Section 6.4

Areas of Surfaces of Revolution

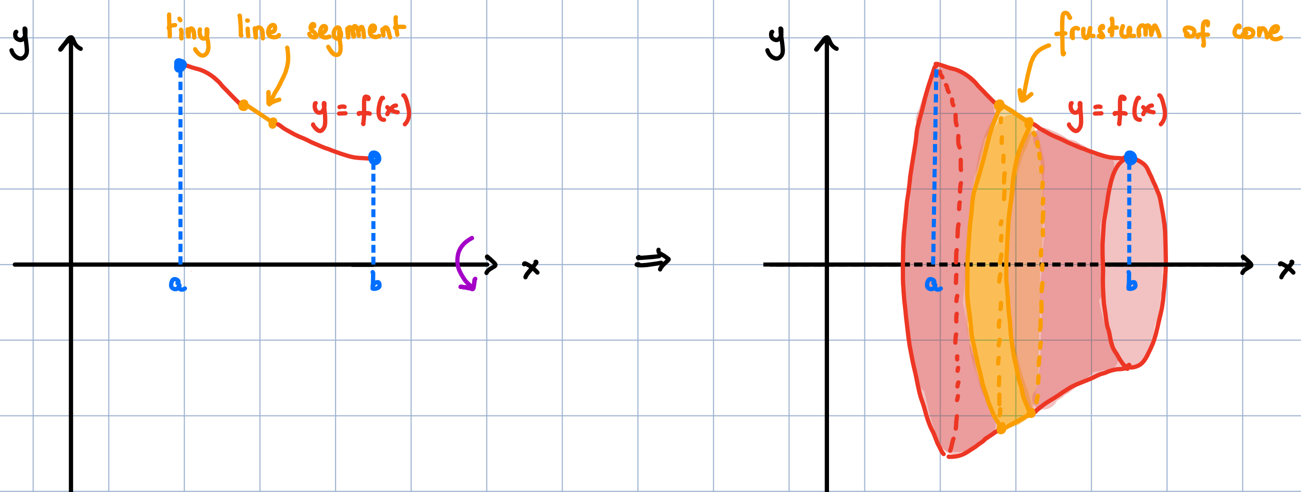
Learning Goals:

	Learning Goal	*Homework Problems*
	6.4.1 Set and evaluate an x or a y integral for a surface of revolution about the x or the y-axis	6.4: 9,13,17,19,21,25,26,29

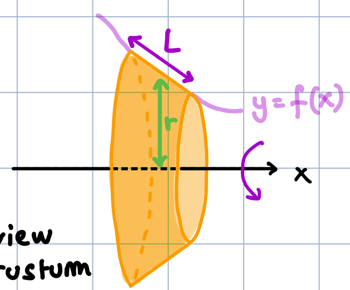
Conceptual introduction:

In the previous sections, we have computed volumes of solids of revolution (i.e. how much water fits inside).
In this section, we compute areas of surfaces of revolution (i.e. how much is needed to coat the sides).

We revolve a curve $y = f(x)$, $a \leq x \leq b$ around the x-axis to form a surface of revolution.



To compute the surface area, we decompose the graph into tiny line segments. Revolving the tiny line segment about the x-axis forms a frustum of cone.

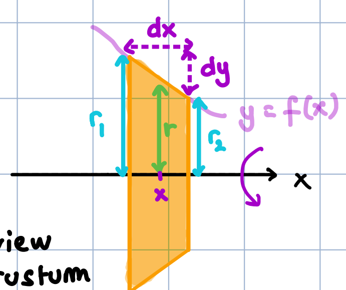


3d view of frustum

$$\boxed{\text{Area of frustum} = 2\pi r L}$$

where L = slant height

r = midpoint radius $= \dfrac{r_1 + r_2}{2}$



side view of frustum

For our surface:

$r = y(x) = f(x)$

$L = ds = \sqrt{dx^2 + dy^2}$ (previous section)

