Rutgers University
Math 152

Sections 5.5, 5.6, 8.1: Review of Integration - Worksheet Solutions

1. Evaluate the following antiderivatives.

(a) $\displaystyle\int \frac{dx}{\sqrt{8x - x^2}}$

Solution. We will be able to use the reference antiderivative

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}(u) + C$$

after completing the square in the square root and using a substitution. Completing the square gives

$$8x - x^2 = -(x^2 - 8x) = -(x^2 - 8x + 16) + 16 = 16 - (x-4)^2.$$

Therefore the integral can be written as

$$\int \frac{dx}{\sqrt{8x - x^2}} = \int \frac{dx}{\sqrt{16 - (x-4)^2}} = \int \frac{dx}{\sqrt{16\left(1 - \frac{(x+4)^2}{16}\right)}} = \int \frac{dx}{4\sqrt{1 - \left(\frac{x-4}{4}\right)^2}}.$$

We substitute $u = \frac{x-4}{4}$, which gives $du = \frac{dx}{4}$. The integral becomes

$$\begin{aligned}\int \frac{dx}{4\sqrt{1 - \left(\frac{x-4}{4}\right)^2}} &= \int \frac{du}{\sqrt{1-u^2}} \\ &= \sin^{-1}(u) + C \\ &= \boxed{\sin^{-1}\left(\frac{x-4}{4}\right) + C}.\end{aligned}$$

(b) $\displaystyle\int \frac{\tan^{-1}(t)^3}{1+t^2}dt$

Solution. We use the substitution $u = \tan^{-1}(t)$, so $du = \frac{dt}{1+t^2}$. This gives

$$\begin{aligned}\int \frac{\tan^{-1}(t)^3}{1+t^2}dt &= \int u^3 du \\ &= \frac{1}{4}u^4 + C \\ &= \boxed{\frac{1}{4}\tan^{-1}(t)^4 + C}.\end{aligned}$$

(c) $\displaystyle\int \frac{\tan(3\ln(x))}{x}dx$

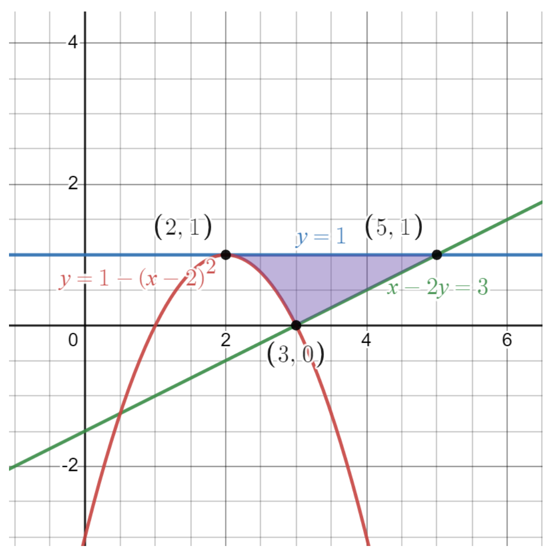
Solution. We use the substitution $u = 3\ln(x)$, so $du = \frac{3dx}{x}$ and the integral becomes

$$\begin{aligned}\int \frac{\tan(3\ln(x))}{x}dx &= \int \frac{\tan(u)}{3}du \\ &= \frac{1}{3}\ln|\sec(u)| + C \\ &= \boxed{\frac{1}{3}\ln|\sec(3\ln(x))| + C}.\end{aligned}$$

2. For each of the regions described below (i) sketch the region, clearly labeling the curves and their intersection points, (ii) calculate the area of the region using an x-integral and (iii) calculate the area of the region using a y-integral.

(a) The region to the right of the parabola $y = 1-(x-2)^2$, below the line $y = 1$ and to the left of the line $x - 2y = 3$.

