Rutgers University
Math 152

Section 6.3: Arc Length - Worksheet Solutions

1. Calculate the arc length of the given curves.

   (a) $y = 11 - 2(x-5)^{3/2}$, $5 \leqslant x \leqslant 6$.

   Solution. We have
$$\frac{dy}{dx} = -2 \cdot \frac{3}{2}(x-5)^{1/2} = -3\sqrt{x-5}.$$
   So
$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(-3\sqrt{x-5}\right)^2 = 1 + 9(x-5) = 9x - 44.$$
   Therefore the arc length is given by
$$\begin{aligned} L &= \int_5^6 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_5^6 \sqrt{9x-44}\, dx. \end{aligned}$$
   We use the substitution $u = 9x - 44$, so that $du = 9dx$. The bounds become
$$\begin{aligned} x = 5 &\Rightarrow u = 1, \\ x = 6 &\Rightarrow u = 10. \end{aligned}$$
   So the integral becomes
$$\begin{aligned} L &= \int_1^{10} \frac{1}{9}\sqrt{u}\, du \\ &= \frac{1}{9}\left[\frac{2}{3}u^{3/2}\right]_1^{10} \\ &= \boxed{\frac{2}{27}\left(10^{3/2} - 1\right) \text{ units}}. \end{aligned}$$

   (b) $x = \frac{1}{4}\sqrt[3]{y} - \frac{9}{5}\sqrt[3]{y^5}$, $1 \leqslant y \leqslant 2$.

   Solution. We have
$$\frac{dx}{dy} = \frac{1}{4} \cdot \frac{1}{3}y^{-2/3} - \frac{9}{5} \cdot \frac{5}{3}y^{2/3} = \frac{1}{12}y^{-2/3} - 3y^{2/3}.$$
   So
$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(\frac{1}{12}y^{-2/3} - 3y^{2/3}\right)^2$$

$$\begin{aligned}
&= 1 + \frac{1}{144} y^{-4/3} + 9y^{4/3} - 2 \cdot \frac{1}{12} y^{-2/3} \cdot 3y^{2/3} \\
&= 1 + \frac{1}{144} y^{-4/3} + 9y^{4/3} - \frac{1}{2} \\
&= \frac{1}{144} y^{-4/3} + 9y^{4/3} + \frac{1}{2} \\
&= \left(\frac{1}{12} y^{-2/3}\right)^2 + \left(3y^{2/3}\right)^2 + 2 \cdot \frac{1}{12} y^{-2/3} \cdot 3y^{2/3} \\
&= \left(\frac{1}{12} y^{-2/3} + 3y^{2/3}\right)^2.
\end{aligned}$$

Therefore the arc length is given by

$$\begin{aligned}
L &= \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \\
&= \int_1^2 \sqrt{\left(\frac{1}{12} y^{-2/3} + 3y^{2/3}\right)^2}\, dy \\
&= \int_1^2 \left|\frac{1}{12} y^{-2/3} + 3y^{2/3}\right| dy \\
&= \int_1^2 \left(\frac{1}{12} y^{-2/3} + 3y^{2/3}\right) dy \\
&= \left[\frac{1}{4} y^{1/3} + \frac{9}{5} y^{5/3}\right]_1^2 \\
&= \left(\frac{1}{4} 2^{1/3} + \frac{9}{5} 2^{5/3}\right) - \left(\frac{1}{4} + \frac{9}{5}\right) \\
&= \boxed{\frac{5\sqrt[3]{2} + 72\sqrt[3]{4} - 41}{20} \text{ units}}.
\end{aligned}$$

(c) $x = \sqrt{16y - y^2}$, $4 \leqslant y \leqslant 12$.

Solution. We have

$$\frac{dx}{dy} = \frac{16 - 2y}{2\sqrt{16y - y^2}} = \frac{8 - y}{\sqrt{16y - y^2}}.$$