So the frustum has surface area

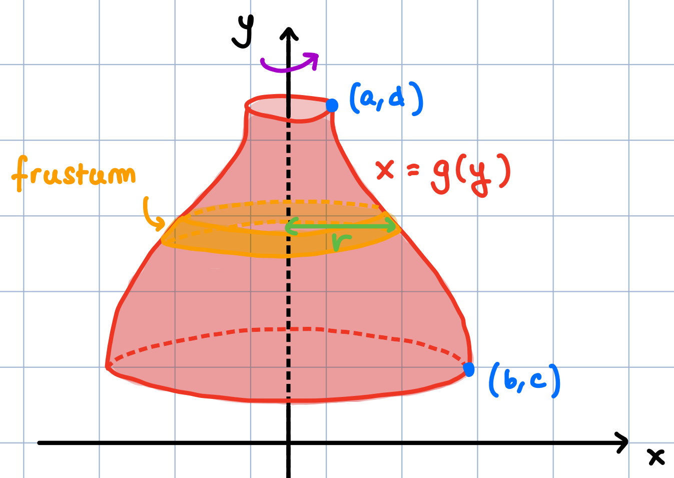
$$dA = 2\pi y\, ds = \begin{cases} 2\pi\, y(x) \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx & (\text{in terms of } x) \\ 2\pi y \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy & (\text{in terms of } y) \end{cases}$$

Total surface area for revolution about the x-axis:

$$\boxed{A = \int_{\text{curve}} 2\pi y\, ds = \begin{cases} \int_a^b 2\pi y(x) \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \\ \int_c^d 2\pi y \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy \end{cases}}$$

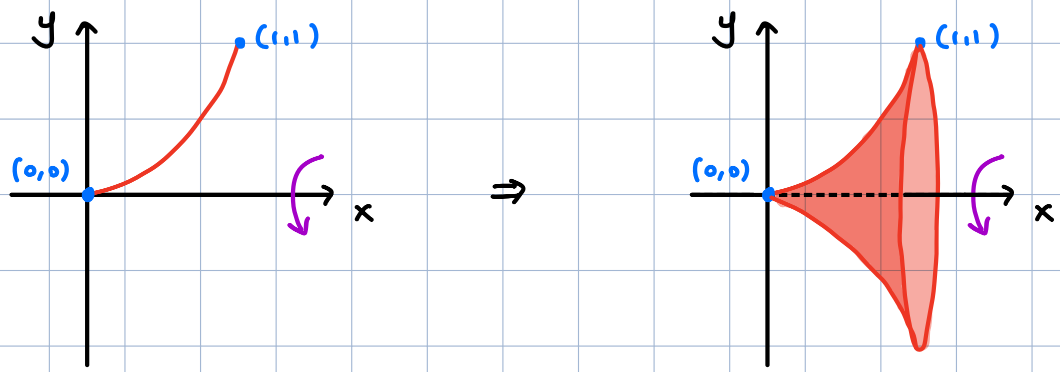
Revolution about the y-axis: we can use the same method to compute areas of surfaces obtained by revolving a curve $x = g(y)$ about the y-axis. The formulas become:

$$\boxed{A = \int_{\text{curve}} 2\pi x\, ds = \begin{cases} \int_c^d 2\pi x(y) \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy \\ \int_a^b 2\pi x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \end{cases}}$$



The radius of the frustum is x when we revolve about the y-axis.

Examples: 1) Find the area obtained by revolving $y = \frac{x^3}{3}$, $0 \le x \le 1$ about the x-axis.



$$A = \int_0^1 2\pi y(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 2\pi \frac{x^3}{3}\sqrt{1+(x^2)^2}\,dx$$

$$= \frac{2\pi}{3}\int_0^1 x^3\sqrt{1+x^4}\,dx$$

u-sub

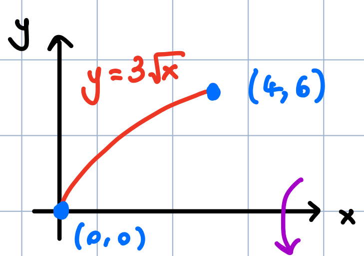
$u = 1+x^4$ $\quad x=0 \Rightarrow u=1$

$du = 4x^3dx$ $\quad x=1 \Rightarrow u=2$

$$= \frac{2\pi}{3}\int_1^2 \frac{1}{4}\sqrt{u}\,du$$

$$= \frac{2\pi}{3}\cdot\frac{1}{4}\cdot\frac{2}{3}\left[u^{3/2}\right]_1^2 = \boxed{\frac{\pi}{9}\left(2^{3/2}-1\right) \text{ square units}}$$

