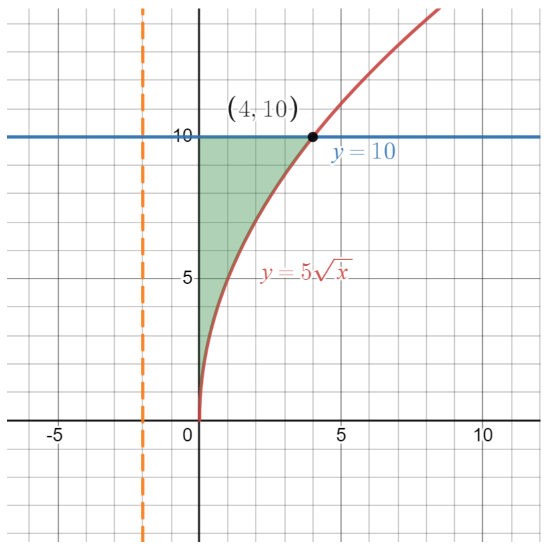
Rutgers University
Math 152

Section 6.2: Volume by Shells - Worksheet

1. Find the volume of the solid of revolution obtained by revolving the given region about the given axis using (i) the method of cylindrical shells and (ii) the method of disks/washers.

   (a) The region bounded by the y-axis, the curve $y = 5\sqrt{x}$ and the line $y = 10$ revolved about the line $x = -2$.

   Solution.



(i) Cylindrical shells are obtained by revolving strips parallel to the axis of revolution. Here the axis is vertical, so we consider the vertical strip at x. The shell radius is the distance between the axis $x = -2$ and the vertical strip at x, so it is $r(x) = x - (-2) = x + 2$. The shell height is the length of strip and is given by $h(x) = y_{\text{top}}(x) - y_{\text{bot}}(x) = 10 - 5\sqrt{x}$. Thus, the volume is

$$\begin{aligned}
V &= \int_0^4 2\pi r(x)h(x)dx \\
&= \int_0^4 2\pi(x+2)(10-5\sqrt{x})dx \\
&= 10\pi \int_0^4 (x+2)(2-\sqrt{x})dx \\
&= 10\pi \int_0^4 \left(2x + 4 - x^{3/2} - 2\sqrt{x}\right) dx \\
&= 10\pi \left[x^2 + 4x - \frac{2}{5}x^{5/2} - \frac{4}{3}x^{3/2}\right]_0^4 \\
&= 10\pi \left(4^2 + 4\cdot 4 - \frac{2}{5}4^{5/2} - \frac{4}{3}4^{3/2}\right)
\end{aligned}$$

$$= \boxed{\frac{256\pi}{3} \text{ cubic units}}.$$

(ii) Washers are obtained by revolving strips perpendicular to the axis of revolution. Here the axis is vertical, so we consider the vertical strip at y. We will need to express the curve as a function of y:

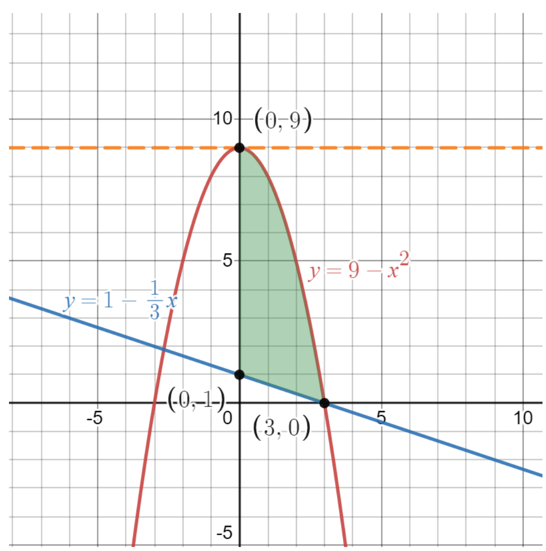
$$y = 5\sqrt{x} \;\Rightarrow\; x = \frac{y^2}{25}$$

The inner radius is $r_{\text{in}}(y) = 0 - (-2) = 2$. The outer radius is $r_{\text{out}} = \frac{y^2}{25} - (-2) = \frac{y^2}{25} + 2$. Therefore, the volume is

$$\begin{aligned} V &= \int_0^{10} \pi \left(r_{\text{out}}(y)^2 - r_{\text{in}}(y)^2\right) dy \\ &= \int_0^{10} \pi \left(\left(\frac{y^2}{25} + 2\right)^2 - 2^2\right) dy \\ &= \pi \int_0^{10} \left(\frac{y^4}{625} + \frac{4y^2}{25}\right) dy \\ &= \pi \left[\frac{y^5}{3125} + \frac{4y^3}{75}\right]_0^{10} \\ &= \pi \left(\frac{10^5}{3125} + \frac{4 \cdot 10^3}{75}\right) \\ &= \boxed{\frac{256\pi}{3} \text{ cubic units}}. \end{aligned}$$

(b) The region in the first quadrant bounded by the curves $y = 9 - x^2$ and $y = 1 - \frac{1}{3}x$ revolved about the line $y = 9$.