Solution. (i)



(ii) The region is not vertically simple, so we will need a sum of x-integrals. For $2 \leqslant x \leqslant 3$, the vertical strip at x is bounded by $y = 1$ on the top and $y = 1-(x-2)^2$ on the bottom. For $3 \leqslant x \leqslant 5$, the vertical strip at x is bounded by $y = 1$ on the top and the line $x - 2y = 3 \Rightarrow y = \frac{x-3}{2}$ on the bottom. Therefore the area is given by

$$\begin{aligned}A &= \int_2^3 \left(1-(1-(x-2)^2)\right)dx + \int_3^5 \left(1 - \frac{x-3}{2}\right)dx \\ &= \int_2^3 (x-2)^2 dx + \frac{1}{2}\int_3^5 (5-x)dx \\ &= \left[\frac{(x-2)^3}{3}\right]_2^3 + \frac{1}{2}\left[5x - \frac{x^2}{2}\right]_3^5\end{aligned}$$

$$= \frac{(3-2)^3}{3} - \frac{(2-2)^3}{3} + \frac{1}{2}\left(5 \cdot 5 - \frac{5^2}{2} - 5 \cdot 3 + \frac{3^2}{2}\right)$$
$$= \boxed{\frac{4}{3} \text{ square units}}.$$

(iii) The region is horizontally simple. The horizontal strip at y is bounded on the right by the line $x - 2y = 3$, which gives $x = 2y + 3$ when expressed as a function of y. The curve bounding on the left is the right branch of the parabola $y = 1 - (x-2)^2$. Expressing this branch as a function of y gives

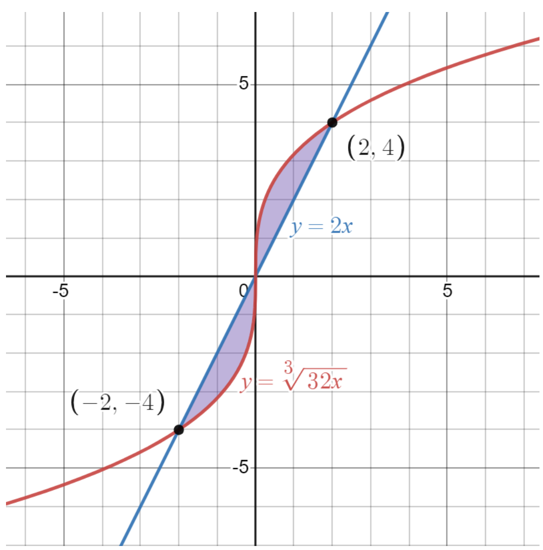
$$y = 1 - (x-2)^2 \;\Rightarrow\; (x-2)^2 = 1 - y \;\Rightarrow |x-2| = \sqrt{1-y} \;\Rightarrow\; x - 2 = \sqrt{1-y} \;\Rightarrow\; x = 2 + \sqrt{1-y}.$$

Note that $|x-2| = x-2$ since $x - 2 \geqslant 0$ on the right branch of the parabola. Therefore the area is

$$A = \int_0^1 \left((2y+3) - \left(2 + \sqrt{1-y}\right)\right) dy$$
$$= \int_0^1 \left(2y + 1 - \sqrt{1-y}\right) dy$$
$$= \left[y^2 + y + \frac{2}{3}(1-y)^{3/2}\right]_0^1$$
$$= 1 + 1 - \frac{2}{3}$$
$$= \boxed{\frac{4}{3} \text{ square units}}.$$

(b) The region bounded by the curves $y = 2x$ and $y = \sqrt[3]{32x}$.

Solution. (i)



(ii) The region is not vertically simple. For $0 \leqslant x \leqslant 2$, the vertical strip at x is bounded on the top by $y = \sqrt[3]{32x}$ and on the bottom by $y = 2x$. For $-2 \leqslant x \leqslant 0$, the vertical strip at x is bounded on the top by $y = 2x$ and on the bottom by $y = \sqrt[3]{32x}$. Therefore

$$\begin{aligned}
A &= \int_{-2}^{0} \left(2x - \sqrt[3]{32x}\right) dx + \int_{0}^{2} \left(\sqrt[3]{32x} - 2x\right) dx \\
&= \left[x^2 - 32^{1/3}\frac{3}{4}x^{4/3}\right]_{-2}^{0} + \left[32^{1/3}\frac{3}{4}x^{4/3} - x^2\right]_{0}^{2} \\
&= -(-2)^2 + 32^{1/3}\frac{3}{4}(-2)^{4/3} + 32^{1/3}\frac{3}{4}2^{4/3} - 2^2 \\
&= \boxed{4 \text{ square units}}.
\end{aligned}$$

(iii) We need to express the curves as functions of y to use a y-integral.

