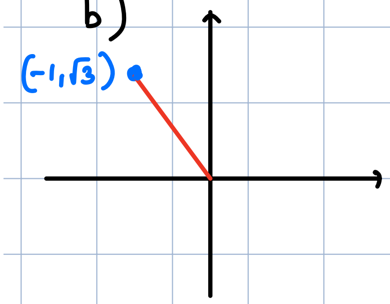
$\Rightarrow$ $\boxed{\left(4\sqrt{2}, -\frac{\pi}{4}\right)}$

Remark:

- Polar coordinates are not unique: $\left(4\sqrt{2}, \frac{7\pi}{4}\right)$, $\left(4\sqrt{2}, -\frac{9\pi}{4}\right)$ also work here. Indeed, $(r,\theta) = (r, \theta + 2\pi k)$ for any integer k.
- We can allow r to be negative: $(-r, \theta) = (r, \theta + \pi)$.

So $\left(-4\sqrt{2}, \frac{3\pi}{4}\right)$ would also be valid polar coordinates for $(-4, 4)$.

b)



$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2.$

$\tan(\theta) = -\sqrt{3} \Rightarrow \theta = -\frac{\pi}{3}$ or $\frac{2\pi}{3}.$

Since the point is in QII, $\theta = \frac{2\pi}{3}.$

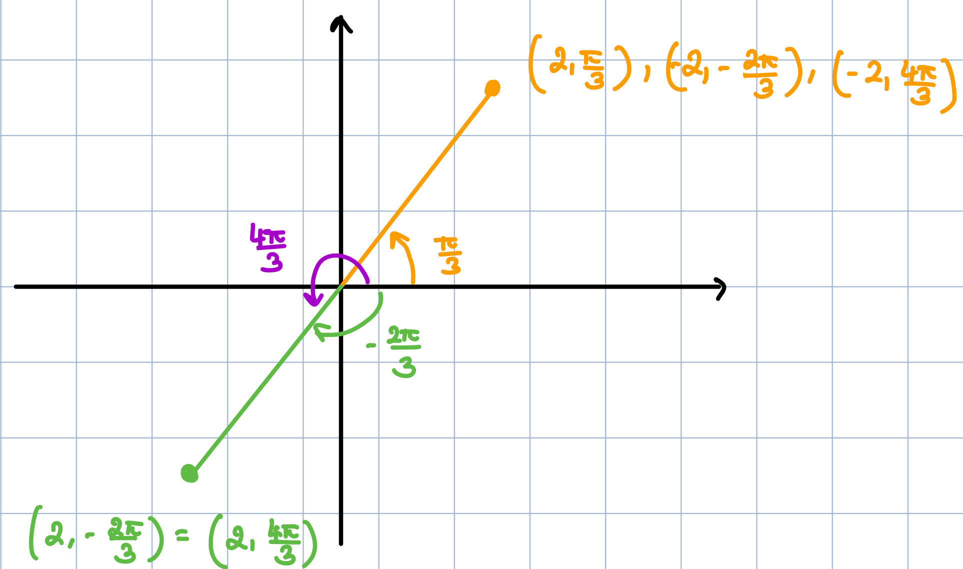
$\Rightarrow \boxed{\left(2, \frac{2\pi}{3}\right)}$

⚠ $\tan^{-1}$ always give an angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (QI, IV).

$$\theta = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & \text{if } x > 0 \quad (\text{QI, IV}) \\ \pi + \tan^{-1}\left(\frac{y}{x}\right) & \text{if } x < 0 \quad (\text{QII, III}) \end{cases}$$

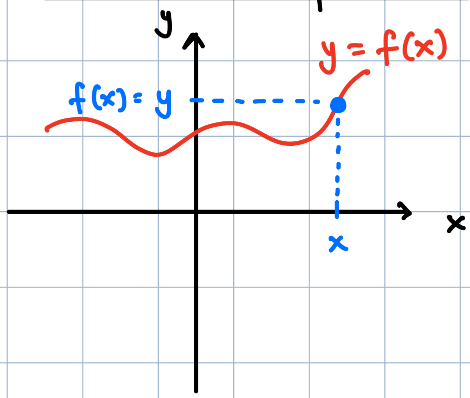
3) Which of these polar coordinates represent the same points?