Solution.



(i) Cylindrical shells are obtained by revolving strips parallel to the axis of revolution. Here the axis is horizontal, so we consider the horizontal strip at y. The shell radius is the distance between the axis $y = 9$ and the horizontal strip at y, which is below $y = 9$, so it is $r(y) = 9 - y$. to find the shell height, we need to express the curves as functions of y:

$$y = 9 - x^2 \;\Rightarrow\; |x| = \sqrt{9-y} \;\Rightarrow\; x = \sqrt{9-y}, y = 1 - \frac{x}{3} \;\Rightarrow\; x = 3 - 3y.$$

The shell height is given by

$$h(y) = \begin{cases} \sqrt{9-y} & \text{if } 1 \leqslant y \leqslant 9, \\ \sqrt{9-y} - (3-3y) = \sqrt{9-y} - 3 + 3y & \text{if } 0 \leqslant y \leqslant 1. \end{cases}$$

Therefore, the volume is

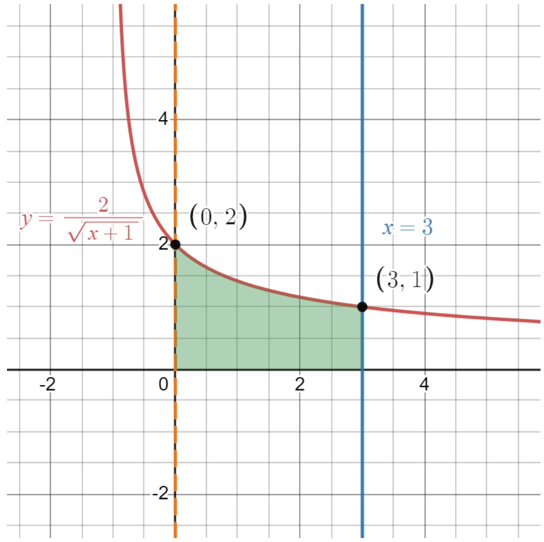
$$\begin{aligned} V &= \int_0^9 2\pi r(y)h(y)dy \\ &= \int_0^1 2\pi(9-y)\left(\sqrt{9-y} - 3 + 3y\right)dy + \int_1^9 2\pi(9-y)\sqrt{9-y}dy \\ &= 2\pi\left(\int_0^1 \left((9-y)^{3/2} + 30y - 3y^2 - 27\right)dy + \int_1^9 (9-y)^{3/2}dy\right) \\ &= 2\pi\left(\int_0^9 (9-y)^{3/2}dy + \int_0^1 \left(30y - 3y^2 - 27\right)dy\right) \\ &= 2\pi\left(\left[-\frac{2}{5}(9-y)^{5/2}\right]_0^9 + \left[15y^2 - y^3 - 27y\right]_0^1\right) \\ &= 2\pi\left(\frac{2}{5}9^{5/2} + 15 - 1 - 27\right) \\ &= \boxed{\frac{842\pi}{5}\text{ cubic units}}. \end{aligned}$$

(ii) Washers are obtained by revolving strips perpendicular to the axis of revolution. Here the axis is horizontal, so we consider the vertical strip at x. The inner radius is $r_{\text{in}}(x) = 9 - (9 - x^2) = x^2$. The outer radius is $r_{\text{out}}(x) = 9 - \left((1 - \frac{x}{3}\right) = 8 + \frac{x}{3}$. Therefore, the volume is

$$\begin{aligned} V &= \int_0^3 \pi\left(r_{\text{out}}(x)^2 - r_{\text{in}}(x)^2\right)dx \\ &= \int_0^3 \pi\left(\left(8 + \frac{x}{3}\right)^2 - (x^2)^2\right)dx \\ &= \pi\int_0^3 \left(64 + \frac{16x}{3} + \frac{x^2}{9} - x^4\right)dx \\ &= \pi\left[64x + \frac{8x^2}{3} + \frac{x^3}{27} - \frac{x^5}{5}\right]_0^3 \\ &= \pi\left(64\cdot 3 + \frac{8\cdot 3^2}{3} + \frac{3^3}{27} - \frac{3^5}{5}\right) \\ &= \boxed{\frac{842\pi}{5}\text{ cubic units}}. \end{aligned}$$

(c) The region below the graph of $y = \dfrac{2}{\sqrt{x+1}}$ for $0 \leqslant x \leqslant 3$ revolved about the y-axis.

