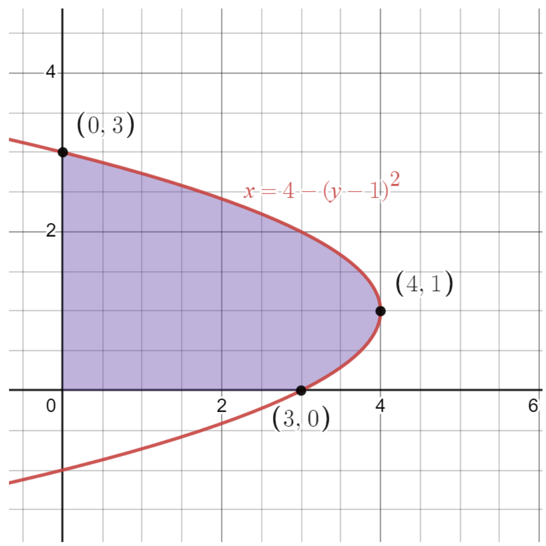
Rutgers University
Math 152

Section 6.1: Volume by Cross-Sections - Worksheet Solutions

1. Consider the region $\mathcal{R}$ in the first quadrant bounded by the curve $x = 4 - (y-1)^2$.

   (a) Sketch the region. Make sure to clearly label the curve and its intercepts.

   Solution.



   (b) A solid has base $\mathcal{R}$ and cross-sections perpendicular to the y-axis. Calculate the volume of the solid if the cross-sections are (i) semi-circles with diameter in the base and (ii) equilateral triangles with a side in the base.

   Solution. The horizontal strip at y in the region is bounded on the right by $x = 4 - (y-1)^2$ and on the left by $x = 0$. Therefore it has length $\ell(y) = 4 - (y-1)^2 - 0 = 4 - (y^2 - 2y + 1) = 3 + 2y - y^2$.

   (i) A semi-circle of diameter ℓ has area $\frac{1}{2}\pi\left(\frac{\ell}{2}\right)^2 = \frac{\pi}{8}\ell^2$. With the length of the strip $\ell(y)$ we found above, we can express the area of the cross-section at y as

$$A(y) = \frac{\pi}{8}\ell(y)^2 = \frac{\pi}{8}\left(3 + 2y - y^2\right)^2 = \frac{\pi}{8}\left(9 + 12y - 2y^2 - 4y^3 + y^4\right).$$

   So the volume of the solid is

$$\begin{aligned} V &= \int_0^3 A(y)dy \\ &= \int_0^3 \frac{\pi}{8}\left(9 + 12y - 2y^2 - 4y^3 + y^4\right) dy \end{aligned}$$

$$
\begin{aligned}
&= \frac{\pi}{8}\left[9y + 6y^2 - \frac{2}{3}y^3 - y^4 + \frac{1}{5}y^5\right]_0^3 \\
&= \frac{\pi}{8}\left(9 \cdot 3 + 6(3)^2 - \frac{2}{3}(3)^3 - 3^4 + \frac{1}{5}(3)^5\right) \\
&= \boxed{\frac{63\pi}{40} \text{ cubic units}}.
\end{aligned}
$$

(ii) An equilateral triangle with side length ℓ has area $\frac{\sqrt{3}}{4}\ell^2$. With the length of the strip $\ell(y)$ we found above, we can express the area of the cross-section at y as

$$A(y) = \frac{\sqrt{3}}{4}\ell(y)^2 = \frac{\sqrt{3}}{4}\left(3 + 2y - y^2\right)^2 = \frac{\sqrt{3}}{4}\left(9 + 12y - 2y^2 - 4y^3 + y^4\right).$$

So the volume of the solid is

$$
\begin{aligned}
V &= \int_0^3 A(y)dy \\
&= \int_0^3 \frac{\sqrt{3}}{4}\left(9 + 12y - 2y^2 - 4y^3 + y^4\right) dy \\
&= \frac{\sqrt{3}}{4}\left[9y + 6y^2 - \frac{2}{3}y^3 - y^4 + \frac{1}{5}y^5\right]_0^3 \\
&= \frac{\sqrt{3}}{4}\left(9 \cdot 3 + 6(3)^2 - \frac{2}{3}(3)^3 - 3^4 + \frac{1}{5}(3)^5\right) \\
&= \boxed{\frac{63\sqrt{3}}{20} \text{ cubic units}}.
\end{aligned}
$$