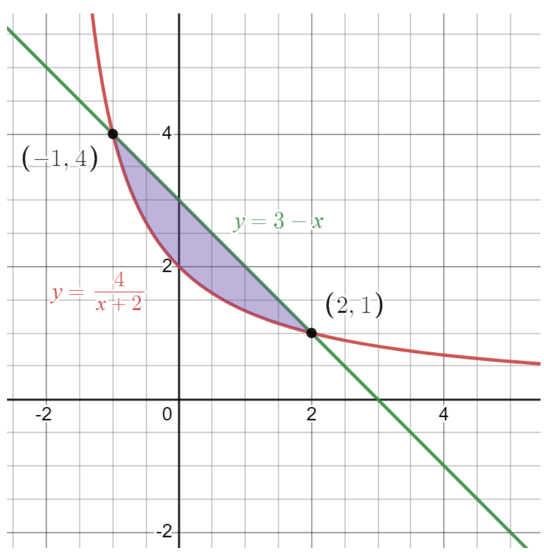
$$\begin{aligned}
&y = \sqrt[3]{32x} \;\Rightarrow\; 32x = y^3 \;\Rightarrow\; x = \frac{y^3}{32}, \\
&y = 2x \;\Rightarrow\; x = \frac{y}{2}.
\end{aligned}$$

The region is not horizontally simple. For $0 \leqslant y \leqslant 4$, the horizontal strip at y is bounded on the right by $x = \frac{y}{2}$ and on the left by $x = \frac{y^3}{32}$. For $-4 \leqslant y \leqslant 0$, the horizontal strip at y is bounded on the right by $x = \frac{y^3}{32}$ and on the left by $x = \frac{y}{2}$. Therefore

$$\begin{aligned}
A &= \int_{-4}^{0} \left(\frac{y^3}{32} - \frac{y}{2}\right) dy + \int_{0}^{4} \left(\frac{y}{2} - \frac{y^3}{32}\right) dy \\
&= \left[\frac{y^4}{128} - \frac{y^2}{4}\right]_{-4}^{0} + \left[\frac{y^2}{4} - \frac{y^4}{128}\right]_{0}^{4} \\
&= -\frac{(-4)^4}{128} + \frac{(-4)^2}{4} + \frac{4^2}{4} - \frac{4^4}{128} \\
&= \boxed{4 \text{ square units}}.
\end{aligned}$$

(c) The region bounded by the curves $y = \dfrac{4}{x+2}$ and $y = 3 - x$.

Solution. (i)



(ii) The region is vertically simple. The vertical strip at x is bounded on the top by $y = 3 - x$ and on the bottom by $y = \frac{4}{x+2}$. Therefore the area is

$$\begin{aligned}
A &= \int_{-1}^{2} \left((3 - x) - \frac{4}{x+2} \right) dx \\
&= \left[3x - \frac{1}{2}x^2 - 4\ln|x+2| \right]_{-1}^{2} \\
&= \left(3 \cdot 2 - \frac{1}{2}2^2 - 4\ln(4) \right) - \left(-3 - \frac{1}{2} - 4\ln(1) \right) \\
&= \boxed{\frac{15}{2} - 8\ln(2) \text{ units}^2}.
\end{aligned}$$

(iii) We need to express the curves as functions of y.

$$\begin{aligned}
y &= 3 - x \;\Rightarrow\; x = 3 - y, \\
y &= \frac{4}{x+2} \;\Rightarrow\; x + 2 = \frac{4}{y} \;\Rightarrow\; x = \frac{4}{y} - 2.
\end{aligned}$$

