Rutgers University
Math 152

Sections 11.3, 11.4: Polar Coordinates - Worksheet Solutions

1. Convert the following Cartesian equations to polar.

(a) $y = 11$.

Solution. Since $y = r\sin(\theta)$, the equation becomes $r\sin(\theta) = 11$ or $\boxed{r = 11\csc(\theta)}$.

(b) $x + y = 0$.

Solution. The graph of this equation is a line through the origin of slope -1. So it has polar equation $\theta = \theta_0$, where θ_0 is a constant such that $\tan(\theta_0) = -1$. This gives $\theta_0 = \frac{3\pi}{4}$, so the polar equation is $\boxed{\theta = \frac{3\pi}{4}}$.

(c) $(x - 3)^2 + y^2 = 9$.

Solution. Replacing $x = r\cos(\theta)$ and $y = r\sin(\theta)$ gives

$$\begin{aligned}
&(r\cos(\theta) - 3)^2 + (r\sin(\theta))^2 = 9\\
&r^2\cos(\theta)^2 - 6r\cos(\theta) + 9 + r^2\sin(\theta)^2 = 9\\
&r^2 - 6r\cos(\theta) = 0\\
&r(r - 6\cos(\theta)) = 0\\
&r = 0 \text{ or } r = 6\cos(\theta).
\end{aligned}$$

Note that $r = 0$ (which is the polar equation of the origin) is already included in the graph of $r = 6\cos(\theta)$. So we need only keep the equation $\boxed{r = 6\cos(\theta)}$.

(d) $y = 7 + 2x$.

Solution. Replacing $x = r\cos(\theta)$ and $y = r\sin(\theta)$ gives

$$\begin{aligned}
&r\sin(\theta) = 7 + 2r\cos(\theta)\\
&r\sin(\theta) - 2r\cos(\theta) = 7\\
&r(\sin(\theta) - 2\cos(\theta)) = 7\\
&\boxed{r = \frac{7}{\sin(\theta) - 2\cos(\theta)}}.
\end{aligned}$$

(e) $x^2 + y^2 + xy = 2$.

Solution. Replacing $x = r\cos(\theta)$ and $y = r\sin(\theta)$ gives

$$\begin{aligned}
&(r\cos(\theta))^2 + (r\sin(\theta))^2 + r\cos(\theta) r\sin(\theta) = 2 \\
&r^2 + r^2\cos(\theta)\sin(\theta) = 2 \\
&r^2(1 + \cos(\theta)\sin(\theta)) = 2 \\
&r^2 = \frac{2}{1 + \sin(\theta)\cos(\theta)} \\
&r = \pm\sqrt{\frac{2}{1 + \sin(\theta)\cos(\theta)}}
\end{aligned}$$

It may seem like we need two polar equations for this curve, but keeping only one of the two will be enough. Indeed, we can see (using either the Cartesian equation and replacing (x, y) by $(-x, -y)$, or the polar equation and replacing (r, θ) by $(r, \theta + \pi)$) that this curve is symmetric about the origin. Furthermore the two polar equations that we have found trace graphs that are symmetric of each other about the origin. Since the curve they trace is symmetric about the origin, their graphs give the same curve. It follows that a polar equation for this curve is $\boxed{r = \sqrt{\frac{2}{1 + \sin(\theta)\cos(\theta)}}}$.

(f) $y^2 = 3x^2$.

Solution. This Cartesian equation is equivalent to $y = \sqrt{3}x$ or $y = -\sqrt{3}x$. These equations represent lines through the origin of respective slopes $\sqrt{3}$ and $-\sqrt{3}$. So their polar equations are of the form $\theta = \theta_0$, with θ_0 a constant whose tangent is equal to the slope. It follows that the polar equation is $\boxed{\theta = \frac{\pi}{3} \text{ or } \theta = -\frac{\pi}{3}}$.

2. Convert the following polar equations to Cartesian. Then describe the graph.

(a) $r = -7\sec(\theta)$.

Solution. We can write the equation in the form $r = -\frac{7}{\cos(\theta)}$, or $r\cos(\theta) = -7$, so $\boxed{x = -7}$ is the Cartesian equation. The graph is a vertical line.

(b) $r = \dfrac{5}{3\sin(\theta) - 4\cos(\theta)}$.

Solution. We write the equation as

$$r(3\sin(\theta) - 4\cos(\theta)) = 5 \ \Rightarrow\ 3r\sin(\theta) - 4r\cos(\theta) = 5 \ \Rightarrow\ \boxed{3y - 4x = 5}.$$

The corresponding graph is a line of slope $\frac{4}{3}$ and y-intercept $\frac{5}{3}$.