Note: we could have used the integral that we had already compute in part (i) to minimize computations:

$$\int_0^3 \left(9 + 12y - 2y^2 - 4y^3 + y^4\right) dy = \frac{63}{5}.$$

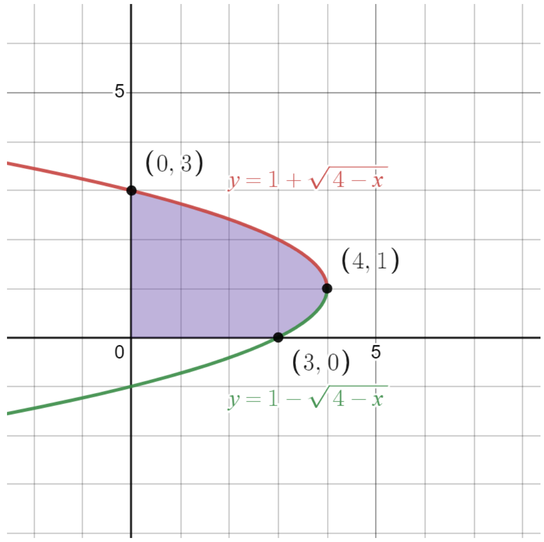
Only the factor in front of the integral changed for part (ii).

(c) A solid has base $\mathcal{R}$ and its cross-sections perpendicular to the x-axis are isosceles right triangles with hypotenuse in the base. Calculate the volume of the solid.

Solution. We need to express the curve as functions of x:

$$x = 4 - (y-1)^2 \;\Rightarrow\; (y-1)^2 = 4 - x \;\Rightarrow |y - 1| = \sqrt{4 - x} \;\Rightarrow\; y = 1 \pm \sqrt{4 - x}.$$

The solution with the positive sign corresponds to the upper branch of the parabola and the negative sign corresponds to the lower branch of the parabola, see figure below.



The vertical strip at x is bounded on the top by $y = 1 + \sqrt{4 - x}$. On the bottom, the vertical strip at x is bounded by $y = 0$ for $0 \leqslant x \leqslant 3$ and $y = 1 - \sqrt{4 - x}$ for $3 \leqslant x \leqslant 4$. Therefore, the length of the strip is

$$\ell(x) = \begin{cases} 1 + \sqrt{4 - x} - 0 = 1 + \sqrt{4 - x} & \text{if } 0 \leqslant x \leqslant 3, \\ (1 + \sqrt{4 - x}) - (1 - \sqrt{4 - x}) = 2\sqrt{4 - x} & \text{if } 3 \leqslant x \leqslant 4. \end{cases}$$

The length of an isosceles right triangle with hypotenuse ℓ is $\frac{1}{4}\ell^2$, so the area of the cross-section at x is

$$A(x) = \frac{1}{4}\ell(x)^2 = \begin{cases} \frac{1}{4}\left(1 + \sqrt{4 - x}\right)^2 = \dfrac{1}{4}\left(5 - x + 2\sqrt{4 - x}\right) & \text{if } 0 \leqslant x \leqslant 3, \\ \frac{1}{4}\left(2\sqrt{4 - x}\right)^2 = 4 - x & \text{if } 3 \leqslant x \leqslant 4. \end{cases}$$

So the volume of the solid is given by

$$\begin{aligned} V &= \int_0^4 A(x)dx \\ &= \int_0^3 \frac{1}{4}\left(5 - x + 2\sqrt{4 - x}\right)dx + \int_3^4 (4 - x)dx \\ &= \frac{1}{4}\left[5x - \frac{1}{2}x^2 - \frac{4}{3}(4 - x)^{3/2}\right]_0^3 + \left[4x - \frac{1}{2}x^2\right]_3^4 \\ &= \frac{1}{4}\left(15 - \frac{9}{2} - \frac{4}{3} + \frac{4}{3}4^{3/2}\right) + \left(16 - \frac{1}{2}16 - 12 + \frac{9}{2}\right) \\ &= \boxed{\frac{131}{24} \text{ cubic units}}. \end{aligned}$$

(d) Calculate the volume of the solid of revolution obtained by revolving $\mathcal{R}$ about (i) the y-axis and (ii) the line $y = -2$.

