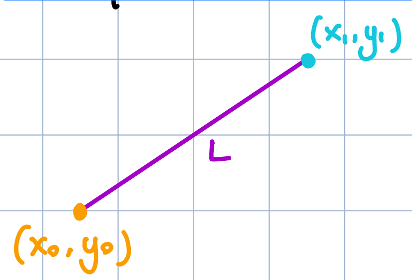
Section 6.3

Arc Length

Learning Goals

	Learning Goal	*Homework Problems*
	6.3.1 Set and evaluate an x or a y integral for the length of a given curve	6.3: 1,2,3,5,7,9,15,29,27,28
	6.3.2 Find a curve with a given length integral or value	6.3: 25

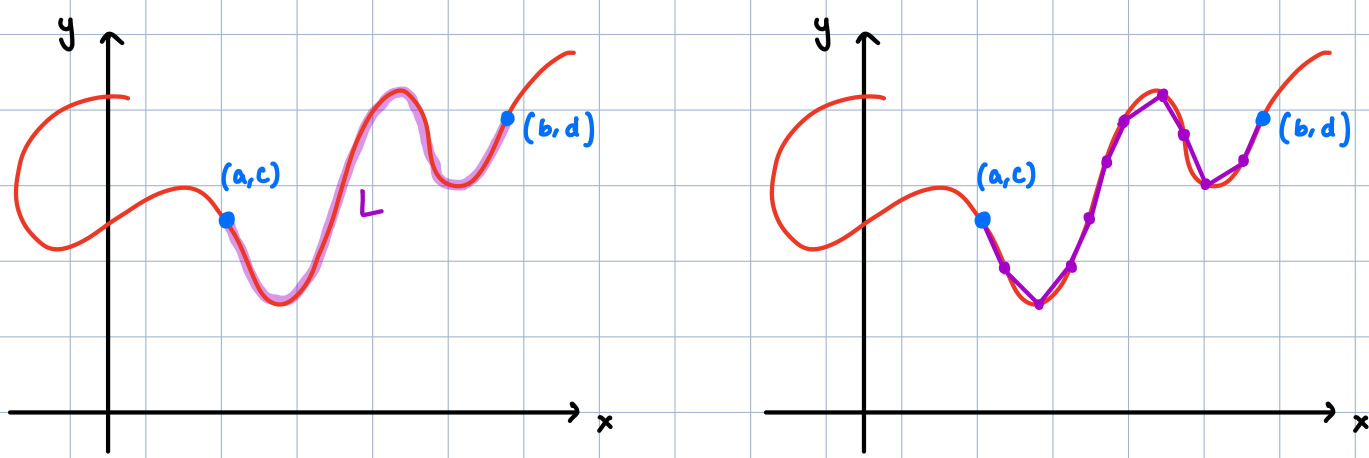
Conceptual introduction : We know how to compute the length of a line segment using the distance formula.



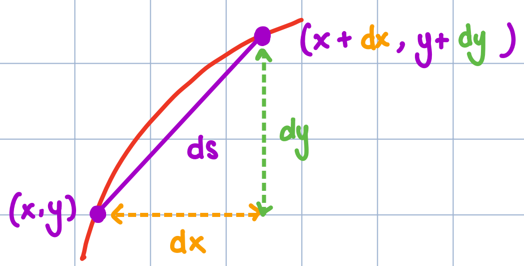
$$L = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

In this section, we want to compute the length of an arc on a smooth curve.

Idea : break-up the curve into tiny line segments and sum their lengths.



A typical tiny line segment looks like

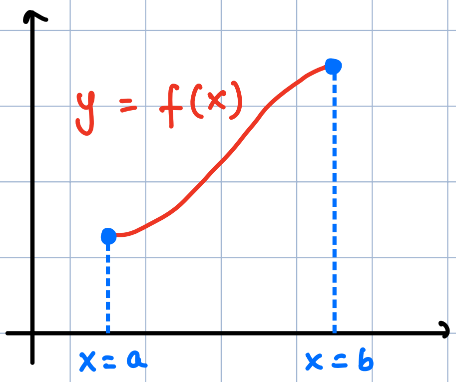


$$ds = \sqrt{dx^2 + dy^2}$$

$$ds = \begin{cases} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx & \text{(factor out } dx\text{; } y \text{ as function of } x\text{)} \\ \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy & \text{(factor out } dy\text{; } x \text{ as function of } y\text{)} \end{cases}$$

And $L = \int_{curve} ds$.