(c) $\theta = \dfrac{\pi}{6}$.

Solution. The graph of this polar equation is a line through the origin of slope $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$, so it has Cartesian equation $\boxed{y = \dfrac{x}{\sqrt{3}}}$.

(d) $r = 3\cot(\theta)\csc(\theta)$.

Solution. Let us start by expressing the trigonometric functions in terms of sine and cosine. We have

$$r = 3\frac{\cos(\theta)}{\sin(\theta)} \cdot \frac{1}{\sin(\theta)} \quad \Rightarrow \quad r\sin(\theta)^2 = 3\cos(\theta).$$

To find the Cartesian equation, we can multiply this last polar equation by r to get $r^2\sin(\theta)^2 = 3r\cos(\theta)$, or $\boxed{y^2 = 3x}$. The graph is a sideways parabola opening towards the right.

(e) $r = 5\sin(\theta)$.

Solution. We can multiply the equation by r to get $r^2 = 5r\sin(\theta)$, or $\boxed{x^2 + y^2 = 5y}$. This equation can be rearranged as $x^2 + y^2 - 5y = 0$, or $x^2 + \left(y - \frac{5}{2}\right)^2 = \frac{25}{4}$. From this we see that the graph is a circle of center $\left(0, \frac{5}{2}\right)$ and radius $\frac{5}{2}$.

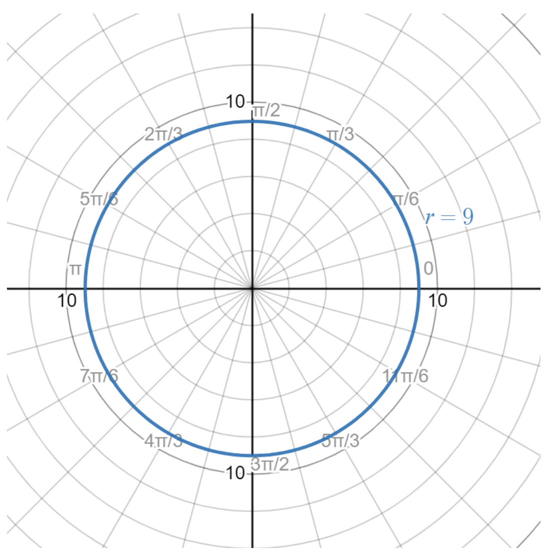
(f) $r = 2\cos(\theta) + 6\sin(\theta)$.

Solution. We can again multiply by r to get $r^2 = 2r\cos(\theta) + 6r\sin(\theta)$, or $\boxed{x^2 + y^2 = 2x + 6y}$. Writing the equation as $(x-1)^2 + (y-3)^2 = 10$, we see that the graph is a circle of center $(1, 3)$ and radius $\sqrt{10}$.

3. For each the following polar curves, identify the symmetries and sketch the graph.

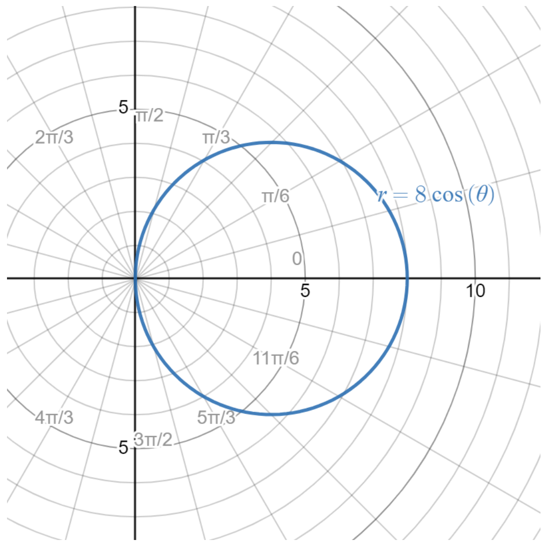
(a) $r = 9$.

Solution. We recognize the polar equation of a circle of radius 9. In particular the graph is symmetric about the origin, the x-axis and the y-axis (or about any line through the origin, which we can see by noting that r does not vary with θ).