Solution. (i) Revolving the horizontal strip at y about the y-axis creates a disk of radius $r(y) = 4 - (y - 1)^2 = 3 + 2y - y^2$. So the volume is given by

$$V = \int_0^3 \pi r(y)^2 dy$$

$$
\begin{aligned}
&= \pi \int_0^3 (3 + 2y - y^2)^2 dy \\
&= \boxed{\frac{63\pi}{5} \text{ units}^3}.
\end{aligned}
$$

(We have used the integral already computed in part (b)(i).)

(ii) Revolving the vertical strip at x about the line $y = -2$ creates a washer. The outer radius of the washer is

$$r_{\text{out}}(x) = 1 + \sqrt{4-x} - (-2) = 3 + \sqrt{4-x}.$$

The inner radius is

$$r_{\text{in}}(x) = \begin{cases} 0 - (-2) = 2 & \text{if } 0 \leqslant x \leqslant 3, \\ 1 - \sqrt{4-x} - (-2) = 3 - \sqrt{4-x} & \text{if } 3 \leqslant x \leqslant 4. \end{cases}$$

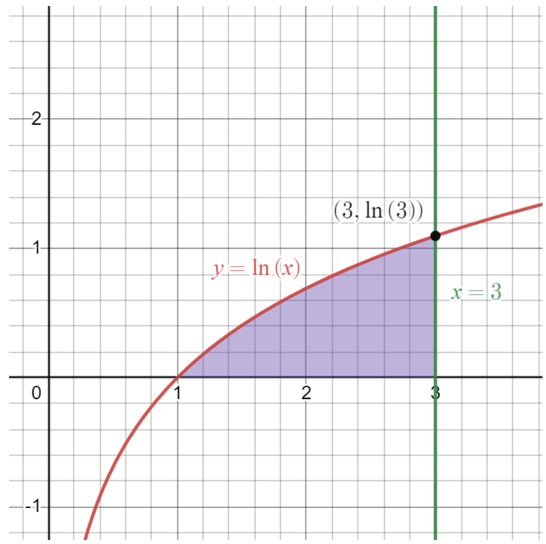
So the volume is given by

$$
\begin{aligned}
V &= \int_0^4 \pi \left(r_{\text{out}}(x)^2 - r_{\text{in}}(x)^2\right) dx \\
&= \int_0^3 \pi \left((3 + \sqrt{4-x})^2 - 2^2\right) dx + \int_3^4 \pi \left((3 + \sqrt{4-x})^2 - (3 - \sqrt{4-x})^2\right) dx \\
&= \pi \int_0^3 \left(9 + 6\sqrt{4-x} - x\right) dx + \pi \int_3^4 12\sqrt{4-x} dx \\
&= \pi \left(\left[9x - 4(4-x)^{3/2} - \frac{1}{2}x^2 \right]_0^3 + \left[-8\sqrt{4-x} \right]_3^4 \right) \\
&= \boxed{\frac{117\pi}{2} \text{ cubic units}}.
\end{aligned}
$$

2. Use the method of disks/washers to calculate the volume of the solids of revolutions obtained by revolving the regions described below about the given axis.

   (a) The region below the graph of $y = \ln(x)$ on $1 \leqslant x \leqslant 3$ revolved about the line $x = 3$.

   Solution. The region and axis of revolution are sketched below.



To use the method of disks/washers, we need to revolve strips perpendicular to the axis of revolution. Since the axis is vertical here, we revolve horizontal strips and use integration with respect to y. We need to express the curve as a function of y:

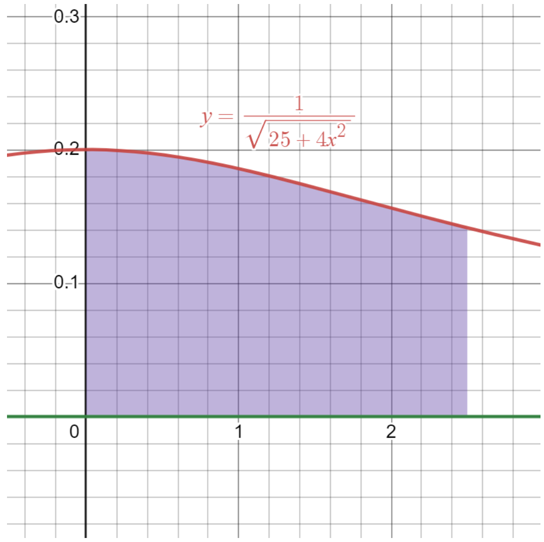
$$y = \ln(x) \;\Rightarrow\; x = e^y.$$

Revolving the horizontal strip at y in the region around $x = 3$ creates a disk with radius $r(y) = 3 - e^y$. So the volume is given by

