Sections 5.5, 5.6, 8.1

Review of Integration

Learning Goals:

§5.3 The Definite Integral

	Learning Goal	*Homework Problems*
	5.3.1 Use even/odd symmetry to integrate functions on the interval [-a,a]	5.6: 65
	5.3.2 Use the Mean Value Theorem for Integrals to find the average value of a function over an interval	5.3: 55, 59

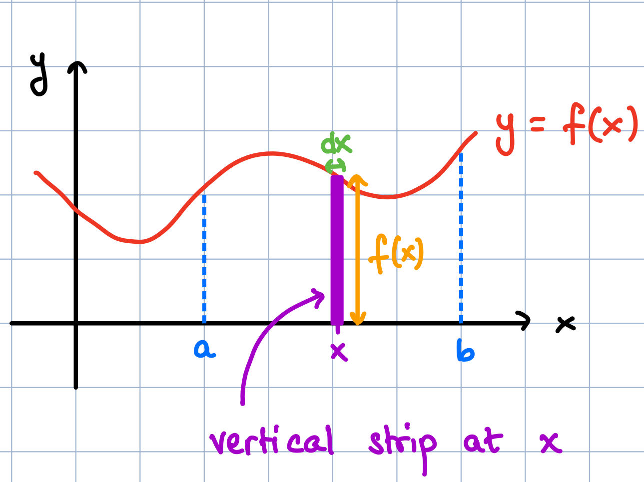
§5.5 Indefinite Integrals and the Substitution Method

	Learning Goal	*Homework Problems*
	5.5.1 Use substitution to simplify an indefinite integral of a composite function	5.5: 3,4,6,7,8,9,21,22,23,25,43,45,46,47,48,51,55,61,64,66
	5.5.2 Use substitution to evaluate an indefinite/a definite integral with mixed trigonometric functions	5.5: 9,22,25
	5.5.3. Use substitution to find an indefinite/a definite integral of ratio of functions	5.5: 4,6,55,61,64,66
	5.5.4 Use substitution to find an indefinite/a definite integral involving composite exponential functions	5.5: 51,61
	5.5.5 Use substitution to evaluate an indefinite/a definite integral with ln(x) and 1/x	5.5: 55,
	5.5.6 Use substitution to evaluate an indefinite/a definite integral with radicals and exponents	5.5: 3,6,21,43,45,46,47,64,66
	5.5.7 Use substitution to evaluate an indefinite/a definite integral with an inverse trigonometric function	5.5: 61,64,66
	5.5.8 Use substitution to evaluate an indefinite/a definite integral with a composite function with a polynomial	5.5: 4,23,443,45,46,47,48

§5.6 Definite Integral Substitutions and the Area Between Curves

	Learning Goal	*Homework Problems*
	5.6.1 Draw the given curves/lines and indicate the specific region in a given problem	5.6: 75,98,101,107
	5.6.2 Find the area of a region bounded between curves/lines on a given interval	5.6: 49,53,55,57,59,60,62,75,87,98,101,109
	5.6.3 Find the area of a region bounded by two functions that cross twice	5.6: 55,57,62
	5.6.4 Find the area of a region bounded by two functions that cross more than twice	5.6: 55,62,71
	5.6.5 Solve area problems where on some interval f>g and on other interval, g>f.	5.6: 62, 111,103
	5.6.6 Solve area problem with integration with respect to y.	5.6: 57,59,85

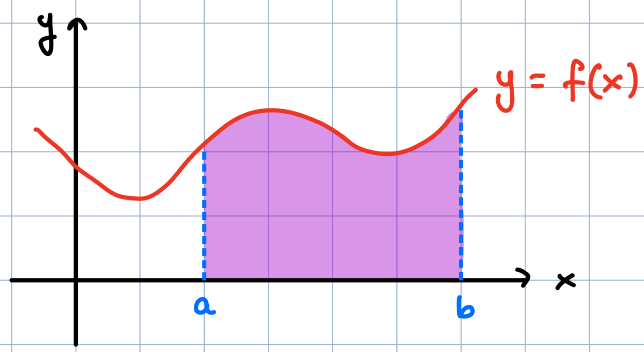
Conceptual introduction: What is an integral?



$\int_a^b f(x)\,dx$ represents the sum of areas of thin vertical strips under the graph of f.