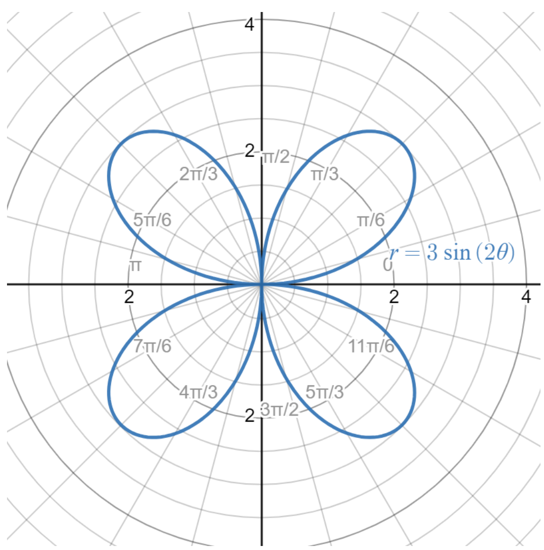
(b) $r = 8\cos(\theta)$.

Solution. If (r, θ) is on the graph, then $8\cos(-\theta) = 8\cos(\theta) = r$, so $(r, -\theta)$ is also on the graph. We deduce that the graph is symmetric about the x-axis. It is not symmetric about the y-axis or the origin since $\cos(\pi + \theta) = \cos(\pi - \theta) = -\cos(\theta)$. The sketch reveals that the graph is a circle of radius 4 centered at $(4, 0)$.



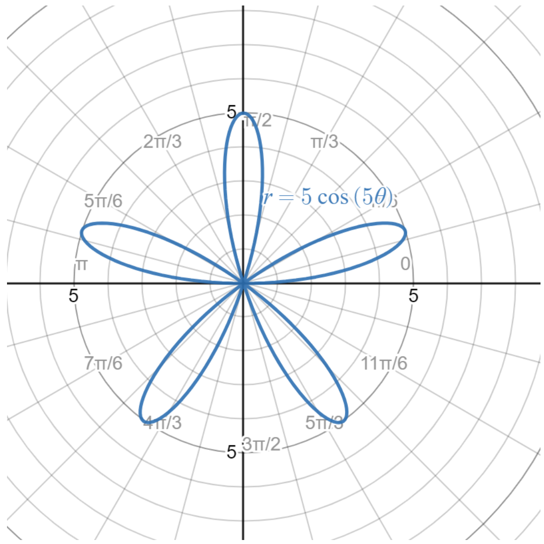
(c) $r = 3\sin(2\theta)$.

Solution. If (r, θ) is on the graph, then $3\sin(-2\theta)) = -3\sin(2\theta) = -r$, so $(-r, -\theta) = (r, \pi - \theta)$ is also on the graph, revealing a symmetry about the y-axis. Likewise, $3\sin(2(\pi - \theta)) = -3\sin(2\theta) = -r$, so $(-r, \pi - \theta) = (r, -\theta)$ is also on the graph, indicating a symmetry about the x-axis. Since the graph is symmetric about both the x-axis and the y-axis, it is also symmetric about the origin. Sketching the graph gives a four-leaved rose.



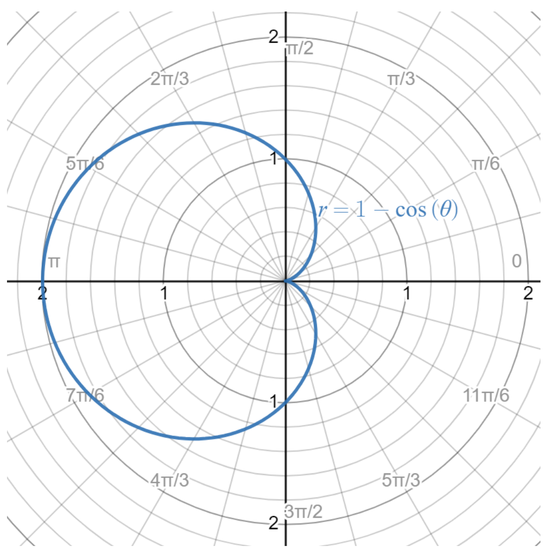
(d) $r = 5\cos(5\theta)$.

Solution. If (r, θ) is on the graph, then $5\cos(5(-\theta)) = 5\cos(5\theta) = r$, so $(r, -\theta)$ is also on the graph, indicating a symmetry about the x-axis. However, $5\cos(5(\pi - \theta)) = -5\cos(5\theta) = -r$ and $5\cos(5(\pi + \theta)) = -5\cos(5\theta) = -r$, so there is no symmetry about the y-axis or the origin. Sketching the graph gives a five-leaved rose.



