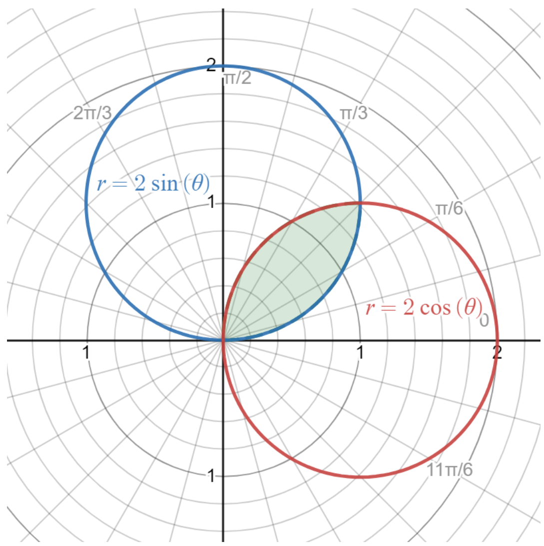
Rutgers University
Math 152

Sections 11.5: Areas and Lengths in Polar Coordinates - Worksheet Solutions

1. Find the areas of the given regions.

   (a) The region shared by the circles $r = 2\sin(\theta)$ and $r = 2\cos(\theta)$.

   Solution. The circles intersect at the origin and when $\theta = \frac{\pi}{4}$, see the figure below.

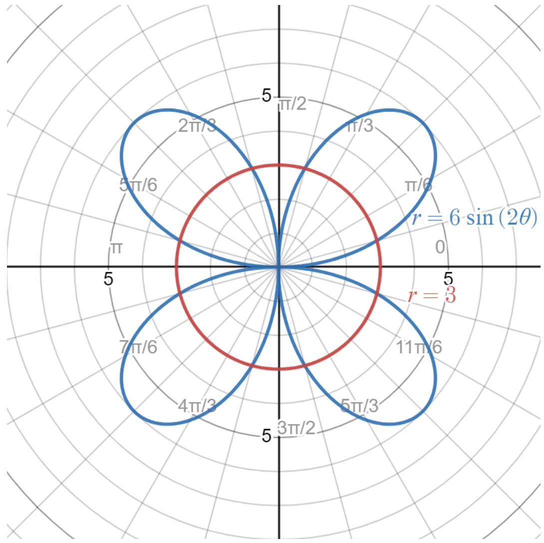


   Note that the region is not radially simple. For $0 \leqslant \theta \leqslant \frac{\pi}{4}$, the ray at θ in the region is bounded by the origin and $r = 2\sin(\theta)$. For $\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{2}$, the ray at θ in the region is bounded by the origin and $r = 2\cos(\theta)$. Note that this divides the region into two regions of equal area. Therefore, the area is given by

$$\begin{aligned}
A &= \int_0^{\pi/4} \frac{1}{2}(2\sin(\theta))^2 d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}(2\cos(\theta))^2 d\theta \\
&= 4\int_0^{\pi/4} \sin(\theta)^2 d\theta \\
&= 4\int_0^{\pi/4} \frac{1-\sin(2\theta)}{2} d\theta \\
&= 2\left[\theta - \frac{\cos(2\theta)}{2}\right]_0^{\pi/4} \\
&= \boxed{\frac{\pi}{2} + 1 \text{ square units}}.
\end{aligned}$$

(b) The region contained inside the leaves of the rose $r = 6\sin(2\theta)$ and outside the circle $r = 3$.

Solution. The curves are sketched below.



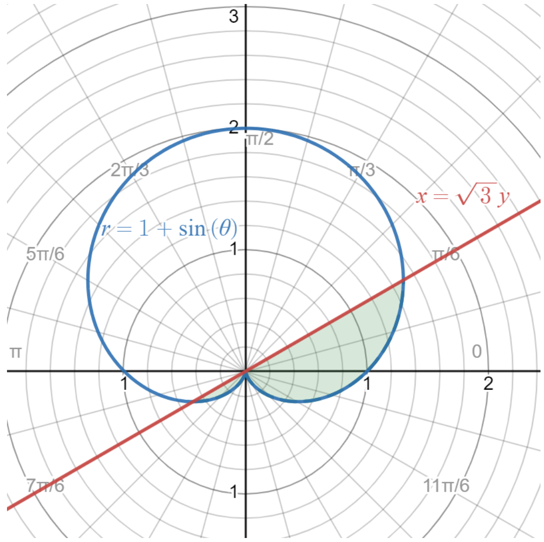
Right off the bat, we observe that by symmetry, it suffices to compute the area of the part of the region in the leaf located in the first quadrant and multiply it by 4. Note that the curves intersect when $6\sin(2\theta) = 3$ or $\sin(2\theta) = \frac{1}{2}$, which gives $\theta = \frac{\pi}{12}, \frac{5\pi}{12}$ as solutions in the first quadrant.