The typical strip at x has:

- length $f(x)$
- width dx
- net area $= f(x)\,dx$



$$\int_a^b \overbrace{f(x)\,dx}^{\text{area of strip}}$$

$=$ "sum" of net areas of all strips for $a \leq x \leq b$.

$=$ net area between $y = f(x)$ and x-axis on $[a,b]$.

To compute definite integrals in practice, we often use the Fundamental Theorem of Calculus:

If f is continuous on $[a,b]$ and F is an antiderivative of F on $[a,b]$, then:

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

Examples: 1) Calculate the average value of $f(x) = \dfrac{1}{4+x^2}$ on $[0,2]$.

Reminder: average value of f on $[a,b]$

$$\boxed{av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

Here, we get

$$av(f) = \frac{1}{2-0}\int_0^2 \frac{dx}{4+x^2}$$

$$= \frac{1}{2}\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2 \qquad \left(\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C\right)$$

$$= \frac{1}{4}\left(\tan^{-1}\left(\frac{2}{2}\right) - \tan^{-1}\left(\frac{0}{2}\right)\right)$$

$$= \frac{1}{4}\cdot\frac{\pi}{4} = \boxed{\frac{\pi}{16}}$$

2) Evaluate $\displaystyle\int \frac{dx}{\sqrt{8-2x-x^2}}$.

$$\int \frac{dx}{\sqrt{8-2x-x^2}} = \int \frac{dx}{\sqrt{9-(x+1)^2}} \qquad \text{(complete the square)}$$

u-sub

$$\boxed{u = x+1 \qquad du = dx}$$

$$= \int \frac{du}{\sqrt{9-u^2}}$$

$$= \sin^{-1}\left(\frac{u}{3}\right) + C \qquad \left(\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right)+C\right)$$

$$= \boxed{\sin^{-1}\left(\frac{x+1}{3}\right) + C}$$

3) Evaluate $\displaystyle\int \frac{dx}{(7+\sqrt{x})^3}$.

We can use the substitution $u = 2+\sqrt{x}$.

$$\int \frac{\overset{2\sqrt{x}\,du}{dx}}{(\underbrace{7+\sqrt{x}}_{u})^3}$$

u - sub

$$\boxed{\begin{aligned} &u = 7+\sqrt{x} \\ &du = \frac{dx}{2\sqrt{x}} \Rightarrow dx = 2\sqrt{x}\,du \end{aligned}}$$

$$= \int \frac{2\sqrt{x}\,du}{u^3}$$

needs to be expressed in terms of u

$u = 7+\sqrt{x} \Rightarrow \sqrt{x} = u-7$

$$= \int \frac{2(u-7)}{u^3}\,du$$

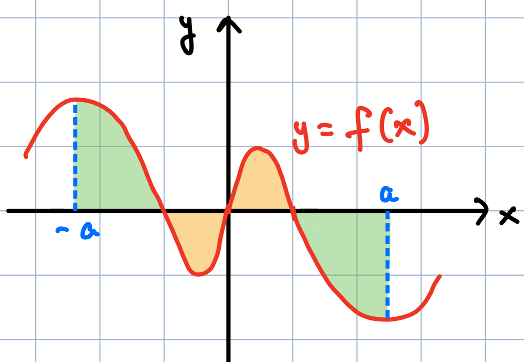
$$= \int 2(u^{-2} - 7u^{-3})\,du$$

$$= 2\left(\frac{u^{-1}}{-1} - 7\,\frac{u^{-2}}{-2}\right) + C$$

$$= \boxed{2\left(\frac{7}{2(7+\sqrt{x})^2} - \frac{1}{7+\sqrt{x}}\right) + C}$$

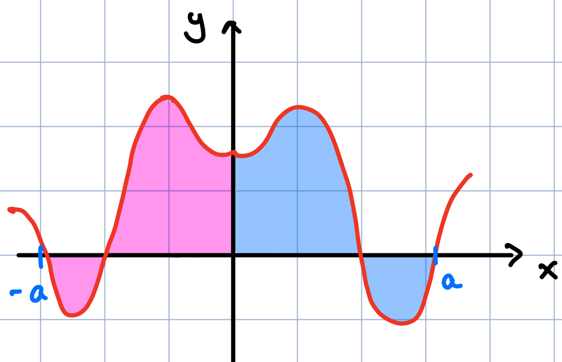
4) $\displaystyle\int_{-1}^{1} \frac{x^5}{x^2+1}\,dx$

Observe :
- the integrand is odd : $f(-x) = -f(x)$
- the interval of integration is centered at 0.