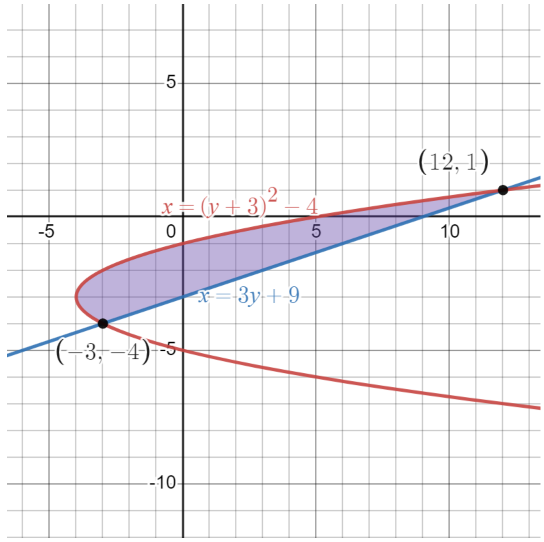
The region is horizontally simple. The horizontal strip at y is bounded on the right by $x = 3 - y$ and on the left by $x = \dfrac{4}{y} - 2$. Therefore

$$\begin{aligned}
A &= \int_{1}^{4} \left((3 - y) - \left(\frac{4}{y} - 2 \right) \right) dy \\
&= \int_{1}^{4} \left(5 - y - \frac{4}{y} \right) dy \\
&= \left[5y - \frac{1}{2}y^2 - 4\ln|y| \right]_{1}^{4} \\
&= \left(5 \cdot 4 - \frac{1}{2}4^2 - 4\ln(4) \right) - \left(5 - \frac{1}{2} - 4\ln(1) \right) \\
&= \boxed{\frac{15}{2} - 8\ln(2) \text{ square units}}.
\end{aligned}$$

3. Calculate the area of the regions described below.

   (a) The region bounded by the parabola $x = (y+3)^2 - 4$ and the line $x = 3y + 9$.

   Solution. A sketch of the region is included below.

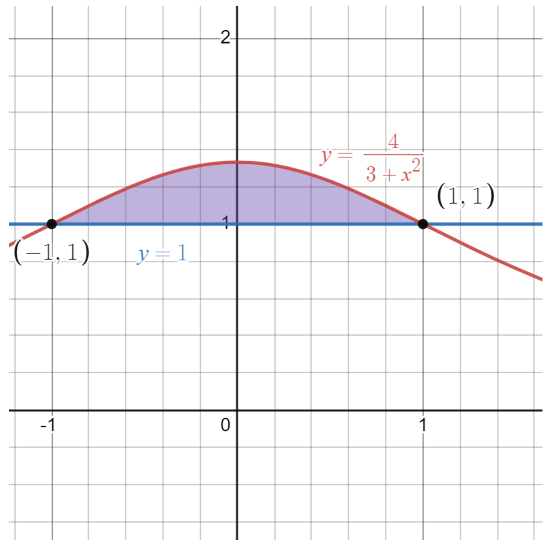


The region is horizontally simple, but not vertically simple. So computing the area using horizontal strips/integration with respect to y will be simpler than using vertical strips/integration with respect to x since we will only need one integral. The horizontal strip at y is bounded on the right by the line $x = 3y + 9$ and on the left by the parabola $x = (y + 3)^2 - 4$. Therefore the area is given by

$$\begin{aligned} A &= \int_{-4}^{1} \left((3y+9) - ((y+3)^2 - 4)\right) dy \\ &= \int_{-4}^{1} \left(3y + 13 - (y^2 + 6y + 9)\right) dy \\ &= \int_{-4}^{1} \left(4 - 3y - y^2\right) dy \\ &= \left[4y - \frac{3}{2}y^2 - \frac{1}{3}y^3\right]_{-4}^{1} \\ &= \left(4 - \frac{3}{2} - \frac{1}{3}\right) - \left(4(-4) - \frac{3}{2}(-4)^2 - \frac{1}{3}(-4)^3\right) \\ &= \boxed{\frac{125}{6} \text{ square units}}. \end{aligned}$$

(b) The region bounded by $y = \dfrac{4}{3+x^2}$ and $y = 1$.

