Section 11.5

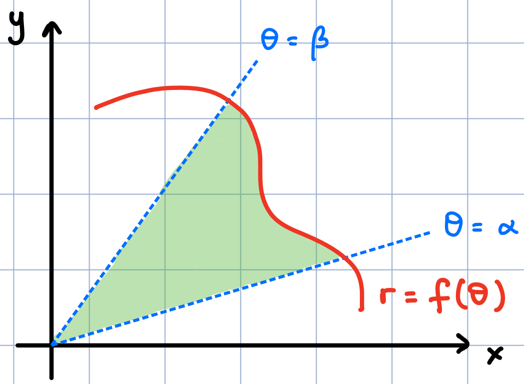
Areas and Lengths in Polar Coordinates

Learning Goals

Learning Goal	*Homework Problems*
11.5.1 Find the area of a region bounded by either a polar curve or part of it	11.5: 2,5,6,9,15,16,17,18
11.5.2 Find the arc length of a polar curve	11.5: 23,28,30

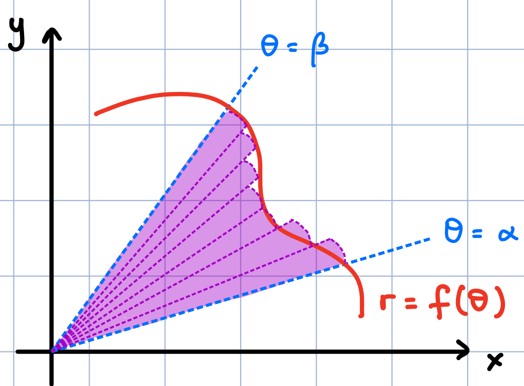
① Areas :

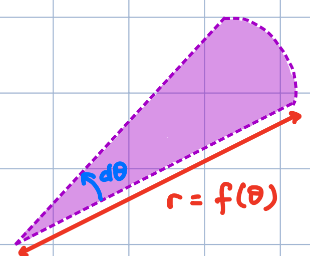
Consider a region defined by $0 \leq r \leq f(\theta)$, $\alpha \leq \theta \leq \beta$. The region is bounded by the rays $\theta = \alpha$, $\theta = \beta$ and the polar curve $r = f(\theta)$.



Such a region is said to be <u>radially simple</u>.

We compute the area of the region by decomposing it into tiny circular sectors.



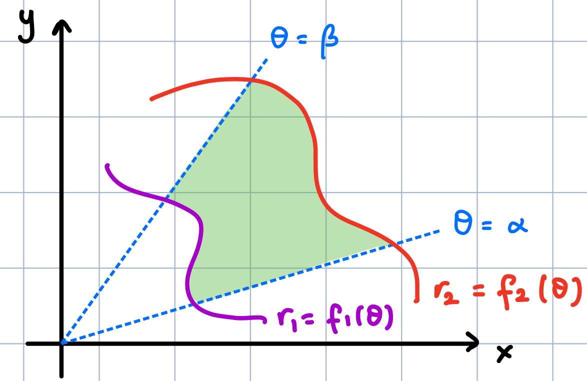


A typical tiny circular sector has central angle $d\theta$ and radius $r = f(\theta)$

So the area of the circular sector is $dA = \frac{1}{2} r^2 d\theta = \frac{1}{2} f(\theta)^2 d\theta$.
To obtain the total area, we "sum" (= integrate) the areas of all sectors between $\theta = \alpha$ and $\theta = \beta$:

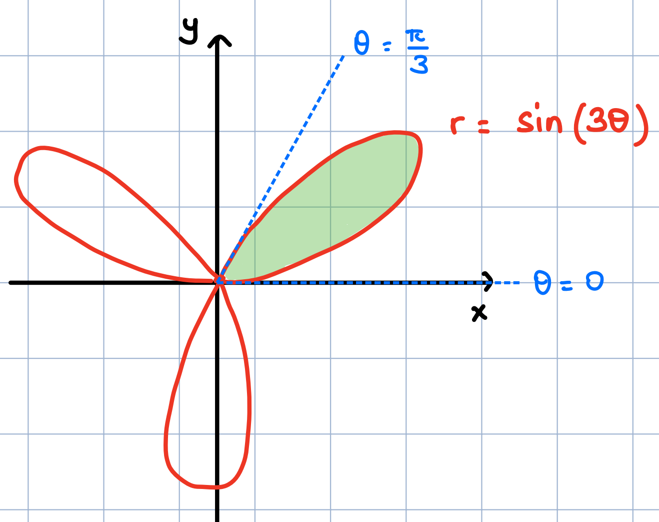
$$\boxed{A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta = \int_{\alpha}^{\beta} \frac{1}{2} f(\theta)^2 d\theta}$$

Remark: for a region $f_1(\theta) \leq r \leq f_2(\theta)$, $\alpha \leq \theta \leq \beta$ bounded by two polar curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$, the area is given by:



$$\boxed{\begin{aligned} A &= \int_\alpha^\beta \frac{1}{2}(r_2^2 - r_1^2)\,d\theta \\ &= \int_\alpha^\beta \frac{1}{2}(f_2(\theta)^2 - f_1(\theta)^2)\,d\theta \end{aligned}}$$

Examples: 1) Find the area inside one leaf of the three-leaved rose $r = \sin(3\theta)$.



The leaf is the radially simple region:

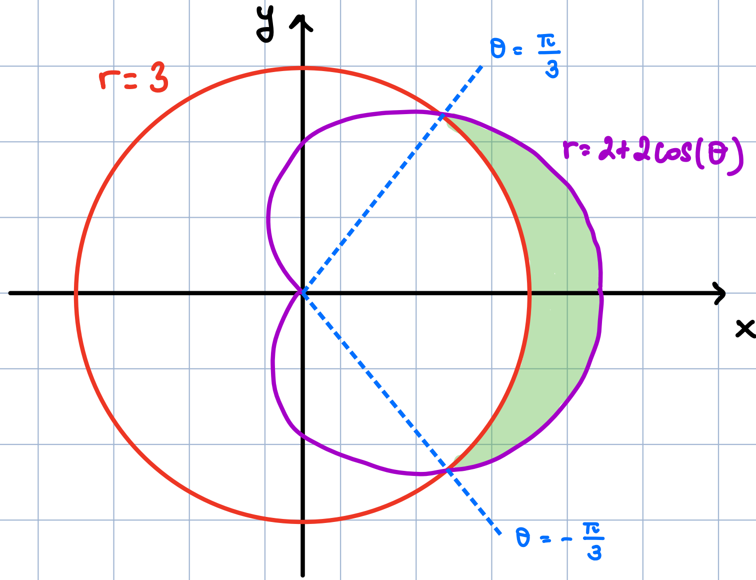
$0 \leq r \leq \sin(3\theta)$, $0 \leq \theta \leq \frac{\pi}{3}$.

$\left(\sin(3\theta) = 0 \Rightarrow 3\theta = 0, \pi \Rightarrow \theta = 0, \frac{\pi}{3}\right)$

So $A = \int_0^{\frac{\pi}{3}} \frac{1}{2}\sin(3\theta)^2 d\theta = \frac{1}{2}\int_0^{\frac{\pi}{3}} \frac{1-\cos(6\theta)}{2}\,d\theta$

$= \frac{1}{4}\left[\theta - \frac{\sin(6\theta)}{6}\right]_0^{\frac{\pi}{3}} = \boxed{\frac{\pi}{12} \text{ square units}}$.

2) Find the area that lies inside the cardioid $r = 2 + 2\cos(\theta)$ but outside the circle $r = 3$.