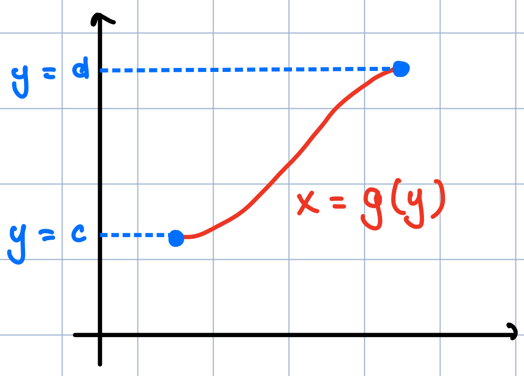
- Using an x-integral:



$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1+f'(x)^2}\, dx$$

valid if $\frac{dy}{dx}$ is continuous on $a \leq x \leq b$.

- Using a y-integral:



$$L = \int_c^d \sqrt{\left(\frac{dx}{dy}\right)^2+1}\, dy = \int_c^d \sqrt{g'(y)^2+1}\, dy$$

valid if $\frac{dx}{dy}$ is continuous on $c \leq y \leq d$

Examples: 1) Find the length of the curve $y = 2x^{3/2}$ for $0 \leq x \leq 1$.

We have $\frac{dy}{dx} = 3x^{1/2}$, continuous on $[0,1]$.
So we can compute L using integration with respect to x.

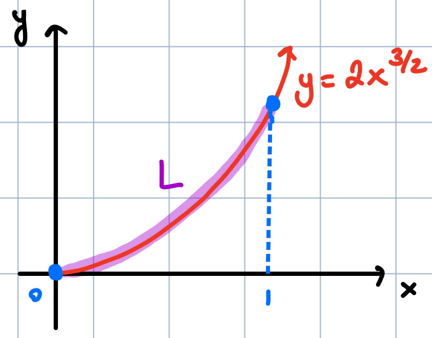
$$L = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1+9x}\, dx$$

$$= \int_1^{10} \sqrt{u}\, \frac{du}{9} = \frac{1}{9}\left[\frac{2}{3}u^{3/2}\right]_1^{10}$$

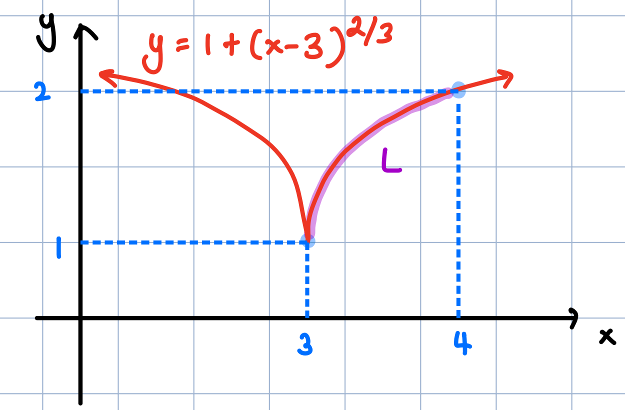
u-sub

$u = 1+9x$	$x=0 \Rightarrow u=1$
$du = 9dx$	$x=1 \Rightarrow u=10$
$\Rightarrow dx = \frac{1}{9}du$	

$$= \boxed{\frac{2}{27}\left(10^{3/2} - 1\right) \text{ units}}$$



2) Find the length of $y = 1 + (x-3)^{2/3}$ for $3 \leq x \leq 4$



$\frac{dy}{dx} = \frac{2(x-3)^{-1/3}}{3}$ is not defined at $x = 3$ (cusp).

So we cannot compute L using an x-integral. We use a y-integral instead.

$$y = 1 + (x-3)^{2/3} \Rightarrow (x-3)^{2/3} = y - 1$$

$$\Rightarrow x - 3 = (y-1)^{3/2} \quad \text{(because } x - 3 \geq 0 \text{ if } 3 \leq x \leq 4\text{)}$$

$$\Rightarrow x = 3 + (y-1)^{3/2}.$$

So $\frac{dx}{dy} = \frac{3}{2}(y-1)^{1/2}$ is continuous on $1 \leq y \leq 2$.

$$\text{So } L = \int_1^2 \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy = \int_1^2 \sqrt{\frac{9}{4}(y-1) + 1}\, dy = \int_1^{\frac{13}{4}} \frac{4}{9}\sqrt{u}\, du$$

u-sub $u = \frac{9}{4}(y-1) + 1$

$$= \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_1^{\frac{13}{4}} = \boxed{\frac{8}{27}\left[\left(\frac{13}{4}\right)^{3/2} - 1\right] \text{ units}}.$$

3) Find the length of the curve $y = \frac{x^2}{2} - \frac{\ln(x)}{4}$ for $1 \leq x \leq 3$.

We have $\frac{dy}{dx} = x - \frac{1}{4x}$ continuous on $[1,3]$, so we can compute L using integration with respect to x.

<u>Trick</u>: try to write $1 + \left(\frac{dy}{dx}\right)^2$ as a perfect square.

