Section 6.2

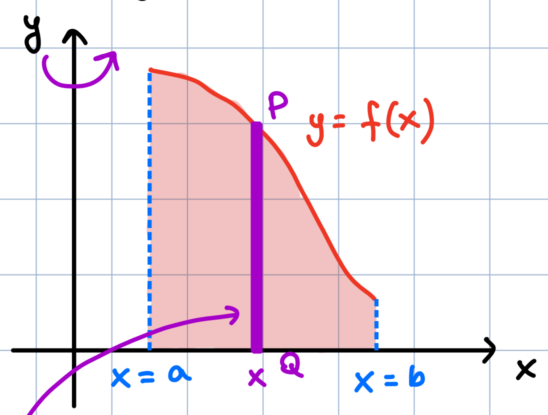
Volume by Shells

Learning Goals:

	Learning Goal	Homework Problems
	6.2.1 Set and evaluate an x or a y integral for a volume of revolution of a region bounded by a unique function on an interval about the x or the y axis using the method of cylindrical shells	6.2: 1,3,5,9,13,17,21,23,27,47
	6.2.2 Set and evaluate an x or a y integral for a volume of revolution of a region bounded by functions that cross about the x or the y axis using the method of cylindrical shells	6.2: 11,35
	6.2.3 Set and evaluate an integral for a volume of revolution of a given region about the line x=nonzero # or y=nonzero# # using the method of cylindrical shells	6.2: 23,27
	6.2.4 Set and evaluate integrals for the same volume of revolution using the disk/washer and the cylindrical shells methods	6.2: 29,37,39
	6.2.5 Set and evaluate an x integral and a y-integral for volume of the same solid	6.2: 37,39
	6.2.6 Set and evaluate the volume of a solid that is described by words and no functions are given	6.2: 41,43,49

Conceptual introduction: in 6.1, we computed volumes using cross-sections. This section: we use nested shells.

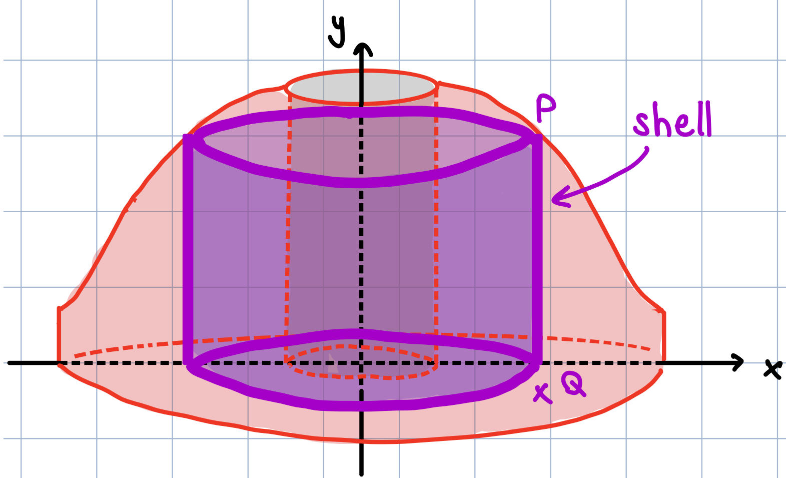
Consider the solid obtained by revolving the region below $y = f(x)$ about the y-axis.

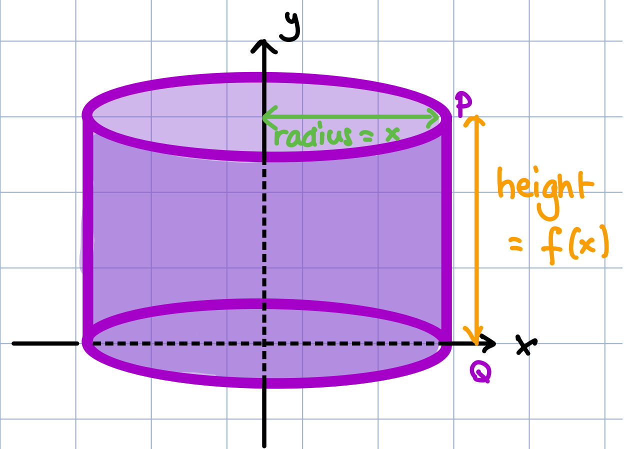


We could try to use washers, but we would need to solve $y = f(x)$ for x (so find $x = f^{-1}(y)$) which could be difficult.