The ray at θ in the region is bounded by $r = 6\sin(2\theta)$ and $r = 3$. Therefore, the area is

$$\begin{aligned}
A &= 4\int_{\pi/12}^{5\pi/12} \frac{1}{2}\left((6\sin(2\theta)^2 - 3^2)\right) d\theta \\
&= 2\int_{\pi/12}^{5\pi/12} \left(36\sin(2\theta)^2 - 9\right) d\theta \\
&= 2\int_{\pi/12}^{5\pi/12} (18(1 - \cos(4\theta)) - 9)\, d\theta \\
&= 18\int_{\pi/12}^{5\pi/12} (1 - 2\cos(4\theta))\, d\theta \\
&= 18\left[\theta - \frac{\sin(4\theta)}{2}\right]_{\pi/12}^{5\pi/12} \\
&= \boxed{6\pi + 9\sqrt{3} \text{ square units}}.
\end{aligned}$$

(c) The region inside the cardioid $r = 1 + \sin(\theta)$ and below the line $x = \sqrt{3}y$.

Solution. The line $x = \sqrt{3}y$ is a line through the origin of slope $\frac{1}{\sqrt{3}}$, so it has polar equation $\theta = \frac{\pi}{6}$ or $\theta = -\frac{5\pi}{6}$. The region is sketched below.

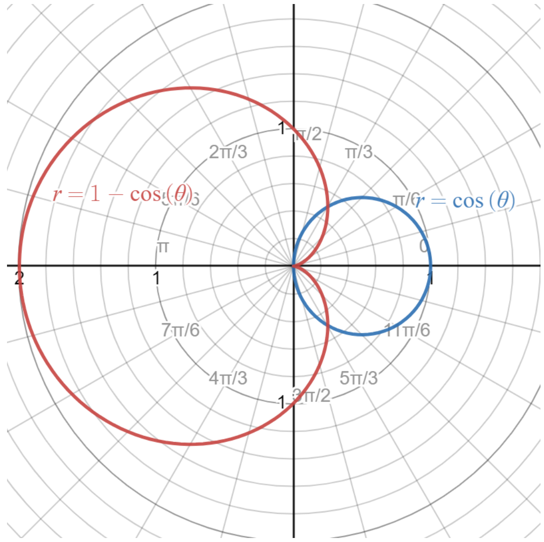


For $-\frac{5\pi}{6} \leqslant \theta \leqslant \frac{\pi}{6}$, the ray at θ in the region is bounded by $r = 1 + \sin(\theta)$. Therefore, the area is

$$
\begin{aligned}
A &= \int_{-5\pi/6}^{\pi/6} \frac{1}{2}(1+\sin(\theta))^2 d\theta \\
&= \frac{1}{2}\int_{-5\pi/6}^{\pi/6} (1 + 2\sin(\theta) + \sin(\theta)^2) d\theta \\
&= \frac{1}{2}\int_{-5\pi/6}^{\pi/6} \left(1 + 2\sin(\theta) + \frac{1-\cos(2\theta)}{2}\right) d\theta \\
&= \frac{1}{4}\int_{-5\pi/6}^{\pi/6} \left(3 + 4\sin(\theta) - \cos(2\theta)\right) d\theta \\
&= \frac{1}{4}\left[3\theta - 4\cos(\theta) - \frac{\sin(2\theta)}{2}\right]_{-5\pi/6}^{\pi/6} \\
&= \boxed{\frac{6\pi - 9\sqrt{3}}{8} \text{ square units}}.
\end{aligned}
$$

(d) The region inside the circle $r = \cos(\theta)$ and outside the cardioid $r = 1 - \cos(\theta)$.

Solution. Observe that the curves intersect when $\cos(\theta) = 1 - \cos(\theta)$, that is $\cos(\theta) = \frac{1}{2}$, which gives $\theta = -\frac{\pi}{3}, \frac{\pi}{3}$. The region is sketched below.