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(x - \frac{1}{4x}\right)^2 = 1 + x^2 + \left(\frac{1}{4x}\right)^2 - 2x\frac{1}{4x} = x^2 + \left(\frac{1}{4x}\right)^2 + \frac{1}{2}$$

$$= x^2 + \left(\frac{1}{4x}\right)^2 + 2x\frac{1}{4x} = \left(x + \frac{1}{4x}\right)^2.$$

So $L = \int_1^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_1^3 \sqrt{\left(x + \frac{1}{4x}\right)^2}\,dx = \int_1^3 \left(x + \frac{1}{4x}\right)dx$

$$= \left[\frac{x^2}{2} + \frac{\ln(|x|)}{4}\right]_1^3 = \boxed{4 + \frac{\ln(3)}{4} \text{ units}}$$

4) Find the length of the curve $x = \frac{e^{3y} + e^{-3y}}{6}$ for $0 \leq y \leq 2$.

We have $\frac{dx}{dy} = \frac{3e^{3y} - 3e^{-3y}}{6} = \frac{e^{3y} - e^{-3y}}{2}$ continous for $0 \leq y \leq 2$.
so we can use a y- integral.

Before we compute L, we try to write $1 + \left(\frac{dx}{dy}\right)^2$ as a perfect square.

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(\frac{e^{3y} - e^{-3y}}{2}\right)^2 = 1 + \frac{e^{6y} + e^{-6y} - 2}{4} = \frac{e^{6y} + e^{-6y} + 2}{4}$$

$$= \left(\frac{e^{3y} + e^{-3y}}{2}\right)^2$$

So $L = \int_0^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_0^2 \sqrt{\left(\frac{e^{3y} + e^{-3y}}{2}\right)^2} = \int_0^2 \frac{e^{3y} + e^{-3y}}{2}\,dy$

$$= \left[\frac{e^{3y} - e^{-3y}}{6} \right]_0^2 = \boxed{\frac{e^6 - e^{-6}}{6} \quad \text{units}}$$

5) Find the length of the curve $y = \ln(\sec(x))$, $0 \leq x \leq \frac{\pi}{3}$.

We have $\frac{dy}{dx} = \tan(x)$, continuous on $\left[0, \frac{\pi}{3}\right]$, so we can use an x- integral.

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan(x)^2 = \sec(x)^2$$

So $$L = \int_0^{\frac{\pi}{3}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{\frac{\pi}{3}} \sqrt{\sec(x)^2}\, dx = \int_0^{\frac{\pi}{3}} |\sec(x)|\, dx = \int_0^{\pi/3} \sec(x)\, dx$$

$$= \Big[\ln(\sec(x) + \tan(x))\Big]_0^{\frac{\pi}{3}} = \boxed{\ln\left(2 + \sqrt{3}\right) \quad \text{units}}$$

Using the formula in reverse:

a) Find a curve $y = f(x)$ passing through $(1, 2)$ with positive derivative whose length integral on $1 \leq x \leq 4$ is

$$L = \int_1^4 \sqrt{1 + \frac{1}{4x}}\, dx$$

$$L = \int_1^4 \sqrt{1 + \frac{1}{4x}}\, dx = \int_1^4 \sqrt{1 + f'(x)^2}\, dx$$

So $f'(x)^2 = \frac{1}{4x}$

take $\sqrt{\cdot}$ on both sides

⚠ $\sqrt{a^2} = |a|$

$$|f'(x)| = \frac{1}{2\sqrt{x}}$$

"positive derivative" so $f'(x) > 0$ and $|f'(x)| = f'(x)$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$f(x) = \sqrt{x} + C, \quad C \text{ constant.}$$

To find C, we use

$$f(1) = 2$$
$$\sqrt{1} + C = 2$$
$$C = 1$$

So $\boxed{f(x) = \sqrt{x} + 1}$

b) Find a curve $y = f(x)$ passing through $(1, -2)$ with negative derivative whose length integral on $0 \leq x \leq 2$ is

$$L = \int_0^2 \sqrt{1 + 9x^8} \, dx$$

$$L = \int_0^2 \sqrt{1 + 9x^8} \, dx$$
$$= \int_0^2 \sqrt{1 + f'(x)^2} \, dx$$

So $f'(x)^2 = 9x^8$

take $\sqrt{\cdot}$ on both sides

⚠ $\sqrt{a^2} = |a|$

$$|f'(x)| = 3x^4$$

"negative derivative"

so $f'(x) < 0$ and $|f'(x)| = -f'(x)$

$$f'(x) = -3x^4$$

$$f(x) = -\frac{3x^5}{5} \quad C, \quad C \text{ constant.}$$

To find C, we use

$$f(1) = -2$$
$$-\frac{3}{5} + C = -2$$
$$C = -\frac{7}{5}$$

So $\boxed{f(x) = -\frac{3x^5}{5} - \frac{7}{5}}$