$\left(2, \frac{\pi}{3}\right), \left(2, -\frac{2\pi}{3}\right), \left(-2, -\frac{2\pi}{3}\right), \left(2, \frac{4\pi}{3}\right), \left(-2, \frac{4\pi}{3}\right)$

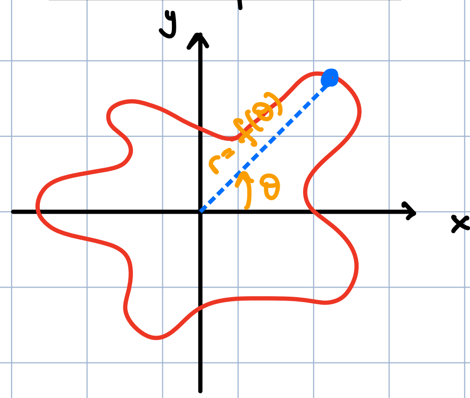


Graphing equations in polar:

Cartesian equations:

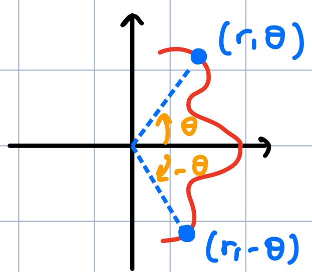


Polar equations:



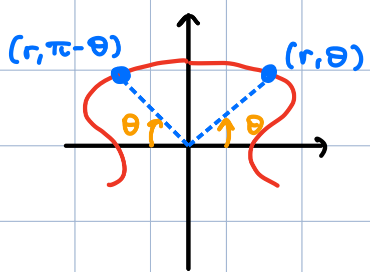
Symmetries of polar curves:

About x-axis



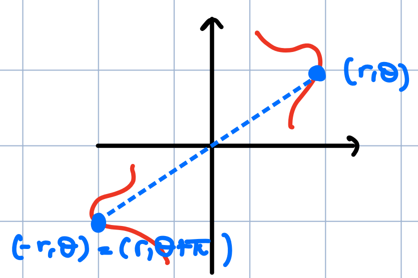
If (r, θ) is on the curve, then so is $(r, -\theta)$

About y-axis



If (r, θ) is on the curve, then so is $(r, \pi - \theta)$

About origin



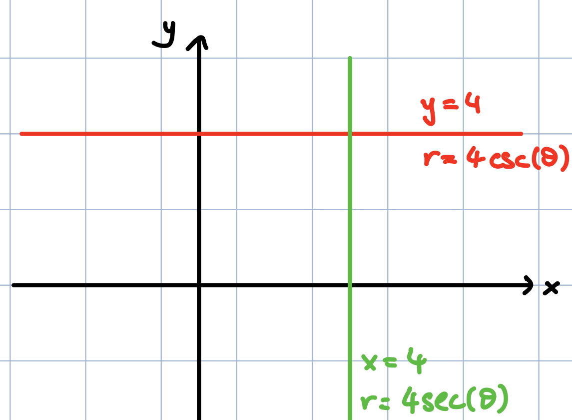
If (r, θ) is on the curve, then so is $(r, \theta + \pi) = (-r, \theta)$

Examples:

1) Convert the polar curve $r = 4\csc(\theta)$ to Cartesian.

$$r = \frac{4}{\csc(\theta)} = \frac{4}{\sin(\theta)} \Rightarrow r\sin(\theta) = 4$$

$$\boxed{y = 4}$$



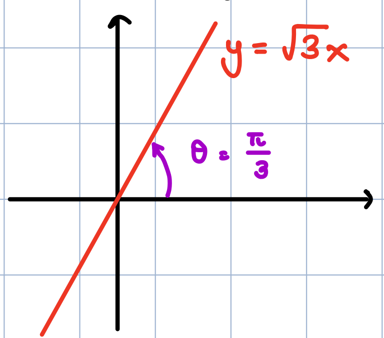
2) Convert $y = x^2$ and $y = \sqrt{3}x$ from Cartesian to polar.