$$\begin{aligned}
V &= \int_0^{\ln(3)} \pi r(y)^2 dy \\
&= \pi \int_0^{\ln(3)} (3 - e^y)^2 dy \\
&= \pi \int_0^{\ln(3)} (9 - 6e^y + e^{2y}) dy \\
&= \pi \left[9y - 6e^y + \frac{1}{2} e^{2y} \right]_0^{\ln(3)} \\
&= \pi \left(9\ln(3) - 6e^{\ln(3)} + \frac{1}{2} e^{2\ln(3)} + 6e^0 - \frac{1}{2} e^0 \right) \\
&= \pi \left(9\ln(3) - 6 \cdot 3 + \frac{1}{2} 3^2 + 6 - \frac{1}{2} \right) \\
&= \boxed{\pi\left(9\ln(3) - 8\right) \text{ cubic units}}.
\end{aligned}$$

(b) The region below the graph of $y = \dfrac{1}{\sqrt{25 + 4x^2}}$ on $0 \leqslant x \leqslant \dfrac{5}{2}$ revolved about the x-axis.

Solution. The region and axis of revolution are sketched below.

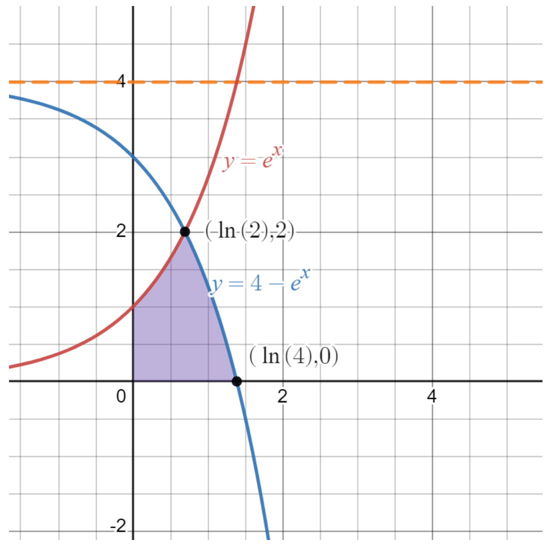


To use the method of disks/washers, we need to revolve strips perpendicular to the axis of revolution. Since the axis is horizontal here, we revolve vertical strips and use integration with respect to x. Revolving the vertical strip at x in the region around the x-axis creates a disk with radius $r(x) = \frac{1}{\sqrt{25+4x^2}}$. So the volume is given by

$$\begin{aligned} V &= \int_0^{5/2} \pi r(x)^2 dx \\ &= \pi \int_0^{5/2} \frac{dx}{25+4x^2} \\ &= \frac{\pi}{25} \int_0^{5/2} \frac{dx}{1+\left(\frac{2x}{5}\right)^2} \\ &= \frac{\pi}{25} \int_0^{1} \frac{5du}{2(1+u^2)} \qquad \left(u = \frac{2x}{5}\right) \\ &= \frac{2\pi}{5} \left[\tan^{-1}(u)\right]_0^1 \\ &= \frac{2\pi}{5} \left(\tan^{-1}(1) - \tan^{-1}(0)\right) \\ &= \frac{2\pi}{5} \left(\frac{\pi}{4} - 0\right) \\ &= \boxed{\frac{\pi^2}{10} \text{ cubic units}}. \end{aligned}$$

(c) The region bounded by $y = e^x$, $y = 4 - e^x$ and the coordinate axes revolved about the line $y = 4$.

Solution. The region and axis of revolution are sketched below.