Solution. A sketch of the region is included below.



Note that the region is both vertically and horizontally simple. So we would need only one integral to compute the area using integration with respect to either x or y. However, integration with respect to x will be simpler here, both to set up the integral and compute the antiderivative. The vertical strip at x is bounded on the top by $y = \frac{4}{3+x^2}$ and on the bottom by $y = 1$. So the area is given by

$$A = \int_{-1}^{1} \left(\frac{4}{3+x^2} - 1 \right) dx = 2\int_{0}^{1} \left(\frac{4}{3+x^2} - 1 \right) dx,$$

the second equality holding because the integrand is even (or equivalently, because the region is symmetric with respect to the y-axis). To compute the antiderivative of the first term of the integrand, we can factor out a 3 from the denominator and use the reference antiderivative

$$\int \frac{du}{1+u^2} = \tan^{-1}(u) + C.$$

This gives

$$\begin{aligned} \int \frac{4dx}{3+x^2} &= \int \frac{4dx}{3\left(1+\frac{x^2}{3}\right)} \\ &= \frac{4}{3}\int \frac{dx}{1+\left(\frac{x}{\sqrt{3}}\right)^2} \\ &= \frac{4}{3}\int \frac{\sqrt{3}du}{1+u^2} \qquad \left(u = \frac{x}{\sqrt{3}}\right) \\ &= \frac{4\sqrt{3}}{3}\tan^{-1}(u) + C \\ &= \frac{4\sqrt{3}}{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C. \end{aligned}$$

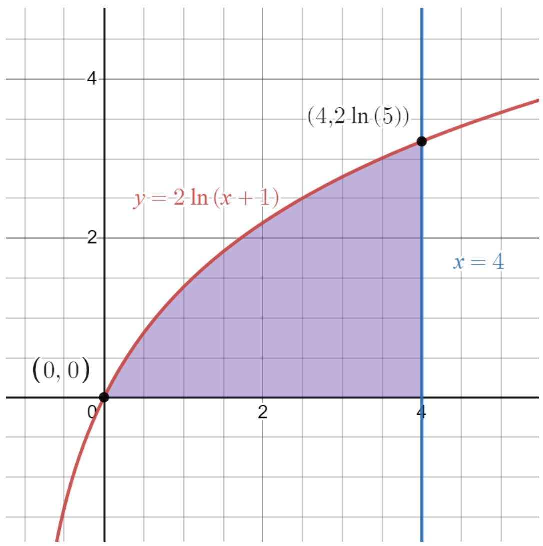
We can now use this to compute the area. We obtain

$$\begin{aligned} A &= 2\int_{0}^{1} \left(\frac{4}{3+x^2} - 1 \right) dx \\ &= 2\left[\frac{4\sqrt{3}}{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - x \right]_0^1 \end{aligned}$$

$$= 2\left(\frac{4\sqrt{3}}{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - 1\right)$$
$$= 2\left(\frac{4\sqrt{3}}{3}\cdot\frac{\pi}{6} - 1\right)$$
$$= \boxed{2\left(\frac{2\sqrt{3}\pi}{9} - 1\right) \text{ square units}}.$$

(c) The region bounded by $y = 2\ln(x+1)$, the x-axis and the line $x = 4$.

Solution. A sketch of the region is included below.