Solution.



(i) Cylindrical shells are obtained by revolving strips parallel to the axis of revolution. Here the axis is vertical, so we consider the vertical strip at x. The shell radius is the distance between the axis $x = 0$ and the vertical strip at x, which is to the right of $x = 0$, so it is $r(x) = x$. The shell height is the length of the strip and is given by $h(x) = \frac{2}{\sqrt{x+1}}$. Therefore, the volume is

$$\begin{aligned} V &= \int_0^3 2\pi r(x)h(x)dx \\ &= \int_0^3 2\pi x \frac{2}{\sqrt{x+1}} dx \\ &= 4\pi \int_0^3 \frac{x}{\sqrt{x+1}} dx. \end{aligned}$$

We can compute this integral using the substitution $u = x + 1$, which gives $du = dx$. The bounds change as follows

$$\begin{aligned} x = 0 &\Rightarrow u = 0 + 1 = 1, \\ x = 3 &\Rightarrow u = 3 + 1 = 4. \end{aligned}$$

The extraneous factor x in the numerator can be expressed in terms of u as $x = u - 1$. We get

$$\begin{aligned} V &= 4\pi \int_1^4 \frac{u-1}{\sqrt{u}} du \\ &= 4\pi \int_1^4 \left(\sqrt{u} - \frac{1}{\sqrt{u}} \right) du \\ &= 4\pi \left[\frac{2}{3} u^{3/2} - 2\sqrt{u} \right]_1^4 \\ &= 4\pi \left(\frac{2}{3} 4^{3/2} - 2\sqrt{4} - \frac{2}{3} + 2 \right) \end{aligned}$$

$$= \boxed{\frac{32\pi}{3} \text{ cubic units}}.$$

(ii) Washers are obtained by revolving strips perpendicular to the axis of revolution. Here the axis is vertical, so we consider the horizontal strip at y. The axis of revolution is the left boundary of the region, so the washers are actually disks. To find the radius of the disk, we need to express the graph as a function of y:

$$y = \frac{2}{\sqrt{x+1}} \Rightarrow \sqrt{x+1} = \frac{2}{y} \Rightarrow x = \frac{4}{y^2} - 1.$$

The radius of the disks is

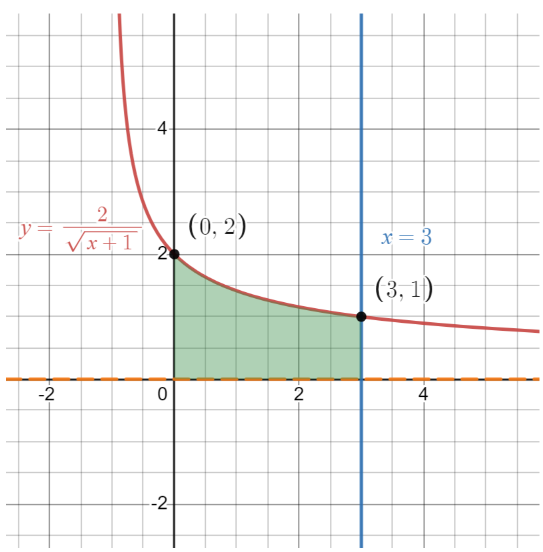
$$r(y) = \begin{cases} 3 & \text{if } 0 \leqslant y \leqslant 1, \\ \frac{4}{y^2} - 1 & \text{if } 1 \leqslant y \leqslant 2. \end{cases}$$

Therefore, the volume is

$$\begin{aligned} V &= \int_0^2 \pi r(y)^2 dy \\ &= \int_0^1 \pi \cdot 3^2 dy + \int_1^2 \pi \left(\frac{4}{y^2} - 1\right)^2 dy \\ &= \pi \left(9 + \int_1^2 \left(\frac{16}{y^4} - \frac{8}{y^2} + 1\right) dy\right) \\ &= \pi \left(9 + \left[-\frac{16}{3y^3} + \frac{8}{y} + y\right]_1^2\right) \\ &= \pi \left(9 - \frac{16}{3 \cdot 2^3} + \frac{8}{2} + 2 + \frac{16}{3} - 8 - 1\right) \\ &= \boxed{\frac{32\pi}{3} \text{ cubic units}}. \end{aligned}$$

(d) The region below the graph of $y = \dfrac{2}{\sqrt{x+1}}$ for $0 \leqslant x \leqslant 3$ revolved about the x-axis.