To use the method of disks/washers, we need to revolve strips perpendicular to the axis of revolution. Since the axis is horizontal here, we revolve vertical strips and use integration with respect to x. Revolving the vertical strip at x in the region creates a washer. The outer radius of the washer is $r_{\text{out}}(x) = 4 - 0 = 4$. the inner radius of the washer is

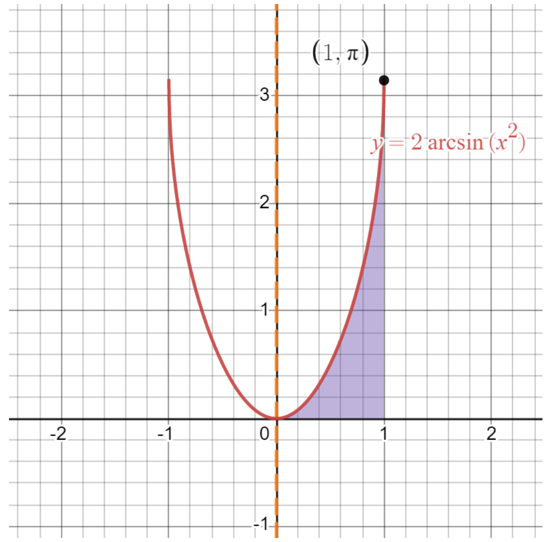
$$r_{\text{in}}(x) = \begin{cases} 4 - e^x & \text{if } 0 \leqslant x \leqslant \ln(2), \\ 4 - (4 - e^x) = e^x & \text{if } \ln(2) \leqslant x \leqslant \ln(4). \end{cases}$$

So the volume is

$$\begin{aligned} V &= \int_0^{\ln(4)} \pi \left(r_{\text{out}}(x)^2 - r_{\text{in}}(x)^2 \right) dx \\ &= \int_0^{\ln(2)} \pi \left(4^2 - (4 - e^x)^2 \right) dx + \int_{\ln(2)}^{\ln(4)} \pi \left(4^2 - (e^x)^2 \right) dx \\ &= \pi \int_0^{\ln(2)} \left(8e^x - e^{2x} \right) dx + \pi \int_{\ln(2)}^{\ln(4)} \left(16 - e^{2x} \right) dx \\ &= \pi \left(\left[8e^x - \frac{1}{2} e^{2x} \right]_0^{\ln(2)} + \left[16x - \frac{1}{2} e^{2x} \right]_{\ln(2)}^{\ln(4)} \right) \\ &= \boxed{\pi \left(16 \ln(2) + \frac{1}{2} \right) \text{ cubic units}}. \end{aligned}$$

(d) The region below the graph of $y = 2 \sin^{-1} \left(x^2 \right)$ on $0 \leqslant x \leqslant 1$ revolved about the y-axis.

Solution. The region and axis of revolution are sketched below.



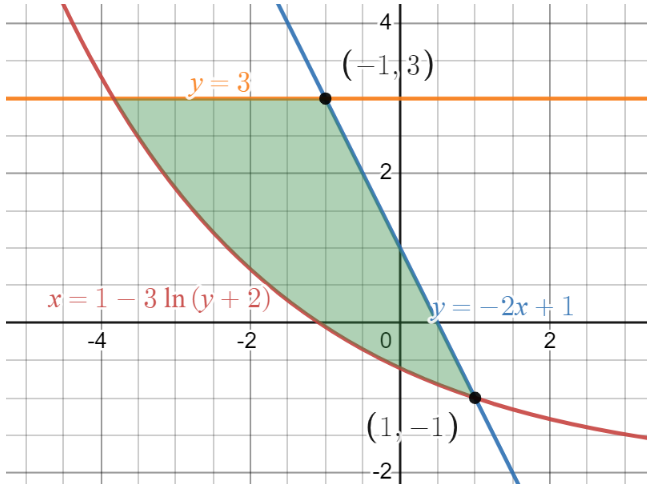
To use the method of disks/washers, we need to revolve strips perpendicular to the axis of revolution. Since the axis is vertical here, we revolve horizontal strips and use integration with respect to y. We need to express the curve as a function of y:

$$y = 2\sin^{-1}(x^2) \;\Rightarrow\; x^2 = \sin\left(\frac{y}{2}\right) \;\Rightarrow\; |x| = \sqrt{\sin\left(\frac{y}{2}\right)} \;\Rightarrow\; x = \sqrt{\sin\left(\frac{y}{2}\right)}$$