So $\boxed{\displaystyle\int_{-1}^{1} \frac{x^5}{x^2+1}\,dx = 0}$

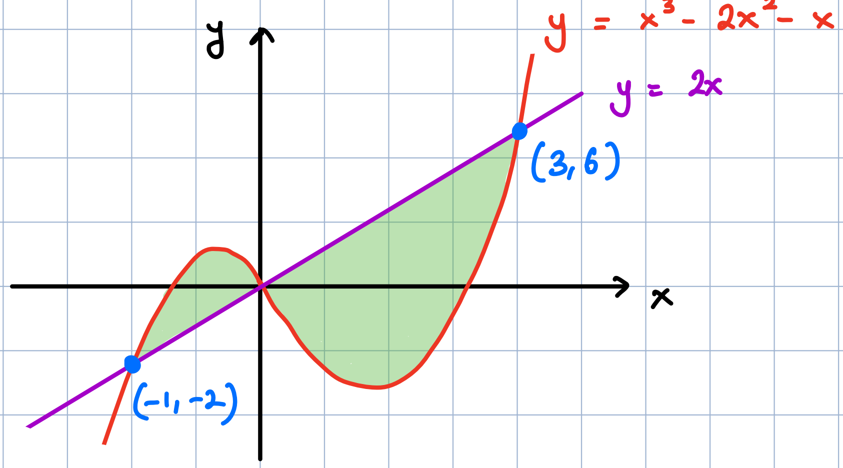
If f is odd, $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$



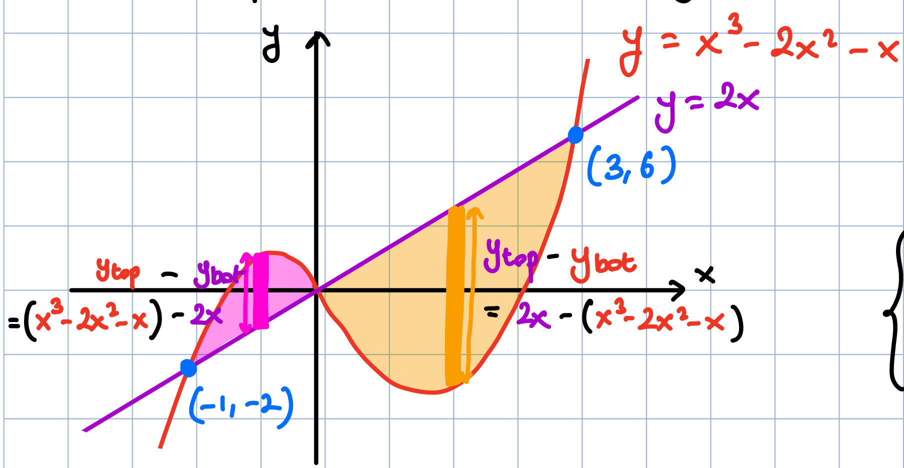
If f is even, $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

Computing areas of regions in the xy-plane:

1) Calculate the area of the region below.



We compute the area using vertical strips.



The vertical strip at x has length:

$$\begin{cases} (x^3 - 2x^2 - x) - 2x & \text{if } -1 \leq x \leq 0 \\ 2x - (x^3 - 2x^2 - x) & \text{if } 0 \leq x \leq 3 \end{cases}$$

So Area $= \displaystyle\int_{-1}^{0} \left((x^3 - 2x^2 - x) - 2x \right) dx + \int_{0}^{3} \left(2x - (x^3 - 2x^2 - x) \right) dx$