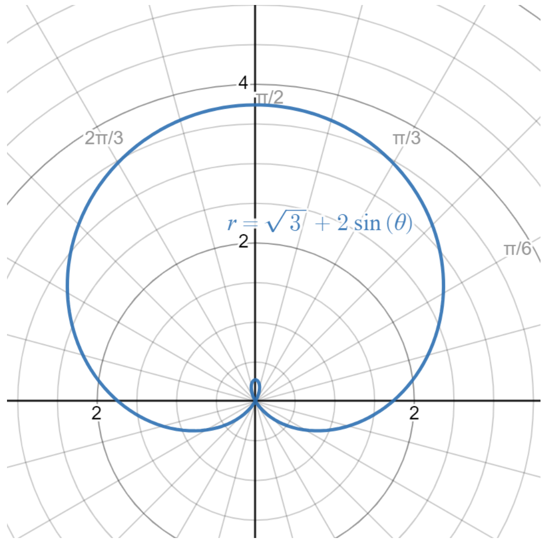
(e) $r = 1 - \cos(\theta)$.

Solution. If (r, θ) is on the graph, then $1 - \cos(-\theta) = 1 - \cos(\theta)$ so $(-r, \theta)$ is also on the graph, indicating a symmetry about the x-axis. However, $1 - \cos(\pi - \theta) = 1 - \cos(\pi + \theta) = 1 + \cos(\theta)$, so the graph is not symmetric about the y-axis or the origin. Sketching the graph gives a cardioid.



(f) $r = \sqrt{3} + 2\sin(\theta)$.

Solution. If (r, θ) is on the graph, then $\sqrt{3} + 2\sin(\pi - \theta) = \sqrt{3} + 2\sin(\theta) = r$, so $(r, \pi - \theta)$ is also on the graph, indicating a symmetry about the y-axis. However, $\sqrt{3} + 2\sin(-\theta) = \sqrt{3} + 2\sin(\pi + \theta) = \sqrt{3} - 2\sin(\theta)$, so the graph is not symmetric about the x-axis or the origin. Sketching the graph gives a looped limaçon.



4. Find an equation of the tangent line to the following polar curves at the given value of θ.

(a) $r = \cos(3\theta)$, $\theta = \dfrac{\pi}{4}$.

Solution. The curve can be parametrized by $x = r\cos(\theta) = \cos(\theta)\cos(3\theta)$ and $y = r\sin(\theta) = \sin(\theta)\cos(3\theta)$. Therefore

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos(\theta)\cos(3\theta) - 3\sin(\theta)\sin(3\theta)}{-\sin(\theta)\cos(3\theta) - 3\cos(\theta)\sin(3\theta)}.$$

So the slope of the tangent line to the curve at $\theta = \frac{\pi}{4}$ is

$$\frac{dy}{dx}_{|\theta=\pi/4} = \frac{\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{3\pi}{4}\right) - 3\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{3\pi}{4}\right)}{-\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{3\pi}{4}\right) - 3\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{3\pi}{4}\right)} = -1.$$

Furthermore, the tangent line passes through the point $\left(x\left(\frac{\pi}{4}\right), y\left(\frac{\pi}{4}\right)\right) = \left(-\frac{1}{2}, -\frac{1}{2}\right)$. So it has equation

$$\boxed{y = -1\left(x + \frac{1}{2}\right) - \frac{1}{2}}.$$

(b) $r = 1 + 2\sin(\theta)$, $\theta = \dfrac{\pi}{6}$.

Solution. The curve can be parametrized by $x = r\cos(\theta) = \cos(\theta)(1+2\sin(\theta))$ and $y = r\sin(\theta) = \sin(\theta)(1+2\sin(\theta))$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\ &= \frac{\cos(\theta)(1+2\sin(\theta)) + 2\sin(\theta)\cos(\theta)}{-\sin(\theta)(1+2\sin(\theta)) + 2\cos(\theta)\cos(\theta)}.\end{aligned}$$

So the slope of the tangent line to the curve at $\theta = \frac{\pi}{6}$ is

$$\begin{aligned}\frac{dy}{dx}_{|\theta=\pi/6} &= \frac{\cos\left(\frac{\pi}{6}\right)\left(1+2\sin\left(\frac{\pi}{6}\right)\right) + 2\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right)}{-\sin\left(\frac{\pi}{6}\right)\left(1+2\sin\left(\frac{\pi}{6}\right)\right) + 2\cos\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right)} \\ &= 3\sqrt{3}.\end{aligned}$$

Furthermore, the tangent line passes through the point $\left(x\left(\frac{\pi}{6}\right), y\left(\frac{\pi}{6}\right)\right) = \left(\sqrt{3}, 1\right)$. So it has equation

$$\boxed{y = 3\sqrt{3}\left(x - \sqrt{3}\right) + 1}.$$