- For $y = x^2$: $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases} \Rightarrow$ $r\sin(\theta) = r^2\cos(\theta)^2$

$$r = \frac{\sin(\theta)}{\cos(\theta)^2}$$

$$\boxed{r = \tan(\theta)\sec(\theta)}$$

- For $y = \sqrt{3}x$: $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases} \Rightarrow$ $r\sin(\theta) = \sqrt{3}\,r\cos(\theta)$

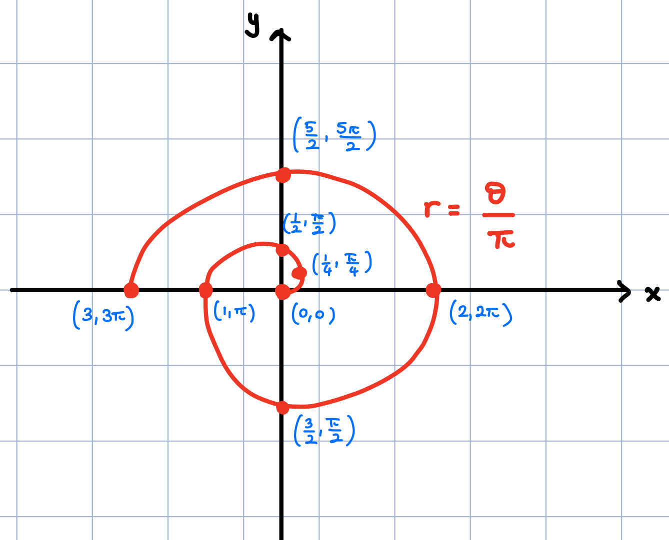


$$\tan(\theta) = \sqrt{3}$$

$$\boxed{\theta = \frac{\pi}{3}}$$

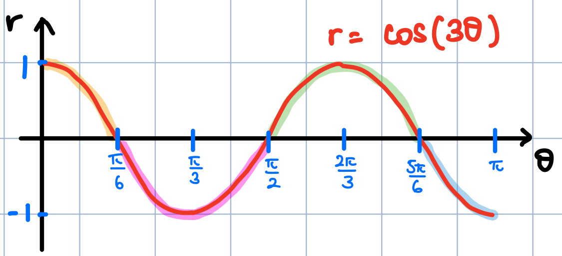
3) Sketch the graph of $r = \frac{\theta}{\pi}$ on $0 \leq \theta \leq 3\pi$.

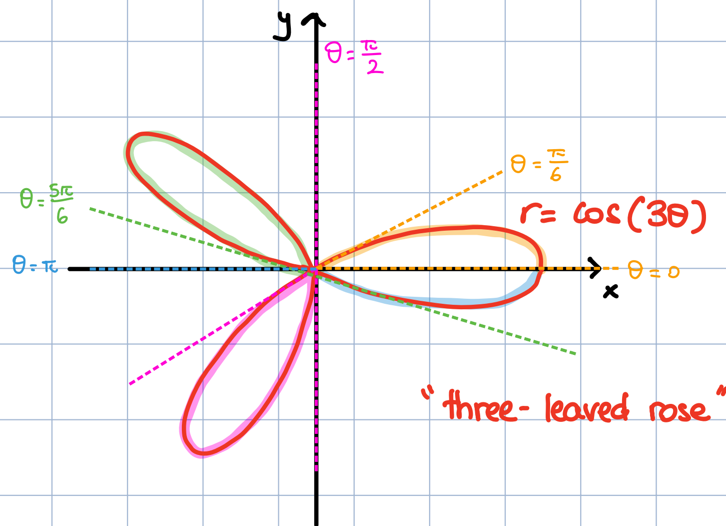
θ	$r = \frac{\theta}{\pi}$
0	0
$\frac{\pi}{4}$	$\frac{1}{4}$
$\frac{\pi}{2}$	$\frac{1}{2}$
π	1
$\frac{3\pi}{2}$	$\frac{3}{2}$
2π	2
3π	3



4) Sketch the graph of $r = \cos(3\theta)$, $0 \leq \theta \leq \pi$.