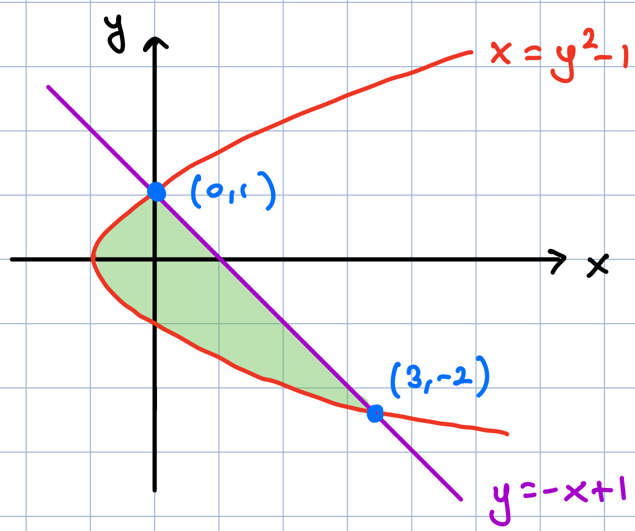
$$= \int_{-1}^{0} (x^3 - 2x^2 - 3x)\,dx + \int_{0}^{3} (3x - x^3 + 2x^2)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{3x^2}{2} \right]_{-1}^{0} + \left[\frac{3x^2}{2} - \frac{x^4}{4} + \frac{2x^3}{3} \right]_{0}^{3}$$

$$= 0 - \left(\frac{(-1)^4}{4} - \frac{2(-1)^3}{3} - \frac{3(-1)^2}{2} \right) + \left(\frac{3(3)^2}{2} - \frac{3^4}{4} + \frac{2 \cdot 3^3}{3} \right) - 0$$

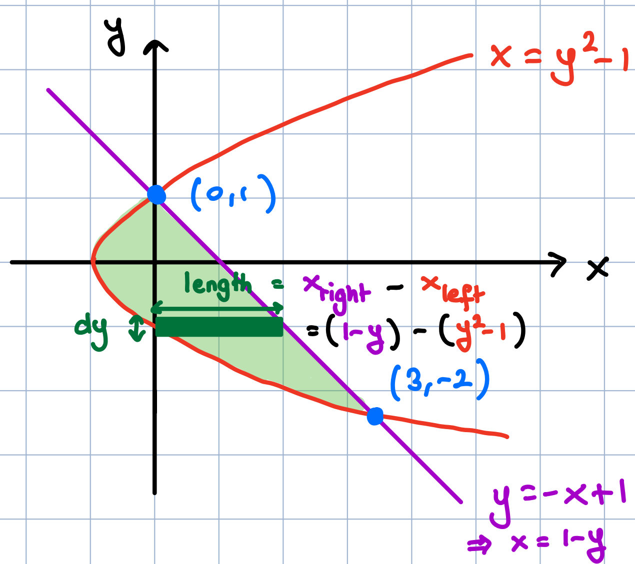
$$= \boxed{\frac{71}{6} \text{ square units}}$$

2) Compute the area of the region below using:



i) integration with respect to y
ii) integration with respect to x

i) Integration with respect to y : we use horizontal strips.



The horizontal strip at y has length $= x_{right} - x_{left}$
$= (1-y) - (y^2-1)$

Upper bound of integration
= largest y-value in region = 1
Lower bound of integration
= least y-value in region = -2

So $\quad A = \int_{-2}^{1} \left[(1-y) - (y^2-1) \right] dy$

$$= \int_{-2}^{1} (2 - y - y^2)\, dy$$

$$= \left[2y - \frac{y^2}{2} - \frac{y^3}{3} \right]_{-2}^{1}$$

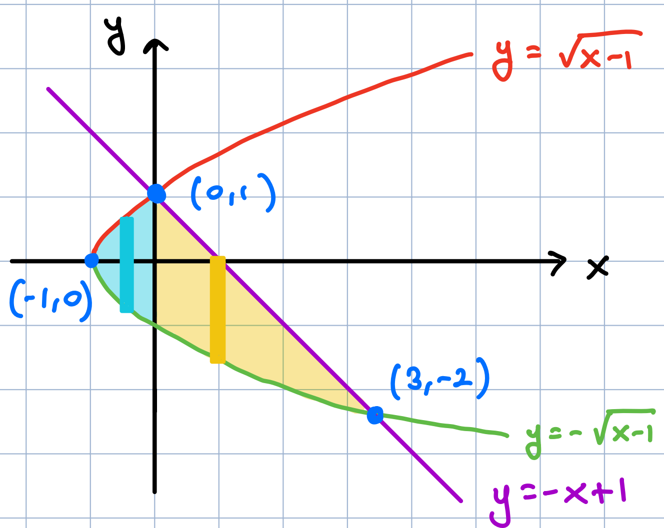
$$= \left(2 - \frac{1^2}{2} - \frac{1^3}{3} \right) - \left(2(-2) - \frac{(-2)^2}{2} - \frac{(-2)^3}{3} \right)$$