We use shells instead.

revolving this strip about the y-axis will create a shell.





Volume of the shell $= 2\pi$ (shell radius)(shell height)(thickness)

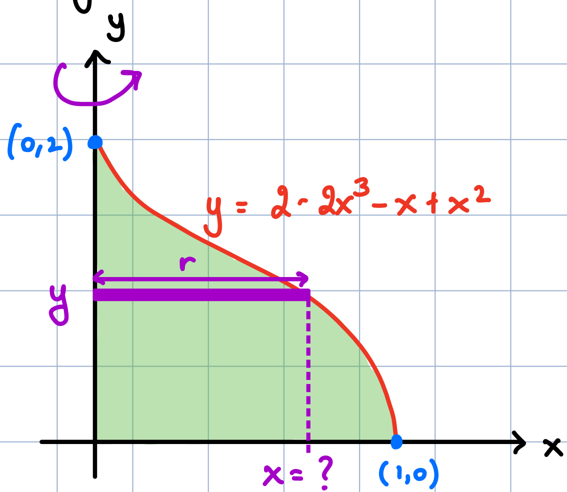
$= 2\pi \, x \, f(x) \, dx$

So the volume of the solid is $\boxed{V = \int_a^b 2\pi x f(x)\, dx}$

General formula to memorize:

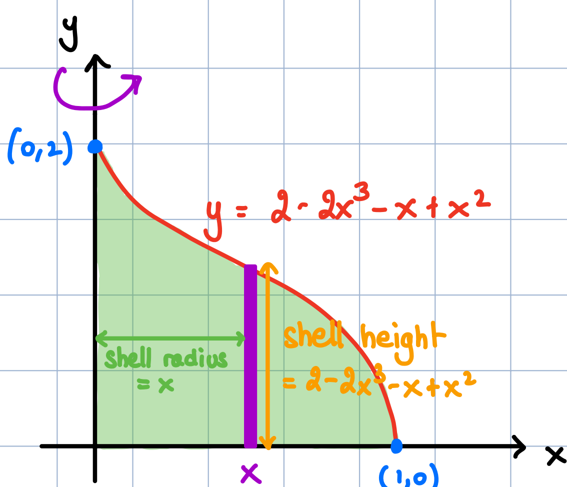
$$\boxed{V = \int_a^b 2\pi \, r(x) \, h(x) \, dx}$$ with $\begin{cases} r(x) = \text{shell radius} \\ h(x) = \text{shell height} \end{cases}$

Examples : 1) Let R be the region bounded by the x-axis, the y-axis and $y = 2 - 2x^3 - x + x^2$. Find the volume of the solid obtained by revolving R about the y-axis.



To use disks, we slice perpendicularly to the axis of revolution (y-axis). We would need to solve the equation $y = 2 - 2x^3 - x + x^2$ for x to find the radius r, but this is not possible !

$\Rightarrow$ So we use the <u>shell method</u> instead.



To find the dimensions of a typical shell, we consider a strip <u>parallel</u> to the axis of revolution (y-axis).

- Shell radius = distance between axis of revolution and strip

  $\Rightarrow r(x) = x$

- Shell height = length of strip

  $\Rightarrow h(x) = 2 - 2x^3 - x + x^2$

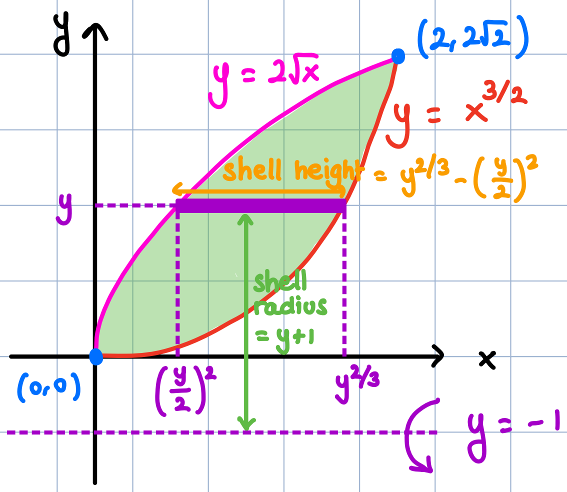
So
$$V = \int_0^1 2\pi\, r(x)\, h(x)\, dx$$
$$= \int_0^1 2\pi x \left(2 - 2x^3 - x + x^2\right) dx$$
$$= 2\pi \int_0^1 \left(2x - 2x^4 - x^2 + x^3\right) dx$$
$$= 2\pi \left[x^2 - \frac{2x^5}{5} - \frac{x^3}{3} + \frac{x^4}{4}\right]_0^1 = \boxed{\frac{31\pi}{30} \text{ cubic units}}.$$