Therefore

$$\begin{aligned}
1 + \left(\frac{dx}{dy}\right)^2 &= 1 + \left(\frac{8 - y}{\sqrt{16y - y^2}}\right)^2 \\
&= 1 + \frac{(8 - y)^2}{16y - y^2} \\
&= \frac{16y - y^2 + (64 - 16y + y^2)}{16y - y^2} \\
&= \frac{64}{16y - y^2} \\
&= \frac{64}{64 - (y - 8)^2}
\end{aligned}$$

where we have completed the square in the denominator for the last step. So the arc length integral becomes

$$
\begin{aligned}
L &= \int_4^{12} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \\
&= \int_4^{12} \sqrt{\frac{64}{64 - (y-8)^2}}\, dy \\
&= 8 \int_4^{12} \frac{dy}{\sqrt{64 - (y-8)^2}} \\
&= 8 \int_4^{12} \frac{dy}{8\sqrt{1 - \left(\frac{y-8}{8}\right)^2}} \\
&= 8 \int_{-1/2}^{1/2} \frac{du}{\sqrt{1-u^2}} \qquad \left(u = \frac{y-8}{8}\right) \\
&= 8 \left[\sin^{-1}(u)\right]_{-1/2}^{1/2} \\
&= 8 \left(\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)\right) \\
&= 8 \left(\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right) \\
&= \boxed{\frac{8\pi}{3} \text{ units}}.
\end{aligned}
$$

(d) $y = \dfrac{1}{6} \ln(\sin(3x)\cos(3x)),\ \dfrac{\pi}{18} \leqslant x \leqslant \dfrac{\pi}{9}$.

Solution. We have

$$
\frac{dy}{dx} = \frac{1}{6} \cdot \frac{3\cos(3x)^2 - 3\sin(3x)^2}{\sin(3x)\cos(3x)} = \frac{3\cos(3x)^2}{6\sin(3x)\cos(3x)} - \frac{3\sin(3x)^2}{6\sin(3x)\cos(3x)} = \frac{1}{2}\cot(3x) - \frac{1}{2}\tan(3x).
$$

So

$$
\begin{aligned}
1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \left(\frac{1}{2}\cot(3x) - \frac{1}{2}\tan(3x)\right)^2 \\
&= 1 + \frac{1}{4}\cot(3x)^2 + \frac{1}{4}\tan(3x)^2 - 2 \cdot \frac{1}{2}\cot(3x) \cdot \frac{1}{2}\tan(3x) \\
&= 1 + \frac{1}{4}\cot(3x)^2 + \frac{1}{4}\tan(3x)^2 - \frac{1}{2} \\
&= \frac{1}{4}\cot(3x)^2 + \frac{1}{4}\tan(3x)^2 + \frac{1}{2} \\
&= \frac{1}{4}\cot(3x)^2 + \frac{1}{4}\tan(3x)^2 + 2 \cdot \frac{1}{2}\cot(3x) \cdot \frac{1}{2}\tan(3x) \\
&= \left(\frac{1}{2}\cot(3x) + \frac{1}{2}\tan(3x)\right)^2.
\end{aligned}
$$

Therefore the arc length is given by

$$
L = \int_{\pi/18}^{\pi/9} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx
$$

$$
\begin{aligned}
&= \int_{\pi/18}^{\pi/9} \sqrt{\left(\frac{1}{2}\cot(3x) + \frac{1}{2}\tan(3x)\right)^2}\, dx \\
&= \int_{\pi/18}^{\pi/9} \left|\frac{1}{2}\cot(3x) + \frac{1}{2}\tan(3x)\right| dx \\
&= \frac{1}{2}\int_{\pi/18}^{\pi/9} (\cot(3x) + \tan(3x))\, dx \\
&= \frac{1}{6}\left[\ln|\sin(3x)| + \ln|\sec(3x)|\right]_{\pi/18}^{\pi/9} \\
&= \frac{1}{6}\left(\ln\left|\sin\left(\frac{\pi}{3}\right)\right| + \ln\left|\sec\left(\frac{\pi}{3}\right)\right| - \ln\left|\sin\left(\frac{\pi}{6}\right)\right| - \ln\left|\sec\left(\frac{\pi}{6}\right)\right|\right) \\
&= \frac{1}{6}\left(\ln\left(\frac{\sqrt{3}}{2}\right) + \ln(2) - \ln\left(\frac{1}{2}\right) - \ln\left(\frac{2}{\sqrt{3}}\right)\right) \\
&= \boxed{\frac{1}{6}\ln(3) \text{ units}}.
\end{aligned}
$$

(e) $y = \dfrac{e^{5x} + e^{-5x}}{10}$, $0 \leqslant x \leqslant \dfrac{1}{5}$.