$$= \boxed{\frac{9}{2} \text{ square units}}$$

ii) Integration <u>with respect to x</u> : we use <u>vertical strips</u>

We need to express the parabola as a function of x :

$$x = y^2 - 1 \Rightarrow \underbrace{y = \sqrt{x+1}}_{\text{upper branch}} \text{ or } \underbrace{y = -\sqrt{x-1}}_{\text{lower branch}}$$



We will have to use a sum of 2 integrals because the curve bounding the top of the strip is not the same in the entire region. (The region is <u>NOT VERTICALLY SIMPLE</u>.)

If $-1 \leq x \leq 0$, top curve is $y = \sqrt{x+1}$,

$\Rightarrow$ length of strip $= y_{top} - y_{bot} = \sqrt{x+1} - (-\sqrt{x+1})$
$= 2\sqrt{x+1}$

If $0 \leq x \leq 3$, the top curve is $y = -x+1$.

$\Rightarrow$ length of strip $= y_{top} - y_{bot} = (-x+1) - (-\sqrt{x+1})$
$= \sqrt{x+1} - x + 1$

So $$A = \int_{-1}^{0} 2\sqrt{x+1}\, dx + \int_{0}^{3} (\sqrt{x+1} - x + 1)\, dx$$

$$= \left[\frac{4}{3}(x+1)^{3/2}\right]_{-1}^{0} + \left[\frac{2}{3}(x+1)^{3/2} - \frac{x^2}{2} + x\right]_{0}^{3}$$

$$= \frac{4}{3}\left(1^{3/2} - 0^{3/2}\right) + \frac{2}{3}\left(4^{3/2} - 1^{3/2}\right) - \frac{3^2}{2} + 3$$

$= \boxed{\frac{9}{2} \text{ square units}}$ we obtain the same result as (i).

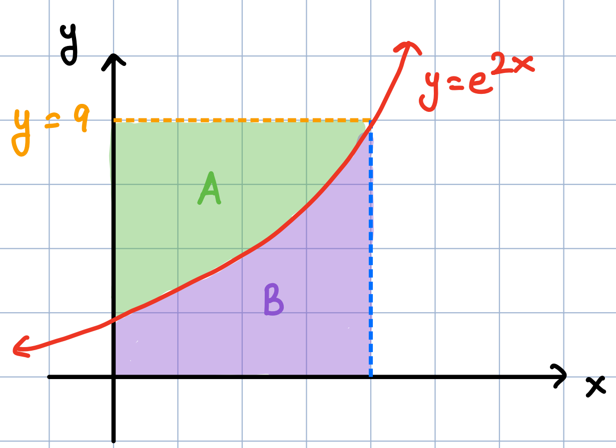
<u>Remark</u>: in this problem, using a y-integral is easier because all horizontal strips are bounded by the same left curve and the same right curve: we say that the region is <u>HORIZONTALLY SIMPLE</u>.

Practice:

1) Calculate the following integrals.

i) $\displaystyle\int_0^{1/4} \frac{e^{\sin^{-1}(2x)}}{\sqrt{1-4x^2}}\,dx$ ii) $\displaystyle\int \tan(3t)^8 \sec(3t)^2\,dt$

2) Express the areas of the regions A, B below using:
i) an x-integral
ii) a y-integral



Solutions

1) i) We substitute $u = \sin^{-1}(2x)$

$$\int_0^{1/4} \frac{e^{\sin^{-1}(2x)}}{\sqrt{1-4x^2}}\,dx \qquad \frac{du}{2}$$

u-sub

$$\boxed{\begin{array}{ll} u = \sin^{-1}(2x) & x=0 \Rightarrow u=0 \\ du = \dfrac{2dx}{\sqrt{1-4x^2}} & x = 1/4 \Rightarrow u = \dfrac{\pi}{6} \\ \Rightarrow \dfrac{dx}{\sqrt{1-4x^2}} = \dfrac{du}{2} & \end{array}}$$

$$= \int_0^{\pi/6} \frac{e^u}{2}\,du$$

$$= \left[\frac{e^u}{2}\right]_0^{\pi/6} = \boxed{\frac{e^{\pi/6}-1}{2}}$$

ii) We substitute $u = \tan(3t)$.