2) Let R be the region bounded $y = 2\sqrt{x}$ and $y = x^{3/2}$.
Set up an integral that computes the volume of the solid obtained by revolving R about the line $y = -1$ using

a) the shell method.

b) the washer method.

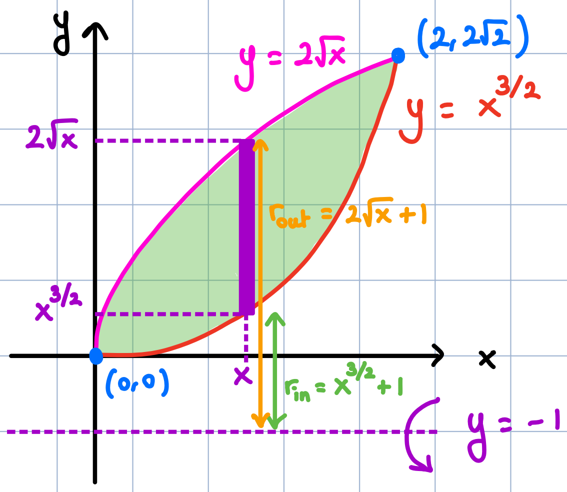
a) Shell method: strip parallel to the axis of revolution.



- Shell radius $= y - (-1) = y+1$
- Shell height $= y^{2/3} - \left(\frac{y}{2}\right)^2$
- Thickness $= dy$

So $\boxed{V = \int_0^{2\sqrt{2}} 2\pi (y+1)\left(y^{2/3} - \left(\frac{y}{2}\right)^2\right) dy}$

b) Washer method: strip perpendicular to the axis of revolution.



- Inner radius: $r_{in}(x) = x^{3/2} + 1$
- Outer radius: $r_{out}(x) = 2\sqrt{x} + 1$

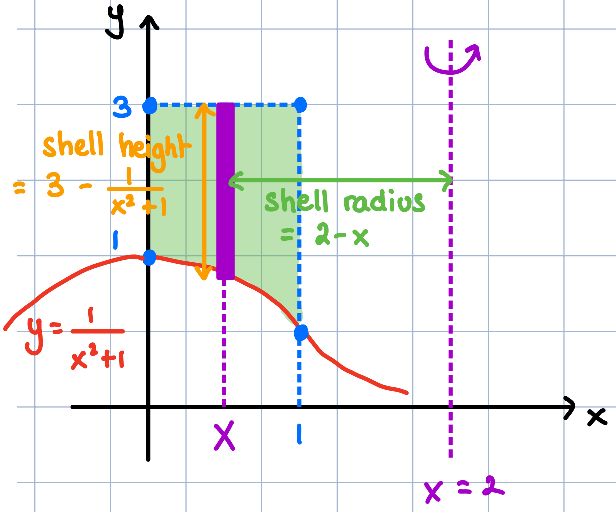
So $\boxed{V = \int_0^2 \pi \left[(2\sqrt{x}+1)^2 - (x^{3/2}+1)^2 \right] dx}$

How to decide between shells and washers?

$\Rightarrow$ Find what is easier for strips between parallel or perpendicular to the axis of revolution.

Parallel $\rightarrow$ use shells, Perpendicular $\rightarrow$ use washers.

3) Consider the region pictured below. Calculate the volume of the solid obtained by revolving R about $x = 2$.



Here, vertical strips are easier because we have a single bounding curve for the top and the bottom each.

$\Rightarrow$ We use shells.