The region is both vertically and horizontally simple. Calculating the area using an x-integral would require finding an antiderivative of ln, which we do not know how to do (yet! we will learn how to do this in section 8.2). So we will prefer a y-integral here. We can express the curve $y = 2\ln(x+1)$ as a function of y as follows

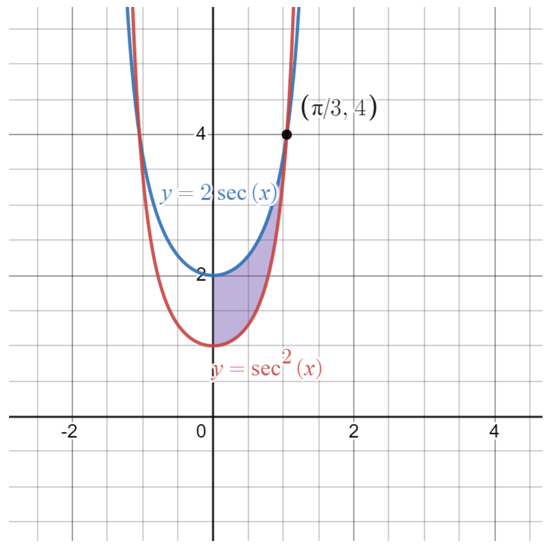
$$y = 2\ln(x+1) \;\Rightarrow\; \frac{y}{2} = \ln(x+1) \;\Rightarrow\; x+1 = e^{y/2} \;\Rightarrow x = e^{y/2} - 1.$$

The horizontal strip at y is bounded by the line $x = 4$ on the right and the curve $x = e^{y/2} - 1$ on the left. Therefore the area is

$$A = \int_0^{2\ln(5)} \left(4 - \left(e^{y/2} - 1\right)\right) dy$$
$$= \int_0^{2\ln(5)} \left(5 - e^{y/2}\right) dy$$
$$= \left[5y - 2e^{y/2}\right]_0^{2\ln(5)}$$
$$= \left(10\ln(5) - 2e^{\ln(5)}\right) - (-2)$$
$$= \boxed{10\ln(5) - 8 \text{ square units}}.$$

(d) The region to the right of the y-axis, above the graph of $y = \sec(x)^2$ and below the graph of $y = 2\sec(x)$.

Solution. A sketch of the region is included below.



The region is vertically simple. The vertical strip at x is bounded on the top by $y = 2\sec(x)$ and on the bottom by $y = \sec(x)^2$. Therefore the area is given by

$$\begin{aligned} A &= \int_0^{\pi/3} \left(2\sec(x) - \sec(x)^2\right) dx \\ &= \left[2\ln|\sec(x) + \tan(x)| - \tan(x)\right]_0^{\pi/3} \\ &= \left(2\ln\left|\sec\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{3}\right)\right| - \tan\left(\frac{\pi}{3}\right)\right) - (2\ln|\sec(0) + \tan(0)| - \tan(0)) \\ &= \left(2\ln\left|2 + \sqrt{3}\right| - \sqrt{3}\right) - (2\ln|1+0| - 0) \\ &= \boxed{2\ln(2+\sqrt{3}) - \sqrt{3} \text{ square units}}. \end{aligned}$$

4. Suppose that f is an **even** function such that

$$\int_{-9}^{5} f(x)dx = -13 \quad \text{and} \quad \int_0^9 f(x)dx = 4.$$

Evaluate the definite integrals below.

(a) $\displaystyle\int_{-9}^{9} f(x)dx$

Solution. Since f is even, by symmetry we have

$$\int_{-9}^{9} f(x)dx = 2\int_0^9 f(x)dx = \boxed{8}.$$

(b) $\displaystyle\int_0^5 (4x - 3f(x))dx$

Solution. Let us start by calculating $\displaystyle\int_0^5 f(x)dx$. By additivity of the integral, we have

$$\int_{-9}^5 f(x)dx = \int_{-9}^0 f(x)dx + \int_0^5 f(x)dx.$$

Since f is even, we have

$$\int_{-9}^0 f(x)dx = \int_0^9 f(x)dx = 4.$$

So we get

$$-13 = 4 + \int_0^5 f(x)dx \;\Rightarrow\; \int_0^5 f(x)dx = -17.$$

Now using the linearity of the integral, we obtain

$$\begin{aligned}\int_0^5 (4x - 3f(x))dx &= 4\int_0^5 xdx - 3\int_0^5 f(x)dx \\ &= 4\left[\frac{1}{2}x^2\right]_0^5 - 3(-17) \\ &= 4\frac{1}{2}(25) + 51 \\ &= \boxed{101}.\end{aligned}$$