Solution. We have

$$
\frac{dy}{dx} = \frac{5e^{5x} - 5e^{5x}}{10} = \frac{e^{5x} - e^{5x}}{2}.
$$

So

$$
\begin{aligned}
1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \left(\frac{e^{5x} - e^{-5x}}{2}\right)^2 \\
&= 1 + \frac{1}{4}e^{10x} + \frac{1}{4}e^{-10x} - 2\cdot\frac{1}{2}e^{5x}\cdot\frac{1}{2}e^{-5x} \\
&= 1 + \frac{1}{4}e^{10x} + \frac{1}{4}e^{-10x} - \frac{1}{2} \\
&= \frac{1}{4}e^{10x} + \frac{1}{4}e^{-10x} + \frac{1}{2} \\
&= \left(\frac{1}{2}e^{5x}\right)^2 + \left(\frac{1}{2}e^{-5x}\right)^2 + 2\cdot\frac{1}{2}e^{5x}\cdot\frac{1}{2}e^{-5x} \\
&= \left(\frac{e^{5x} + e^{-5x}}{2}\right)^2.
\end{aligned}
$$

Therefore the arc length is given by

$$
\begin{aligned}
L &= \int_0^{1/5} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int_0^{1/5} \sqrt{\left(\frac{e^{5x} + e^{-5x}}{2}\right)^2}\, dx \\
&= \int_0^{1/5} \left|\frac{e^{5x} + e^{-5x}}{2}\right| dx \\
&= \int_0^{1/5} \left(\frac{e^{5x} + e^{-5x}}{2}\right) dx
\end{aligned}
$$

$$\begin{aligned}
&= \frac{1}{10}\left[e^{5x} - e^{-5x}\right]_0^{1/5} \\
&= \frac{1}{10}\left(e - e^{-1} - 1 + 1\right) \\
&= \boxed{\frac{e - e^{-1}}{10} \text{ units}}.
\end{aligned}$$

(f) $x = \frac{4}{5}y^{5/4}$, $0 \leqslant y \leqslant 9$.

Solution. We have $\frac{dx}{dy} = y^{1/4}$, so

$$\begin{aligned}
L &= \int_0^9 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \\
&= \int_0^9 \sqrt{1 + y^{1/2}} dy.
\end{aligned}$$

In this last integral, we substitute $u = 1 + y^{1/2}$, so $du = \frac{dy}{2y^{1/2}}$. Observing that $y^{1/2} = u - 1$, we have

$$dy = 2y^{1/2} du = 2(u-1)du.$$

The bounds become

$$\begin{aligned}
y = 0 &\Rightarrow u = 1 + 0^{1/2} = 1, \\
y = 9 &\Rightarrow u = 1 + 9^{1/2} = 4.
\end{aligned}$$

We obtain

$$\begin{aligned}
L &= \int_1^4 \sqrt{u}2(u-1)du \\
&= 2\int_1^4 \left(u^{3/2} - u^{1/2}\right) du \\
&= 2\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^4 \\
&= 4\left(\frac{4^{5/2}}{5} - \frac{4^{3/2}}{3} - \frac{1}{5} + \frac{1}{3}\right) \\
&= \boxed{\frac{232}{15}}.
\end{aligned}$$

2. Find a curve of the form $y = f(x)$ passing through $(4, 13)$, having negative derivative, and whose length integral on $1 \leqslant x \leqslant 7$ is given by

$$L = \int_1^7 \sqrt{1 + \frac{25}{x^3}} dx.$$

Solution. The arc length being given by

$$L = \int_1^7 \sqrt{1 + f'(x)^2} dx,$$

we can deduce that $f'(x)^2 = \frac{25}{x^3}$. Taking square roots on both sides gives

$$\begin{aligned} \sqrt{f'(x)^2} &= \sqrt{\frac{25}{x^3}} \\ \Rightarrow \quad |f'(x)| &= \frac{5}{x^{3/2}}. \end{aligned}$$

Since f has negative derivative, $|f'(x)| = -f'(x)$ and we deduce

$$f'(x) = -\frac{5}{x^{3/2}} = -5x^{-3/2}.$$

Taking an antiderivative gives

$$f(x) = \int -5x^{-3/2} dx = -5(-2)x^{-1/2} + C = \frac{10}{\sqrt{x}} + C.$$

To find the constant C, we can use the fact that the curve passes through $(4, 13)$, which gives

$$\frac{10}{\sqrt{4}} + C = 13 \;\Rightarrow\; C = 8.$$

Therefore, the solution to the problem is the curve

$$\boxed{y = \frac{10}{\sqrt{x}} + 8}.$$