- Shell radius $= 2-x$
- Shell height $= 3 - \frac{1}{1+x^2}$

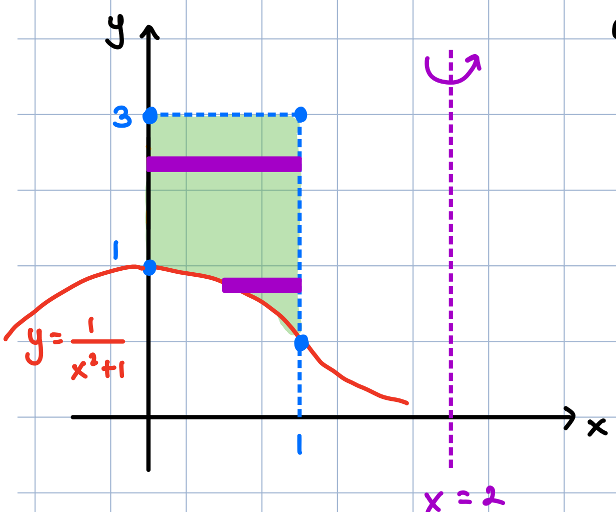
$$\Rightarrow V = \int_0^1 2\pi(2-x)\left(3 - \frac{1}{1+x^2}\right)dx = 2\pi\int_0^1 \left(6 - 3x - \frac{2}{1+x^2} + \frac{x}{1+x^2}\right)dx$$

$$= 2\pi\left[6x - \frac{3x^2}{2} - 2\tan^{-1}(x) + \frac{1}{2}\ln(1+x^2)\right]_0^1$$

$$= \boxed{\pi(9 - \pi + \ln(2)) \text{ cubic units}}.$$

Remark: we can still use washers with horizontal strips, but we need to use two integrals because there are two different curves bounding on the left: $x = 0$ for $1 \leq y \leq 3$

and $y = \frac{1}{x^2+1} \Rightarrow x = \sqrt{\frac{1}{y} - 1}$ for $\frac{1}{2} \leq y \leq 1$.

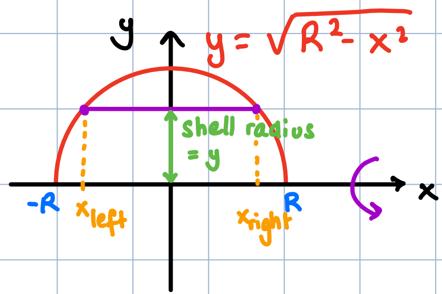


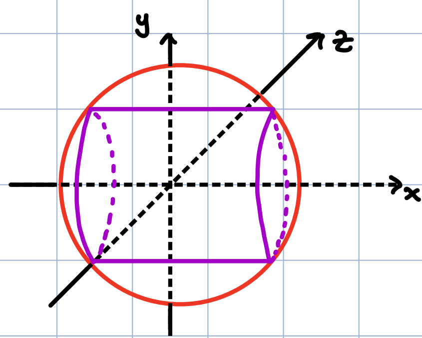
- $r_{in}(y) = 2 - 1 = 1$
- $r_{out}(y) = \begin{cases} 2 - 0 = 2 & \text{if } 1 \leq y \leq 3 \\ 2 - \sqrt{\frac{1}{y} - 1} & \text{if } \frac{1}{2} \leq y \leq 1. \end{cases}$

$$\Rightarrow \boxed{V = \int_{\frac{1}{2}}^1 \pi\left[\left(2 - \sqrt{\frac{1}{y} - 1}\right)^2 - 1^2\right]dy + \int_1^3 \pi(2^2 - 1^2)dy}$$

4) Use the shell method to find the volume of a sphere of radius R.

The sphere is obtained by revolving the semi-circle $y=\sqrt{R^2-x^2}$ about the x-axis.





- Shell height $= x_{right} - x_{left}$

We solve $y=\sqrt{R^2-x^2}$ for x to find x_{left} and x_{right}

$\Rightarrow x = \pm\sqrt{R^2-y^2} \Rightarrow x_{left} = -\sqrt{R^2-y^2}$, $x_{right} = \sqrt{R^2-y^2}$