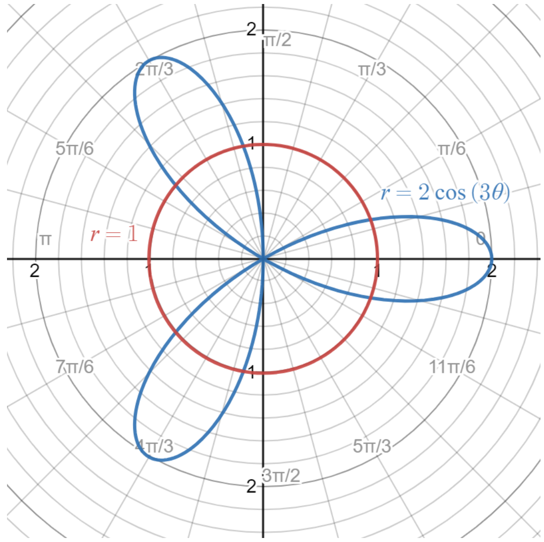


For $-\frac{\pi}{3} \leqslant \theta \leqslant \frac{\pi}{3}$, the ray at θ in the region is bounded between $r = 1 - \cos(\theta)$ and $r = \cos(\theta)$. Also, using the symmetry of the region about the x-axis, it suffices to compute the area for $0 \leqslant \theta \leqslant \frac{\pi}{3}$ and double it. Therefore, the area is

$$\begin{aligned} A &= 2\int_0^{\pi/3} \frac{1}{2}\left(\cos(\theta)^2 - \left(1 - \cos(\theta)^2\right)\right) d\theta \\ &= \int_0^{\pi/3} \left(\cos(\theta)^2 - 1 + 2\cos(\theta) - \cos(\theta)^2\right) d\theta \\ &= \int_0^{\pi/3} (2\cos(\theta) - 1)\, d\theta \\ &= [2\sin(\theta) - \theta]_0^{\pi/3} \\ &= \boxed{\sqrt{3} - \frac{\pi}{3} \text{ square units}}. \end{aligned}$$

(e) The region shared by one leaf of the rose $r = 2\cos(3\theta)$ and the circle $r = 1$.

Solution. The curves are sketched below.

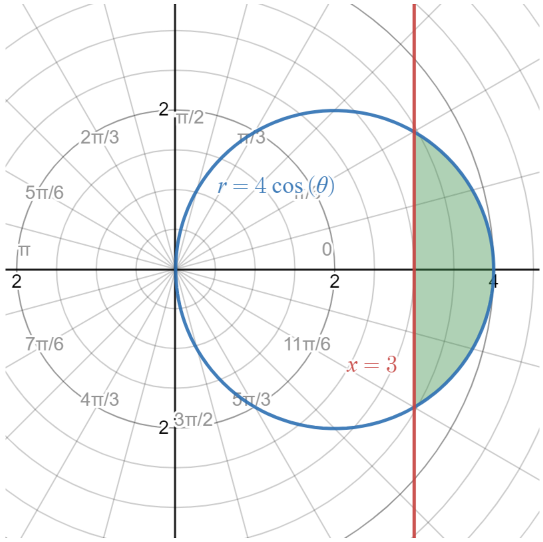


By symmetry, it suffices to find the area of the region in the first quadrant shared by one leaf of the rose and the circle and double it. In the first quadrant, the curves intersect when $2\cos(3\theta) = 1$, that is $\cos(3\theta) = \frac{1}{2}$, which gives $\theta = \frac{\pi}{9}$. Furthermore, the leaf crosses the origin when $\cos(3\theta) = 0$, which gives $\theta = \frac{\pi}{6}$.

For $0 \leqslant \theta \leqslant \frac{\pi}{9}$, the ray at θ is the region is bounded by $r = 1$. For $\frac{\pi}{9} \leqslant \theta \leqslant \frac{\pi}{6}$, the ray at θ is the region is bounded by $r = 2\cos(3\theta)$. Therefore the area is

$$\begin{aligned}
A &= 2\left(\int_0^{\pi/9} \frac{1}{2}1^2 d\theta + \int_{\pi/9}^{\pi/6} \frac{1}{2}(2\cos(3\theta))^2 d\theta\right) \\
&= \int_0^{\pi/9} d\theta + 4\int_{\pi/9}^{\pi/6} \cos(3\theta)^2 d\theta \\
&= \frac{\pi}{9} + 4\int_{\pi/9}^{\pi/6} \frac{1+\cos(6\theta)}{2} d\theta \\
&= \frac{\pi}{9} + 2\left[\theta + \frac{\sin(6\theta)}{6}\right]_{\pi/9}^{\pi/6} \\
&= \frac{\pi}{9} + 2\left(\frac{\pi}{18} - \frac{\sqrt{3}}{12}\right) \\
&= \boxed{\frac{2\pi}{9} - \frac{\sqrt{3}}{6} \text{ square units}}.
\end{aligned}$$