2) Find the area obtained by revolving $y = 3\sqrt{x}$, $0 \le x \le 4$ about the x-axis using a) an x-integral b) a y-integral



a) $$A = \int_0^4 2\pi y(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^4 2\pi(3\sqrt{x})\sqrt{1+\left(\frac{3}{2\sqrt{x}}\right)^2}\,dx$$

$$= 6\pi\int_0^4 \sqrt{x}\sqrt{1+\frac{9}{4x}}\,dx = 6\pi\int_0^4\sqrt{x+\frac{9}{4}}\,dx = 6\pi\left[\frac{2}{3}\left(x+\frac{9}{4}\right)^{3/2}\right]_0^4$$

$$= 4\pi\left[\left(\frac{25}{4}\right)^{3/2} - \left(\frac{9}{4}\right)^{3/2}\right] = \boxed{49\pi \text{ square units}}.$$

b) $$A = \int_0^6 2\pi y \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy \qquad y = 3\sqrt{x} \Rightarrow x = \frac{y^2}{9} \Rightarrow \frac{dx}{dy} = \frac{2y}{9}$$

$$= \int_0^6 2\pi y \sqrt{\left(\frac{2y}{9}\right)^2 + 1}\, dy$$

u-sub

$$u = \left(\frac{2y}{9}\right)^2 + 1 \qquad y = 0 \Rightarrow u = 1$$

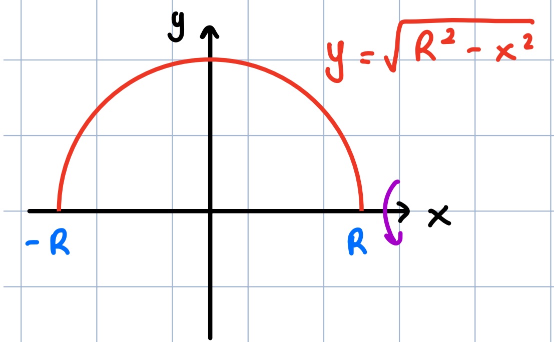
$$du = \frac{8y}{81}\, dy \qquad y = 6 \Rightarrow u = \frac{25}{9}$$

$$= 2\pi \int_1^{\frac{25}{9}} \frac{81}{8} \sqrt{u}\, du$$

$$= 2\pi \cdot \frac{81}{8} \cdot \left[\frac{2}{3} u^{3/2}\right]_1^{25/9} = \frac{27\pi}{2}\left[\left(\frac{25}{9}\right)^{3/2} - 1\right] = \boxed{49\pi \text{ square units}}$$

3) Compute the surface area of a sphere of radius R using the method from this section.

The sphere is obtained by revolving the semi-circle $y = \sqrt{R^2 - x^2}$, $-R \leq x \leq R$, about the x-axis.



$$y = \sqrt{R^2 - x^2} \Rightarrow \frac{dy}{dx} = \frac{-2x}{2\sqrt{R^2 - x^2}} = \frac{-x}{\sqrt{R^2 - x^2}}$$

So $$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{R^2 - x^2} = \frac{R^2}{R^2 - x^2}$$

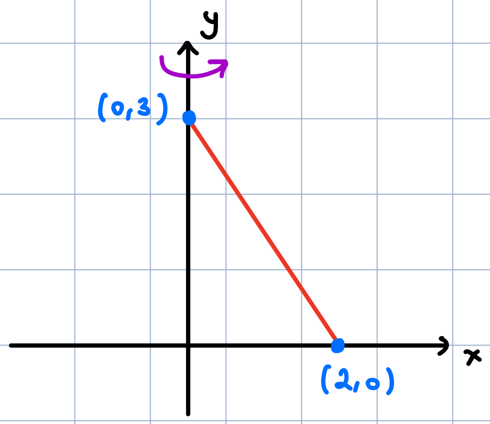
$$A = \int_{-R}^{R} 2\pi y(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{-R}^{R} 2\pi \cancel{\sqrt{R^2 - x^2}} \sqrt{\frac{R^2}{\cancel{R^2 - x^2}}}\, dx$$