(c) $\displaystyle\int_{-3}^3 xf(x)dx$

Solution. Since f is even, the function $g(x) = xf(x)$ is odd, as shown below:

$$g(-x) = (-x)f(-x) = -xf(x) = -g(x).$$

Since the interval of integration $[-3, 3]$ is centered at 0, we deduce

$$\boxed{\int_{-3}^3 xf(x) = 0}.$$

(d) $\displaystyle\int_0^3 xf(x^2)dx$

Solution. We can evaluate this integral using the substitution $u = x^2$, which gives $du = 2xdx$, or $xdx = \frac{du}{2}$. The bounds become

$$\begin{aligned}x = 0 &\;\Rightarrow\; u = 0^2 = 0, \\ x = 3 &\;\Rightarrow\; u = 3^2 = 9.\end{aligned}$$

Therefore

$$\int_0^3 xf(x^2)dx = \int_0^9 \frac{1}{2}f(u)du$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^9 f(u)du \\
&= \frac{1}{2}4 \\
&= \boxed{2}.
\end{aligned}
$$

5. Find the average value of the following functions on the given interval.

   (a) $f(x) = \dfrac{3}{\sqrt{100 - x^2}}$ on $[0, 5]$.

   Solution. The average value is given by

$$
\begin{aligned}
\text{av}(f) &= \frac{1}{5-0}\int_0^5 \frac{3}{\sqrt{100-x^2}}dx \\
&= \frac{3}{5}\int_0^5 \frac{dx}{\sqrt{100\left(1 - \frac{x^2}{100}\right)}} \\
&= \frac{3}{5}\int_0^5 \frac{dx}{10\sqrt{1 - \left(\frac{x}{10}\right)^2}} \\
&= \frac{3}{5}\int_0^{1/2} \frac{du}{\sqrt{1-u^2}} \qquad \left(u = \frac{x}{10}\right) \\
&= \frac{3}{5}\left[\sin^{-1}(u)\right]_0^{1/2} \\
&= \frac{3}{5}\left(\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)\right) \\
&= \frac{3}{5}\cdot\frac{\pi}{6} \\
&= \boxed{\frac{\pi}{10}}.
\end{aligned}
$$

   (b) $f(x) = x\sqrt[3]{3x-7}$ on $[2, 5]$.

   Solution. The average value is given by

$$
\text{av}(f) = \frac{1}{5-2}\int_2^5 x\sqrt[3]{3x-7}dx = \frac{1}{3}\int_2^5 x\sqrt[3]{3x-7}dx.
$$

   We can calculate the integral using the substitution $u = 3x - 7$. This will give $du = 3dx$, or $dx = \frac{du}{3}$. The bounds will become

$$
\begin{aligned}
x = 2 &\Rightarrow u = 3\cdot 2 - 7 = -1, \\
x = 5 &\Rightarrow u = 3\cdot 5 - 7 = 8.
\end{aligned}
$$

   Finally, the extraneous factor x in the integrand can be expressed in terms of u as $x = \frac{u+7}{3}$. We obtain

$$
\text{av}(f) = \frac{1}{3}\int_{-1}^8 \frac{u+7}{3}\sqrt[3]{u}\frac{du}{3}
$$

$$
\begin{aligned}
&= \frac{1}{27}\int_{-1}^{8} \left(u^{4/3} + 7u^{1/3}\right) du \\
&= \frac{1}{27}\left[\frac{3}{7}u^{7/3} + \frac{21}{4}u^{4/3}\right]_{-1}^{8} \\
&= \frac{1}{27}\left(\left(\frac{3}{7}8^{7/3} + \frac{21}{4}8^{4/3}\right) - \left(\frac{3}{7}(-1)^{7/3} + \frac{21}{4}(-1)^{4/3}\right)\right) \\
&= \boxed{\frac{139}{28}}.
\end{aligned}
$$