So shell height $= \sqrt{R^2-y^2} - \left(-\sqrt{R^2-y^2}\right)$

$= 2\sqrt{R^2-y^2}$

- Shell radius $= y$

So $V = \int_0^R 2\pi y \, 2\sqrt{R^2-y^2}\, dy$

u-sub

$u = R^2 - y^2$	$y=0 \Rightarrow u=R^2$
$du = -2y\,dy$	$y=R \Rightarrow u=0$

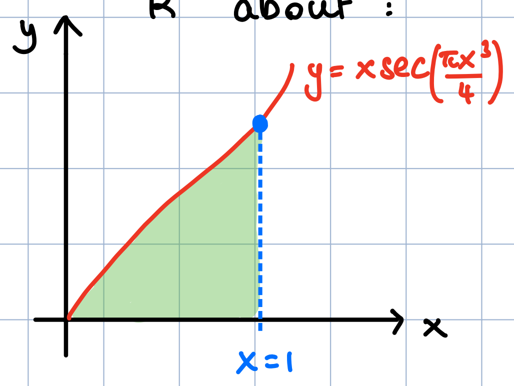
$= \int_{R^2}^{0} 2\pi\sqrt{u}\,(-du)$

$= 2\pi\left[\frac{2}{3}u^{3/2}\right]_0^{R^2}$

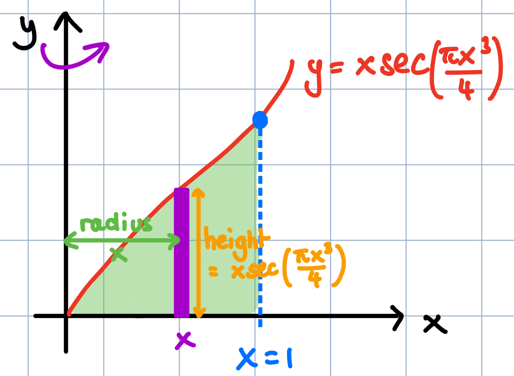
$= \boxed{\frac{4\pi R^3}{3}}$

5) Let R be the region below the graph of
$y = x\sec\left(\frac{\pi x^3}{4}\right)$ on the interval $[0,1]$.
Find the volume of the region obtained by revolving R about:

a) the y-axis
b) the x-axis



a) Revolution about the y-axis: we use shells.



- Shell radius: x
- Shell height: $x\sec\left(\frac{\pi x^3}{4}\right)$.

$$V = \int_0^1 2\pi x \, x\sec\left(\frac{\pi x^3}{4}\right)dx$$

$$= 2\pi \int_0^1 \sec\left(\frac{\pi x^3}{4}\right)x^2 dx$$

$u = \frac{\pi x^3}{4}$, $du = \frac{3\pi x^2}{4}dx$
$\Rightarrow x^2 dx = \frac{4du}{3\pi}$

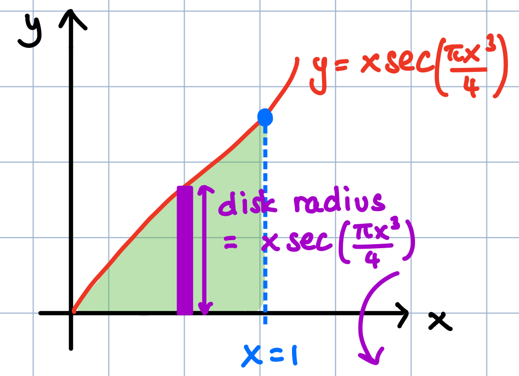
$$= 2\pi \int_0^{\pi/4} \sec(u)\,\frac{4du}{3\pi}$$