The rays bounding the region are at the intersection of the curves:

$$2+2\cos(\theta) = 3$$
$$\cos(\theta) = \frac{1}{2}$$
$$\theta = \frac{\pi}{3}, -\frac{\pi}{3}$$

$$A = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}\left((2+2\cos(\theta))^2 - 3^2\right) d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(4+4\cos(\theta)+4\cos(\theta)^2 - 9\right) d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(4\cos(\theta) + 4\,\frac{1+\cos(2\theta)}{2} - 5\right) d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(4\cos(\theta) + 2\cos(2\theta) - 3\right) d\theta$$

$$= \frac{1}{2}\Big[4\sin(\theta) + \sin(2\theta) - 3\theta\Big]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= \boxed{\frac{5\sqrt{3}}{2} - \pi \text{ square units}}.$$

3) Find the area inside the circle $x^2+(y-2)^2=4$ and below the horizontal line $y=3$.

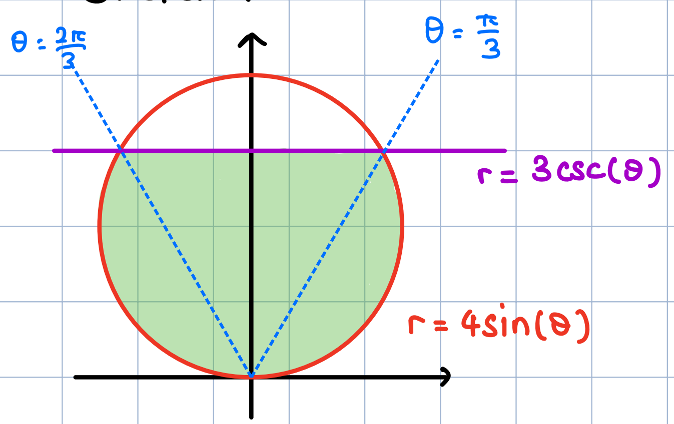
Convert to polar: $y=3 \Rightarrow r\sin(\theta)=3 \Rightarrow r=3\csc(\theta)$

$x^2+(y-2)^2=4 \Rightarrow (r\cos(\theta))^2+(r\sin(\theta)-2)^2=4$

$r^2\cos(\theta)^2 + r^2\sin(\theta)^2 - 4r\sin(\theta) + 4 = 4$

$$r^2 - 4r\sin(\theta) = 0$$
$$r = 4\sin(\theta)$$

Sketch :



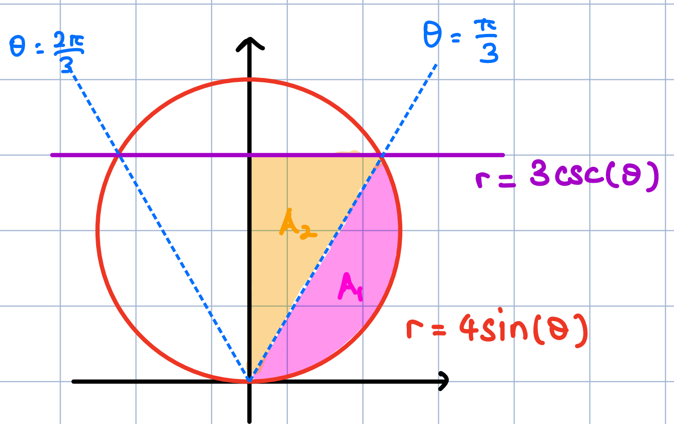
Intersection : $4\sin(\theta) = 3\csc(\theta)$

$\sin(\theta)^2 = \frac{3}{4}$

$\Rightarrow \sin(\theta) = \frac{\sqrt{3}}{2}$ (can restrict to $\sin(\theta) > 0$)

$\theta = \frac{\pi}{3}, \frac{2\pi}{3}$

The region is not radially simple : we must use a sum of integrals .