2. Consider the region $\mathcal{R}$ contained in the circle $r = 4\cos(\theta)$ to the right of the line $x = 3$.



(a) Find the area of the region $\mathcal{R}$ using integration with respect to x.

Solution. The circle has Cartesian equation $(x-2)^2+y^2 = 4$. If we solve for y, we get the two solutions $y = \pm\sqrt{4-(x-2)^2}$. The equation $y = \sqrt{4-(x-2)^2}$ corresponds to the top-half semi-circle, while the equation $y = -\sqrt{4-(x-2)^2}$ corresponds to the bottom-half semi-circle. For $3 \leqslant x \leqslant 4$, the vertical strip at x in the region is bounded by the top-half semi-circle and the bottom-half semi circle, so it has length $\ell(x) = \sqrt{4-(x-2)^2} - (-\sqrt{4-(x-2)^2}) = 2\sqrt{4-(x-2)^2}$. Therefore, the area is

$$A = \int_3^4 \ell(x)dx = 2\int_3^4 \sqrt{4-(x-2)^2}dx.$$

We can compute this integral using a trigonometric substitution. We will want $4-(x-2)^2 = 4-4\sin(\theta)^2$, so we will substitute $x = 2+2\sin(\theta)$. Then we get $dx = 2\cos(\theta)d\theta$ and $\sqrt{4-(x-2)^2} = \sqrt{4-4\sin(\theta)^2} = 2\cos(\theta)$. The bounds become

$$\begin{aligned} x = 3 \;\Rightarrow\; \theta &= \sin^{-1}\left(\frac{x-2}{2}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}, \\ x = 4 \;\Rightarrow\; \theta &= \sin^{-1}\left(\frac{x-2}{2}\right) = \sin^{-1}(1) = \frac{\pi}{2}. \end{aligned}$$

So the integral becomes

$$\begin{aligned} A &= 2\int_{\pi/6}^{\pi/2} (2\cos(\theta))2\cos(\theta)d\theta \\ &= 8\int_{\pi/6}^{\pi/2} \cos(\theta)^2 d\theta \\ &= 4\int_{\pi/6}^{\pi/2} (1+\cos(2\theta))d\theta \\ &= 4\left[\theta + \frac{\sin(2\theta)}{2}\right]_{\pi/6}^{\pi/2} \\ &= \boxed{\frac{4\pi}{3} - \sqrt{3} \text{ square units}}. \end{aligned}$$

(b) Find the area of the region $\mathcal{R}$ using integration with respect to y.

Solution. If we solve the Cartesian equation of the circle for x, we get $x = 2 \pm \sqrt{4-y^2}$. The equation corresponding the the right semi-circle, which bounds the region, is $x = 2 + \sqrt{4-y^2}$. The horizontal strip at y in the region is bounded on the left by $x = 3$ and on the right by $x = 2 + \sqrt{4-y^2}$, so it has length $\ell(y) = 2 + \sqrt{4-y^2} - 3 = \sqrt{4-y^2} - 1$. Plugging-in $x = 3$ in the equation of the circle gives $y = \pm\sqrt{3}$ as the boundaries of the region. Note that by symmetry, it suffices to calculate the area above the x-axis, so for $0 \leqslant y \leqslant \sqrt{3}$, and double it. Therefore, the area is

$$A = 2\int_0^{\sqrt{3}} \ell(y)dy = 2\int_0^{\sqrt{3}} \left(\sqrt{4-y^2} - 1\right) dy = 2\int_0^{\sqrt{3}} \sqrt{4-y^2}dy - 2\sqrt{3}.$$

We can again compute this integral with the trigonometric substitution $y = 2\sin(\theta)$ to get

$$\begin{aligned} A &= 2\int_0^{\pi/3} (2\cos(\theta))2\cos(\theta)d\theta - 2\sqrt{3} \\ &= 8\int_0^{\pi/3} \cos(\theta)^2 d\theta - 2\sqrt{3} \\ &= 4\int_0^{\pi/3} (1+\cos(2\theta))d\theta - 2\sqrt{3} \\ &= 4\left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{\pi/3} - 2\sqrt{3} \\ &= 4\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right) - 2\sqrt{3} \\ &= \boxed{\frac{4\pi}{3} - \sqrt{3} \text{ square units}}. \end{aligned}$$

(c) Find the area of the region $\mathcal{R}$ using integration with respect to θ.