$$\int \tan(3t)^8 \sec(3t)^2\,dt \qquad \frac{du}{3}$$

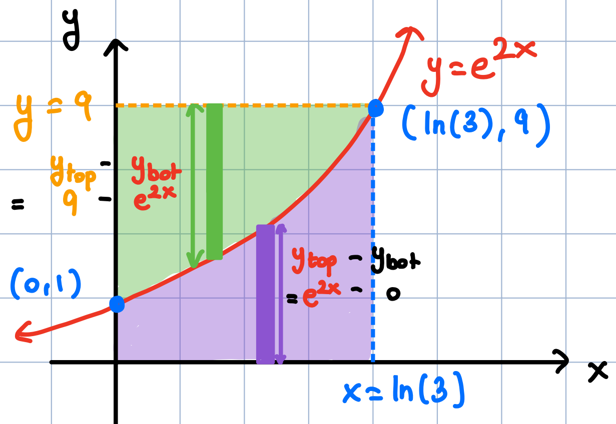
u-sub

$$\boxed{\begin{array}{l} u = \tan(3t) \\ du = 3\sec(3t)^2 dt \\ \Rightarrow \sec(3t)^2 dt = \dfrac{du}{3} \end{array}}$$

$$= \int \frac{1}{3} u^8\,du$$

$$= \frac{1}{27}u^9 + C = \boxed{\frac{1}{27}\tan(3t)^9 + C}$$

2) i) Using an <u>x-integral / vertical strips</u>



For A, the vertical strip at x has length $y_{top} - y_{bot} = 9 - e^{2x}$

So $\boxed{A = \int_0^{\ln(3)} (9 - e^{2x})\,dx}$

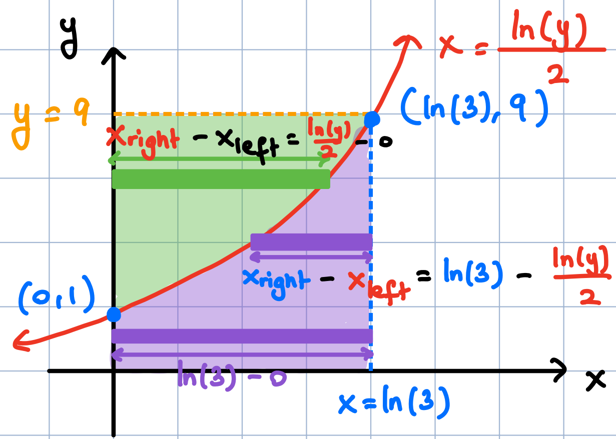
For B, the vertical strip at x has length $y_{top} - y_{bot} = e^{2x} - 0$

So $\boxed{B = \int_0^{\ln(3)} e^{2x}\,dx}$

ii) Using a <u>y-integral / horizontal strips:</u>

Express the curve as a function of y:

$$y = e^{2x} \Rightarrow 2x = \ln(y) \Rightarrow x = \frac{\ln(y)}{2}.$$



For A, the horizontal strip at y has length $x_{right} - x_{left}$
$= \frac{\ln(y)}{2} - 0$

So $\boxed{A = \int_1^9 \frac{\ln(y)}{2}\,dy}$.

For B, we have 2 bounding curves on the left:

- for $0 \leq y \leq 1$, the curve bounding on the left is $x = 0$, so the strip has length $x_{right} - x_{left} = \ln(3) - 0$

- for $1 \leq y \leq 9$, the curve bounding on the left is $x = \frac{\ln(y)}{2}$, so the strip has length $x_{right} - x_{left}$
$= \ln(3) - \frac{\ln(y)}{2}$

So $\boxed{B = \int_0^1 \ln(3)\,dy + \int_1^9 \left(\ln(3) - \frac{\ln(y)}{2}\right) dy}$.