Solution.



(i) Cylindrical shells are obtained by revolving strips parallel to the axis of revolution. Here the axis is horizontal, so we consider the horizontal strip at y. The shell radius is the distance between the axis $y = 0$ and the horizontal strip at y, which is above $y = 0$, so it is $r(y) = y$. To find the shell height, we need to express the graph as a function of y:

$$y = \frac{2}{\sqrt{x+1}} \quad \Rightarrow \quad \sqrt{x+1} = \frac{2}{y} \quad \Rightarrow \quad x = \frac{4}{y^2} - 1.$$

The shell height is given by

$$h(y) = \begin{cases} \frac{y^2}{4} - 1 & \text{if } 1 \leqslant y \leqslant 2, \\ 3 & \text{if } 0 \leqslant y \leqslant 1. \end{cases}$$

Therefore, the volume is

$$\begin{aligned} V &= \int_0^2 2\pi r(y)h(y)dy \\ &= \int_0^1 2\pi 3y dy + \int_1^2 2\pi y \left(\frac{4}{y^2} - 1\right) dy \\ &= 2\pi \left(\int_0^1 3y dy + \int_1^2 \left(\frac{4}{y} - y\right) dy \right) \\ &= 2\pi \left(\left[\frac{3y^2}{2}\right]_0^1 + \left[4\ln|y| - \frac{y^2}{2}\right]_1^2 \right) \\ &= 2\pi \left(\frac{3}{2} + 4\ln(2) - \frac{2^2}{2} + \frac{1}{2} \right) \\ &= \boxed{8\pi \ln(2) \text{ cubic units}}. \end{aligned}$$

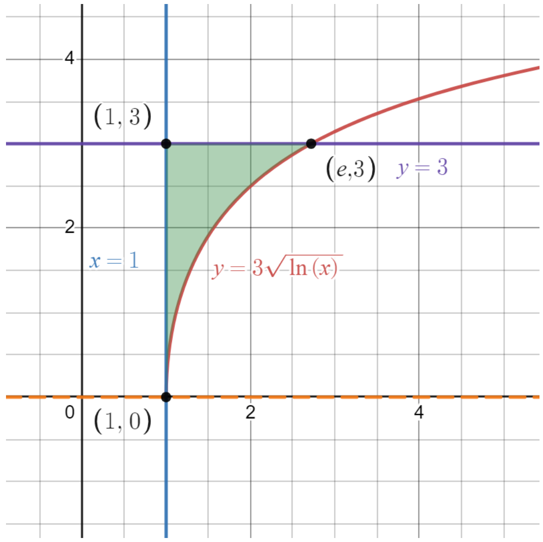
(ii) Washers are obtained by revolving strips perpendicular to the axis of revolution. Here the axis is horizontal, so we consider the vertical strip at x. The axis of revolution is the bottom boundary of the region, so the washers are actually disks of radius $r(x) = \frac{2}{\sqrt{x+1}}$. Therefore, the volume is

$$\begin{aligned} V &= \int_0^3 \pi r(x)^2 dx \\ &= \int_0^3 \pi \left(\frac{2}{\sqrt{x+1}}\right) dx \\ &= 4\pi \int_0^3 \frac{dx}{x+1} \\ &= 4\pi \left[\ln(x+1)\right]_0^3 \\ &= 4\pi \ln(4) \\ &= \boxed{8\pi \ln(2) \text{ cubic units}}. \end{aligned}$$

2. Find the volume of the solid of revolution obtained by revolving the given region about the given axis using the method of cylindrical shells.

   (a) The region bounded by the curve $y = 3\sqrt{\ln(x)}$, the line $y = 3$ and the line $x = 1$ revolved about the x-axis.

   Solution.