Solution. The vertical line $x = 3$ has polar equation $r = 3\sec(\theta)$. The two curves intersect when $3\sec(\theta) = 4\cos(\theta)$, which gives $\cos(\theta)^2 = \frac{3}{4}$ or $\theta = \pm\frac{\pi}{6}$. Note that by symmetry, it will suffice to compute the area of the region above the x-axis, that is for $0 \leqslant \theta \leqslant \frac{\pi}{6}$, and double it. When $0 \leqslant \theta \leqslant \frac{\pi}{6}$, the ray at θ in the region is bounded between $r = 4\cos(\theta)$ and $r = 3\sec(\theta)$. So the area is

$$\begin{aligned} A &= 2\int_0^{\pi/6} \frac{1}{2}\left((4\cos(\theta))^2 - (3\sec(\theta))^2\right) d\theta \\ &= \int_0^{\pi/6} \left(16\cos(\theta)^2 - 9\sec(\theta)^2\right) d\theta \\ &= \int_0^{\pi/6} \left(8 + 8\cos(2\theta) - 9\sec(\theta)^2\right) d\theta \\ &= [8\theta + 4\sin(2\theta) - 9\tan(\theta)]_0^{\pi/6} \\ &= \frac{8\pi}{6} + 4\frac{\sqrt{3}}{2} - 9\frac{\sqrt{3}}{3} \\ &= \boxed{\frac{4\pi}{3} - \sqrt{3} \text{ square units}}. \end{aligned}$$

3. Find the lengths of the given polar curves.

(a) $r = \sqrt{1+\cos(2\theta)}$, $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

Solution. We have

$$\frac{dr}{d\theta} = \frac{-2\sin(2\theta)}{2\sqrt{1+\cos(2\theta)}} = -\frac{\sin(2\theta)}{\sqrt{1+\cos(2\theta)}}.$$

So

$$\begin{aligned}
\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} &= \sqrt{1+\cos(2\theta) + \frac{\sin(2\theta)^2}{1+\cos(2\theta)}} \\
&= \sqrt{\frac{(1+\cos(2\theta))^2 + \sin(2\theta)^2}{1+\cos(2\theta)}} \\
&= \sqrt{\frac{1+2\cos(2\theta)+\cos(2\theta)^2+\sin(2\theta)^2}{1+\cos(2\theta)}} \\
&= \sqrt{\frac{1+2\cos(2\theta)+1}{1+\cos(2\theta)}} \\
&= \sqrt{\frac{2+2\cos(2\theta)}{1+\cos(2\theta)}} \\
&= \sqrt{2}.
\end{aligned}$$

Therefore, the length of the curve is

$$\begin{aligned}
L &= \int_0^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\
&= \int_0^{\pi/2} \sqrt{2} d\theta \\
&= \boxed{\frac{\pi\sqrt{2}}{2} \text{ units}}.
\end{aligned}$$

(b) $r = \dfrac{2}{1-\cos(\theta)}$, $\frac{\pi}{2} \leqslant \theta \leqslant \pi$.

Solution. The trained eye notices that this expression looks like something we'd get from a double-angle formula. Indeed, we know that $\sin(x)^2 = \frac{1-\cos(2x)}{2}$, so taking reciprocals and replacing x by $\frac{\theta}{2}$ gives

$$\frac{2}{1-\cos(\theta)} = \frac{1}{\sin\left(\frac{\theta}{2}\right)^2} = \csc\left(\frac{\theta}{2}\right)^2.$$

Working with the form $r = \csc\left(\frac{\theta}{2}\right)^2$ of the polar equation of the curve will simply computations a little. We have

$$\frac{dr}{d\theta} = 2\csc\left(\frac{\theta}{2}\right)\left(-\frac{1}{2}\csc\left(\frac{\theta}{2}\right)\cot\left(\frac{\theta}{2}\right)\right) = -\csc\left(\frac{\theta}{2}\right)^2\cot\left(\frac{\theta}{2}\right).$$