We can use the graph in the $r\theta$- plane to help us graph in the xy- plane.





The curve has a symmetry about the x- axis since $\cos(3(-\theta)) = \cos(3\theta)$.

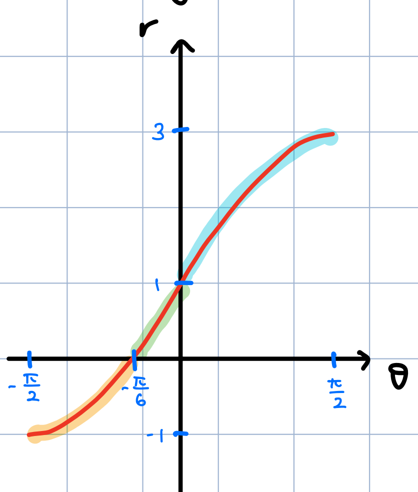
No symmetry about the y- axis or the origin

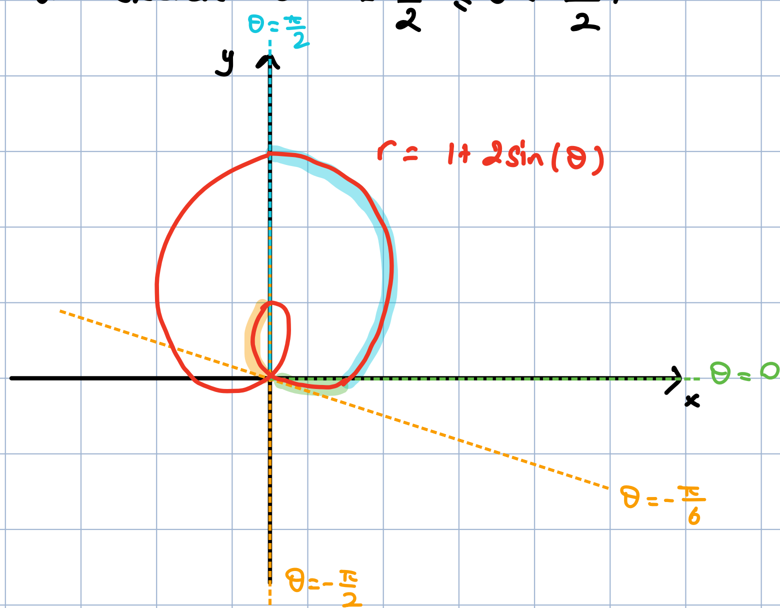
$$\cos(3(\pi-\theta)) = \cos(3\pi - 3\theta) = -\cos(3\theta) \neq \cos(3\theta)$$

$$\cos(3(\theta+\pi)) = \cos(3\pi + 3\theta) = -\cos(3\theta) \neq \cos(3\theta)$$

5) Sketch $r = 1 + 2\sin(\theta)$ for $-\pi \leq \theta \leq \pi$.

Since $\sin(\pi - \theta) = \sin(\theta)$, the curve is symmetric about the y- axis, so it suffices to sketch on $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.



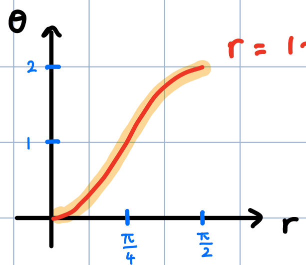


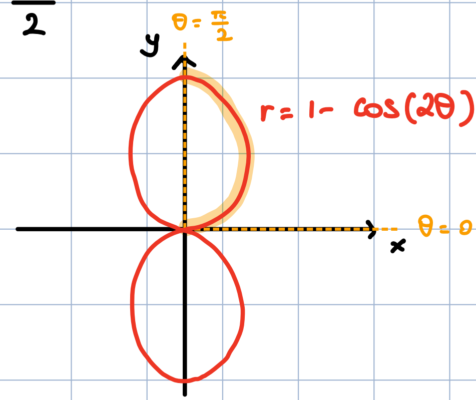
6) Sketch $r = 1 - \cos(2\theta)$ for $0 \leq \theta \leq 2\pi$.