where the simplification $|x| = x$ is because $x \geqslant 0$ for the portion of the curve considered. Revolving the horizontal strip at y in the region around the y-axis creates a washer with outer radius $r_{\text{out}}(y) = 1$ and inner radius $r_{\text{in}}(y) = \sqrt{\sin\left(\frac{y}{2}\right)}$. So the volume is

$$\begin{aligned} V &= \int_0^\pi \pi\left(r_{\text{out}}(y)^2 - r_{\text{in}}(y)^2\right) dy \\ &= \int_0^\pi \pi\left(1 - \sin\left(\frac{y}{2}\right)\right) dy \\ &= \pi\left[y + 2\cos\left(\frac{y}{2}\right)\right]_0^\pi \\ &= \pi\left(\pi + 2\cos\left(\frac{\pi}{2} - 2\cos(0)\right)\right) \\ &= \boxed{\pi\left(\pi - 2\right)\ \text{cubic units}}. \end{aligned}$$

3. Consider the region $\mathcal{R}$ shaded in the figure below.



Use the method of washers to set-up integrals that compute the volume of the solid obtained by revolving $\mathcal{R}$ about the line

(a) $x=2$

Solution. We need to express the blue line as a function of y:

$$y=-2x+1 \;\Rightarrow\; x=\frac{1-y}{2}$$

Revolving the horizontal strip at y in the region about the line $x=2$ creates a washer with inner radius $r_{\text{in}}(y)=2-\frac{1-y}{2}=\frac{3+y}{2}$ and outer radius $r_{\text{out}}(y)=2-(1-3\ln(y+2))=1+3\ln(y+2)$. So the volume of the solid is given by

$$\boxed{V=\int_{-1}^{3}\pi\left((1+3\ln(y+2))^2-\left(\frac{3+y}{2}\right)^2\right)dy}.$$

(b) $y=3$

Solution. We need to express the red curve as a function of x:

$$x=1-3\ln(y+2) \;\Rightarrow\; \ln(y+2)=\frac{1-x}{3} \;\Rightarrow\; y=e^{(1-x)/3}-2.$$

Revolving the vertical strip at x in the region about $y=3$ creates a disk of radius

$$r(x)=\begin{cases}3-\left(e^{(1-x)/3}-2\right)=5-e^{(1-x)/3} & \text{if } 1-3\ln(5)\leqslant x\leqslant -1,\\ 3-(-2x+1)=2+2x & \text{if } -1\leqslant x\leqslant 1.\end{cases}$$

So the volume is given by

$$\boxed{V=\int_{1-3\ln(5)}^{-1}\pi\left(5-e^{(1-x)/3}\right)^2dx+\int_{-1}^{1}\pi\,(2+2x)^2\,dx}.$$

(c) $x = -4$

Solution. Revolving the horizontal strip at y in the region about the line $x = -4$ creates a washer with outer radius $r_{\text{out}}(y) = \frac{1-y}{2} - (-4) = \frac{9-y}{2}$ and inner radius $r_{\text{in}}(y) = (1 - 3\ln(y+2)) - (-4) = 5 - 3\ln(y+2)$. So the volume of the solid is given by

$$\boxed{V = \int_{-1}^{3} \pi \left(\left(\frac{9-y}{2} \right)^2 - (5 - 3\ln(y+2))^2 \right) dy}.$$

(d) $y = -2$

Solution. Revolving the vertical strip at x in the region creates a washer with inner radius $r_{\text{in}}(x) = e^{(1-x)/3} - 2 - (-2) = e^{(1-x)/3}$. The outer radius is given by

$$r_{\text{out}}(x) = \begin{cases} 3 - (-2) = 5 & \text{if } 1 - 3\ln(5) \leqslant x \leqslant -1, \\ -2x + 1 - (-2) = 3 - 2x & \text{if } -1 \leqslant x \leqslant 1. \end{cases}$$

So the volume is given by

$$\boxed{V = \int_{1-3\ln(5)}^{-1} \pi \left(5^2 - \left(e^{(1-x)/3} \right)^2 \right) dx + \int_{-1}^{1} \pi \left((3-2x)^2 - \left(e^{(1-x)/3} \right)^2 \right) dx}.$$