$A = 2A_1 + 2A_2$ (symmetry)

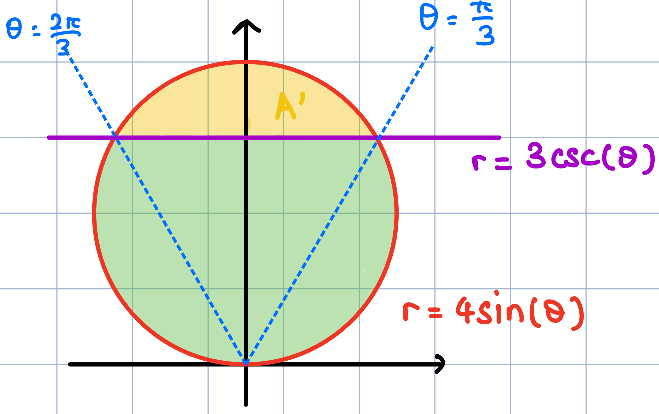
$$A = 2\int_0^{\frac{\pi}{3}} \frac{1}{2}(4\sin(\theta))^2 d\theta + 2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}(3\csc(\theta))^2 d\theta$$

$$= 16\int_0^{\frac{\pi}{3}} \frac{1-\cos(2\theta)}{2} d\theta + 9\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \csc(\theta)^2 d\theta$$

$$= 8\left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{\frac{\pi}{3}} + 9\left[-\cot(\theta)\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= 8\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right) + \frac{9}{\sqrt{3}} = \boxed{\frac{8\pi}{3} + \sqrt{3} \text{ square units}}.$$

Alternative method :



A = Area of full circle $- A'$

$= 4\pi - A'$

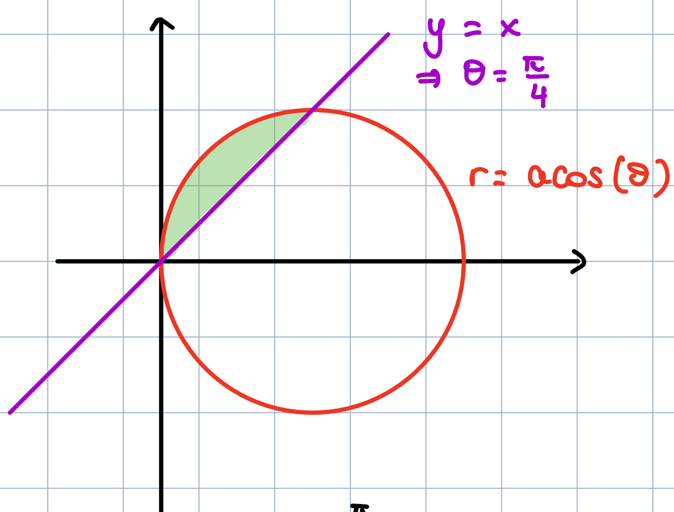
$$A' = 2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}\left(16\sin(\theta)^2 - 9\csc(\theta)^2\right)d\theta$$

$$= \left[8\left(\theta - \frac{\sin(2\theta)}{2}\right) + 9\cot(\theta)\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= \frac{8\pi}{6} + 2\sqrt{3} - \frac{9}{\sqrt{3}} = \frac{4\pi}{3} - \sqrt{3}.$$

So $A = 4\pi - A' = 4\pi - \frac{4\pi}{3} + \sqrt{3} = \boxed{\frac{8\pi}{3} + \sqrt{3} \text{ square units}}$.

4) Find the area inside the circle $r = a\cos(\theta)$ $(a>0)$ and and above the line $y = x$.



Convert to polar:

$y = x \Rightarrow \frac{y}{x} = 1 \Rightarrow \tan(\theta) = 1$

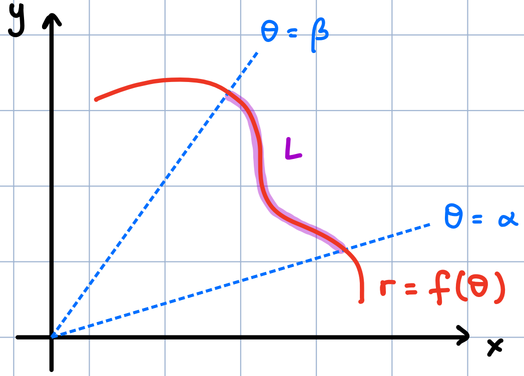
$\Rightarrow \theta = \frac{\pi}{4}.$

$$A = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}(a\cos(\theta))^2 d\theta$$

$$= \frac{a^2}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos(\theta)^2 d\theta = \frac{a^2}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1+\cos(2\theta)}{2} d\theta = \frac{a^2}{4}\left[\theta + \frac{\sin(2\theta)}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \boxed{\frac{a^2}{8}\left(\frac{\pi}{2} - 1\right) \text{ square units}}.$$