(i) Cylindrical shells are obtained by revolving strips parallel to the axis of revolution. Here the axis is horizontal, so we consider the horizontal strip at y. The shell radius is the distance between the axis $y = 0$ and the horizontal strip at y, which is above $y = 0$, so it is $r(y) = y$. To find the shell height, we will need to express the curve as a function of y:

$$y = 3\sqrt{\ln(x)} \;\Rightarrow\; \ln(x) = \frac{y^2}{9} \;\Rightarrow\; x = e^{y^2/9}.$$

The shell height is the length of the strip, which is $h(y) = e^{y^2/9} - 1$. So the volume is

$$\begin{aligned} V &= \int_0^4 2\pi r(y)h(y)dy \\ &= \int_0^4 2\pi y \left(e^{y^2/9} - 1\right) dy. \end{aligned}$$

We can compute this integral using the substitution $u = \frac{y^2}{9}$, which gives $du = \frac{2ydy}{9}$. The bounds become

$$\begin{aligned} y = 0 \;&\Rightarrow\; u = \frac{0^2}{9} = 0, \\ y = 4 \;&\Rightarrow\; u = \frac{4^2}{9} = \frac{16}{9}. \end{aligned}$$

We obtain

$$\begin{aligned} V &= 9\pi \int_0^{16/9} (e^u - 1)\, du \\ &= 9\pi \left[e^u - u\right]_0^{16/9} \\ &= 9\pi \left(e^{16/9} - \frac{16}{9} - 1\right) \\ &= \boxed{\pi\left(9e^{16/9} - 25\right) \text{ cubic units}}. \end{aligned}$$

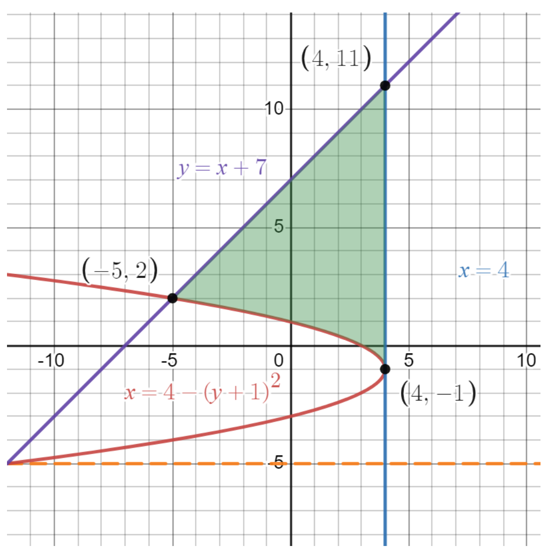
(b) The region below the graph of $y = \dfrac{1}{16+x^4}$ for $0 \leqslant x \leqslant 2$ revolved about the y-axis.

Solution. The cylindrical shell is obtained by revolving the vertical strip at x. It has radius $r(x) = x$ and height $h(x) = \frac{1}{16+x^4}$. Therefore the volume is

$$\begin{aligned} V &= \int_0^2 2\pi r(x)h(x)dx \\ &= \int_0^2 2\pi x \frac{1}{16+x^4} dx \\ &= \int_0^4 \pi \frac{du}{16+u^2} \qquad (u = x^2) \\ &= \pi \left[\frac{1}{4} \tan^{-1}\left(\frac{x}{4}\right) \right]_0^4 \\ &= \frac{\pi}{4} \tan^{-1}(1) \\ &= \boxed{\frac{\pi^2}{16} \text{ cubic units}}. \end{aligned}$$

(c) The region bounded by the curve $x = 4 - (y+1)^2$, the line $x = 4$ and the line $y = x + 7$ revolved about the line $y = -5$.

Solution.



(i) Cylindrical shells are obtained by revolving strips parallel to the axis of revolution. Here the axis is horizontal, so we consider the horizontal strip at y. The shell radius is the distance between the axis $y = -5$ and the horizontal strip at y, which is above $y = -5$, so it is $r(y) = y - (-5) = y + 5$. The shell height is given by

$$h(y) = \begin{cases} 1 - (4 - (y+1)^2) = y^2 + 2y - 2 & \text{if } -1 \leqslant y \leqslant 2, \\ 1 - (y - 7) = 8 - y & \text{if } 2 \leqslant y \leqslant 11. \end{cases}$$

So the volume is

$$\begin{aligned}
V &= \int_{-1}^{11} 2\pi r(y)h(y)dy \\
&= \int_{-1}^{2} 2\pi(y+5)(y^2+2y-2)dy + \int_{2}^{11} 2\pi(y+5)(8-y)dy \\
&= 2\pi\left(\int_{-1}^{2}\left(y^3+7y^2+8y-10\right)dy + \int_{2}^{11}\left(40+3y-y^2\right)dy\right) \\
&= 2\pi\left(\left[\frac{y^4}{4}+\frac{7y^3}{3}+4y^2-10y\right]_{-1}^{2} + \left[40y+\frac{3y^2}{2}-\frac{y^3}{2}\right]_{2}^{11}\right) \\
&= \boxed{\frac{405\pi}{2} \text{ cubic units.}}
\end{aligned}$$