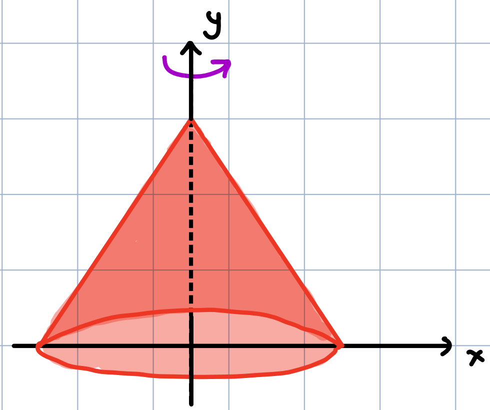
$$= \int_{-R}^{R} 2\pi R\, dx$$

$$= 2\pi R \left[x\right]_{-R}^{R} = 2\pi R(R - (-R)) = \boxed{4\pi R^2}$$

4) Find the area obtained by revolving the line segment $3x + 2y = 6$, $x, y \geq 0$, about the y-axis.



$\Rightarrow$



$$A = \int_0^3 2\pi x(y)\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$$

$$= \int_0^3 2\pi\left(2 - \tfrac{2}{3}y\right)\sqrt{\left(-\tfrac{2}{3}\right)^2 + 1}\, dy$$

$3x + 2y = 6$ so $x = 2 - \frac{2}{3}y$

$\frac{dx}{dy} = -\frac{2}{3}$

$$= \int_0^3 2\pi\left(2 - \tfrac{2}{3}y\right)\frac{\sqrt{13}}{3}\, dy = \frac{2\pi\sqrt{13}}{3}\left[2y - \frac{y^2}{3}\right]_0^3 = \boxed{2\pi\sqrt{13} \text{ square units}}$$

5) Find the area obtained by revolving $x = \sqrt{2y - y^2}$, $0 \leq y \leq 2$, about the y-axis.

$x = \sqrt{2y - y^2}$ so $\frac{dx}{dy} = \frac{2 - 2y}{2\sqrt{2y - y^2}} = \frac{1 - y}{\sqrt{2y - y^2}}$

So $1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{(1-y)^2}{2y - y^2} = \frac{2y - y^2 + (1-y)^2}{2y - y^2} = \frac{2y - y^2 + 1 + y^2 - 2y}{2y - y^2} = \frac{1}{2y - y^2}$

$$A = \int_0^2 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_0^2 2\pi\sqrt{2y - y^2}\sqrt{\frac{1}{2y - y^2}}\, dy = \int_0^2 2\pi\, dy = \boxed{4\pi \text{ square units}}.$$

Practice:

Find the area of the surface obtained by revolving the curve $y = x^2+1$, $0 \leq x \leq 1$, about the y-axis using

1) a y-integral
2) an x-integral

Solutions:

1) The curve can be expressed in terms of y as $x = \sqrt{y-1}$, $1 \leq y \leq 2$.

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y-1}} \quad \text{so} \quad 1+\left(\frac{dx}{dy}\right)^2 = 1+\frac{1}{4(y-1)} = \frac{4y-3}{4(y-1)}$$

So $$A = \int_1^2 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = \int_1^2 2\pi\sqrt{y-1}\,\sqrt{\frac{4y-3}{4(y-1)}}\,dy$$

$$= \pi\int_1^2 \sqrt{4y-3}\,dy = \pi\left[\frac{2}{3}\cdot\frac{1}{4}(4y-3)^{3/2}\right]_1^2$$

$$= \boxed{\frac{\pi}{6}\left(5^{3/2}-1\right) \text{ square units}}.$$

2) $\dfrac{dy}{dx} = 2x$, so

$$A = \int_0^1 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 2\pi x\sqrt{1+4x^2}\,dx$$

$u = 1+4x^2$ $\quad$ $x=0 \Rightarrow u=1$
$du = 8x\,dx$ $\quad$ $x=1 \Rightarrow u=5$
$\Rightarrow x\,dx = \frac{du}{8}$

$$= 2\pi\int_1^5 \sqrt{u}\,\frac{du}{8}$$

$$= \frac{\pi}{4}\left[\frac{2}{3}u^{3/2}\right]_1^5 = \boxed{\frac{\pi}{6}\left(5^{3/2}-1\right) \text{ square units}}.$$