② Length of a polar curve



From 11.2, we know that

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$x = r\cos(\theta) \Rightarrow \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)$$

$$y = r\sin(\theta) \Rightarrow \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)$$

So $\left(\frac{dx}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2\cos(\theta)^2 + r^2\sin(\theta)^2 - 2r\frac{dr}{d\theta}\cos(\theta)\sin(\theta)$

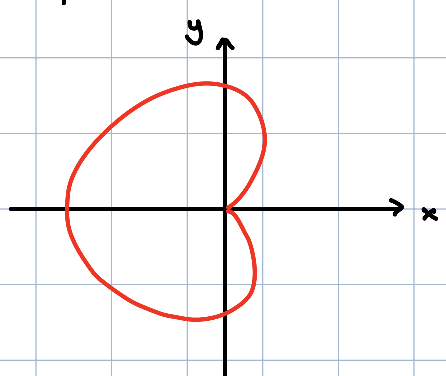
$$\left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2\sin(\theta)^2 + r^2\cos(\theta)^2 + 2r\frac{dr}{d\theta}\cos(\theta)\sin(\theta)$$

$$\Rightarrow \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 + r^2.$$

So $\boxed{L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta}$

provided the curve is traced exactly once as θ runs from $\theta = \alpha$ to $\theta = \beta$.

Examples : 1) Find the length of the cardiod $r = 1 - \cos(\theta)$



The curve is traced once on $0 \leq \theta \leq 2\pi$.

$r = 1 - \cos(\theta)$

$\frac{dr}{d\theta} = \sin(\theta)$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = 1 - 2\cos(\theta) + \cos(\theta)^2 + \sin(\theta)^2 = 2(1-\cos(\theta)) = 4\sin\left(\frac{\theta}{2}\right)^2$$

So $$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

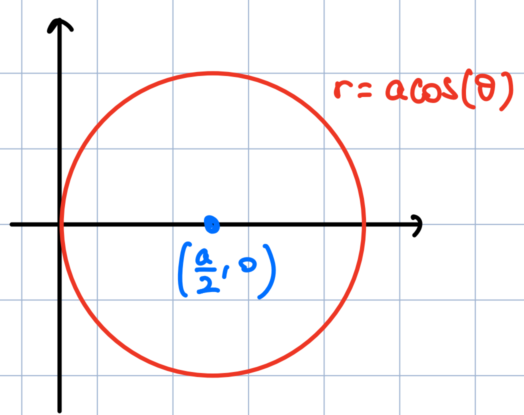
$$= \int_0^{2\pi} 2\left|\sin\left(\frac{\theta}{2}\right)\right| d\theta$$

$\sin\left(\frac{\theta}{2}\right) \geq 0$ when $0 \leq \theta \leq 2\pi$

$$= 2\int_0^{2\pi} \sin\left(\frac{\theta}{2}\right) d\theta$$

$$= -4\left[\cos\left(\frac{\theta}{2}\right)\right]_0^{2\pi} = \boxed{8}.$$

2) Find the length of the circle $r = a\cos(\theta)$, where $a > 0$ is a constant.



The circle is traced once from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$.

$r = a\cos(\theta)$

$\frac{dr}{d\theta} = -a\sin(\theta)$

So $$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2\cos(\theta)^2 + a^2\sin(\theta)^2 = a^2.$$

$$L = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{a^2}\, d\theta = \boxed{\pi a}.$$