So

$$\begin{aligned}
\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} &= \sqrt{\csc\left(\frac{\theta}{2}\right)^4 + \csc\left(\frac{\theta}{2}\right)^4 \cot\left(\frac{\theta}{2}\right)^2} \\
&= \csc\left(\frac{\theta}{2}\right)^2 \sqrt{1 + \cot\left(\frac{\theta}{2}\right)^2} \\
&= \csc\left(\frac{\theta}{2}\right)^2 \sqrt{\csc\left(\frac{\theta}{2}\right)^2} \\
&= \csc\left(\frac{\theta}{2}\right)^3 .
\end{aligned}$$

Therefore, the length of the curve is

$$\begin{aligned}
L &= \int_{\pi/2}^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\
&= \int_{\pi/2}^{\pi} \csc\left(\frac{\theta}{2}\right)^3 d\theta \\
&= 2\int_{\pi/4}^{\pi/2} \csc\left(u\right)^3 du,
\end{aligned}$$

where we have made the substitution $u = \frac{\theta}{2}$ in the last integral. In Section 8.3, we learned that we can compute such integrals with an IBP followed by a trigonometric identity giving a relation that we can solve for the unknown integral. For the IPB, we use the parts

$$\begin{aligned}
u &= \csc(u) \ \Rightarrow\ du = -\csc(u)\cot(u)du, \\
dv &= \csc(u)^2 du \ \Rightarrow\ v = -\cot(u).
\end{aligned}$$

We get

$$\begin{aligned}
\int_{\pi/4}^{\pi/2} \csc\left(u\right)^3 du &= [-\csc(u)\cot(u)]_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2} (-\csc(u)\cot(u))(-\cot(u))du \\
\int_{\pi/4}^{\pi/2} \csc\left(u\right)^3 du &= [-\csc(u)\cot(u)]_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2} (-\csc(u)\cot(u))(-\cot(u))du \\
\int_{\pi/4}^{\pi/2} \csc\left(u\right)^3 du &= \sqrt{2} - \int_{\pi/4}^{\pi/2} \csc(u)\cot(u)^2 du \\
\int_{\pi/4}^{\pi/2} \csc\left(u\right)^3 du &= \sqrt{2} - \int_{\pi/4}^{\pi/2} \csc(u)(\csc(u)^2 - 1)du \\
\int_{\pi/4}^{\pi/2} \csc\left(u\right)^3 du &= \sqrt{2} - \int_{\pi/4}^{\pi/2} \csc(u)^3 du + \int_{\pi/4}^{\pi/2} \csc(u)du \\
\int_{\pi/4}^{\pi/2} \csc\left(u\right)^3 du &= \sqrt{2} - \int_{\pi/4}^{\pi/2} \csc(u)^3 du + [-\ln|\csc(u) + \cot(u)|]_{\pi/4}^{\pi/2} \\
\int_{\pi/4}^{\pi/2} \csc\left(u\right)^3 du &= \sqrt{2} - \int_{\pi/4}^{\pi/2} \csc(u)^3 du + \ln\left(\sqrt{2} + 1\right).
\end{aligned}$$

Moving the term $-\int_{\pi/4}^{\pi/2} \csc(u)^3 du$ to the left-hand side gives

$$2\int_{\pi/4}^{\pi/2} \csc(u)^3 du = \sqrt{2} - \ln\left(\sqrt{2}+1\right),$$

so

$$L = 2\int_{\pi/4}^{\pi/2} \csc(u)^3 du = \boxed{\sqrt{2} - \ln\left(\sqrt{2}+1\right) \text{ units}}.$$

(c) $r = e^{3\theta}$, $0 \leqslant \theta \leqslant \pi$.

Solution. We have

$$\frac{dr}{d\theta} = 3e^{3\theta}.$$

So

$$\begin{aligned}
L &= \int_0^\pi \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \\
&= \int_0^\pi \sqrt{\left(e^{3\theta}\right)^2 + \left(3e^{3\theta}\right)^2} d\theta \\
&= \int_0^\pi \sqrt{e^{6\theta} + 9e^{6\theta}} d\theta \\
&= \int_0^\pi \sqrt{10e^{6\theta}} d\theta \\
&= \int_0^\pi \sqrt{10}e^{3\theta} d\theta \\
&= \left[\frac{\sqrt{10}e^{3\theta}}{3}\right]_0^\pi \\
&= \boxed{\frac{\sqrt{10}(e^{3\pi} - 1)}{3} \text{ units}}.
\end{aligned}$$