$$= \frac{8}{3}\int_0^{\pi/4} \sec(u)\,du$$

$$= \frac{8}{3}\Big[\ln\left(|\sec(u)+\tan(u)|\right)\Big]_0^{\pi/4}$$

$$= \boxed{\frac{8}{3}\ln\left(\sqrt{2}+1\right) \text{ cubic units}}$$

b) Revolution about the x - axis : we use disks



• Disk radius : $x\sec\left(\frac{\pi x^3}{4}\right)$

$$V = \int_0^1 \pi\left(x\sec\left(\frac{\pi x^3}{4}\right)\right)^2 dx$$

$$= \pi \int_0^1 \sec\left(\frac{\pi x^3}{4}\right)^2 x^2 dx$$

$u = \frac{\pi x^3}{4} \Rightarrow du = \frac{3\pi x^2}{4} dx$

$\Rightarrow x^2 dx = \frac{4du}{3\pi}$

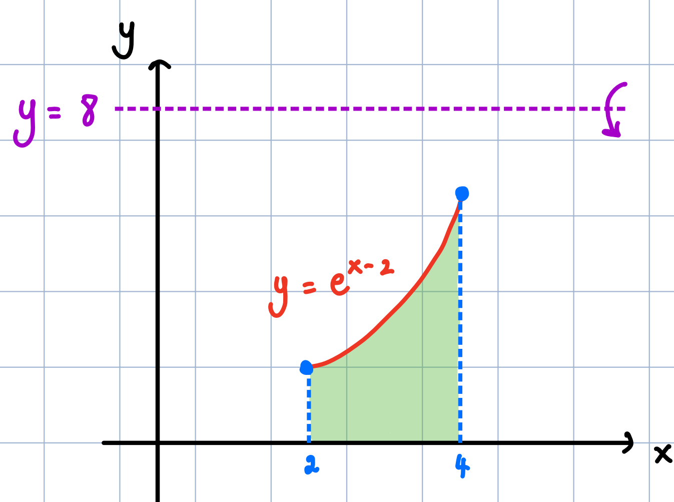
$$= \pi \int_0^{\pi/4} \sec(u)^2 \frac{4du}{3\pi}$$

$$= \frac{4}{3} \int_0^{\pi/4} \sec(u)^2 du$$

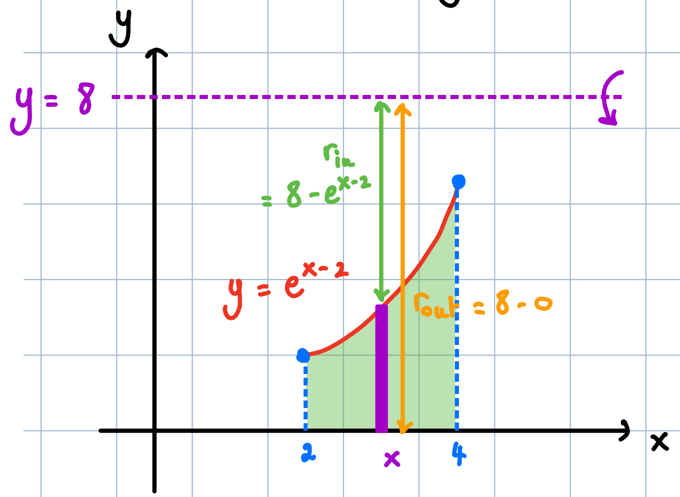
$$= \frac{4}{3}\Big[\tan(u)\Big]_0^{\pi/4}$$

$$= \boxed{\frac{4}{3} \text{ cubic units}}.$$

6) Set-up an integral equal to the volume of the solid obtained by revolving the region below about the the line $y = 8$ using
a) washers
b) shells.



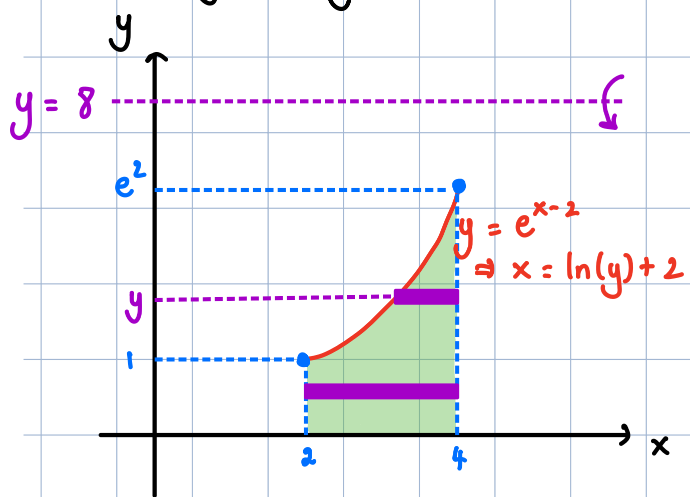
a) Washers : strips perpendicular to the the axis, so we use an x- integral.



- $r_{out} = 8 - 0 = 8$
- $r_{in} = 8 - e^{x-2}$

So $\boxed{V = \int_2^4 \pi \left(8^2 - (8 - e^{x-2})^2\right) dx}$

b) Shells : strips parallel to the the axis, so we use a y- integral.



- Shell radius $= 8 - y$.
- Shell height : we have 2 left bounding curves :
  - if $0 \leq y \leq 1$, height $= 4 - 2 = 2$
  - if $1 \leq y \leq e^2$, height $= 4 - (\ln(y) + 2)$ $= 2 - \ln(y)$

So $\boxed{V = \int_0^1 2\pi (8-y)(2)\,dy + \int_1^{e^2} 2\pi (8-y)(2 - \ln(y))\,dy}$