Since $\cos(2(-\theta)) = \cos(2\theta)$, the curve is symmetric about the x-axis.

Since $\cos(2(\pi - \theta)) = \cos(2\pi - 2\theta) = \cos(2\theta)$, the curve is symmetric about the y-axis.

So it suffices to sketch on $0 \leq \theta \leq \frac{\pi}{2}$





Tangent lines to polar curves:

A polar curve $r = f(\theta)$ can be parametrized using

$$\begin{cases} x = r\cos(\theta) = f(\theta)\cos(\theta) \\ y = r\sin(\theta) = f(\theta)\sin(\theta) \end{cases}$$

Then we can use what we have learned about parametric curves to find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \quad \text{and} \quad \begin{cases} \frac{dy}{d\theta} = f'(\theta)\sin(\theta) + f(\theta)\cos(\theta) \\ \frac{dx}{d\theta} = f'(\theta)\cos(\theta) - f(\theta)\sin(\theta) \end{cases}$$

So $$\boxed{\frac{dy}{dx} = \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}}$$ (provided $\frac{dx}{d\theta} \neq 0$).

Remark: if the curve passes through the origin at $\theta = \theta_0$, then $f(\theta_0) = 0$ and $\frac{dy}{dx}\Big|_{\theta=\theta_0} = \tan(\theta_0)$.

Examples: 1) Find an equation of the tangent line to $r = \sin(2\theta)$ at $\theta = \frac{\pi}{3}$.

We have
$$f\left(\tfrac{\pi}{3}\right) = \sin\left(2\tfrac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$
$$f'(\theta) = 2\cos(2\theta) \Rightarrow f'\left(\tfrac{\pi}{3}\right) = 2\cos\left(\tfrac{2\pi}{3}\right) = -1$$

So
$$\frac{dy}{dx}\Big|_{\theta=\frac{\pi}{3}} = \frac{f'(\frac{\pi}{3})\sin(\frac{\pi}{3}) + f(\frac{\pi}{3})\cos(\frac{\pi}{3})}{f'(\frac{\pi}{3})\cos(\frac{\pi}{3}) - f(\frac{\pi}{3})\sin(\frac{\pi}{3})}$$
$$= \frac{-1\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{1}{2}}{-1\cdot\frac{1}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{4}.$$

At $\theta = \frac{\pi}{3}$, the curve passes through
$$x = f\left(\tfrac{\pi}{3}\right)\cos\left(\tfrac{\pi}{3}\right) = \frac{\sqrt{3}}{4}$$
$$y = f\left(\tfrac{\pi}{3}\right)\sin\left(\tfrac{\pi}{3}\right) = \frac{3}{4}.$$

So the tangent line has equation:
$$\boxed{y = \frac{\sqrt{3}}{4}\left(x - \frac{\sqrt{3}}{4}\right) + \frac{3}{4}}$$

2) Find an equation of the tangent line to $r = 1 - \cos(\theta)$ at $\theta = \frac{\pi}{2}$.

We have
$$f\left(\tfrac{\pi}{2}\right) = 1 - \cos\left(\tfrac{\pi}{2}\right) = 1$$
$$f'(\theta) = \sin(\theta) \Rightarrow f'\left(\tfrac{\pi}{2}\right) = 1.$$

So
$$\frac{dy}{dx}\Big|_{\theta=\frac{\pi}{2}} = \frac{f'(\frac{\pi}{2})\sin(\frac{\pi}{2}) + f(\frac{\pi}{2})\cos(\frac{\pi}{2})}{f'(\frac{\pi}{2})\cos(\frac{\pi}{2}) - f(\frac{\pi}{2})\sin(\frac{\pi}{2})} = \frac{1\cdot 1 + 1\cdot 0}{1\cdot 0 - 1\cdot 1} = -1.$$

At $\theta = \frac{\pi}{2}$, the curve passes through $x = f\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) = 0$

$y = f\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) = 1$

So the tangent line has equation:

$$\boxed{y = -x + 1}$$