Rutgers University
Math 152

Section 6.4: Areas of Surfaces of Revolution - Worksheet Solutions

1. Find the surface area obtained by revolving the given curve about the given axis.

   (a) The curve $y = \sqrt{3x-5}$, $2 \leqslant x \leqslant 3$, revolved about the x-axis.

   Solution. Method 1: we use an x-integral. The area for a surface of revolution about the x-axis is given by

$$\begin{aligned} A &= \int_{\text{curve}} 2\pi y ds \\ &= \int_2^3 2\pi y(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx \\ &= \int_2^3 2\pi\sqrt{3x-5}\sqrt{1+\left(\frac{3}{2\sqrt{3x-5}}\right)^2}dx \\ &= 2\pi\int_2^3\sqrt{(3x-5)\left(1+\frac{9}{4(3x-5)}\right)}dx \\ &= 2\pi\int_2^3\sqrt{3x-5+\frac{9}{4}}dx \\ &= 2\pi\int_2^3\sqrt{3x-\frac{11}{4}}dx. \end{aligned}$$

   We can finish evaluating this integral using the substitution $u = 3x - \frac{11}{4}$, which gives $du = 3dx$. The bounds change as follows

$$\begin{aligned} x = 2 &\Rightarrow u = 6 - \frac{11}{4} = \frac{13}{4}, \\ x = 3 &\Rightarrow u = 9 - \frac{11}{4} = \frac{25}{4}. \end{aligned}$$

   So the integral becomes

$$\begin{aligned} A &= 2\pi\int_{13/4}^{25/4}\frac{1}{3}\sqrt{u}du \\ &= \frac{2\pi}{3}\left[\frac{2}{3}u^{3/2}\right]_{13/4}^{25/4} \\ &= \frac{2\pi}{3}\cdot\frac{2}{3}\left(\left(\frac{25}{4}\right)^{3/2}-\left(\frac{13}{4}\right)^{3/2}\right) \\ &= \boxed{\frac{\pi\left(125-13^{3/2}\right)}{18}\text{ square units}}. \end{aligned}$$

Method 2: we use a y-integral, observing that we can express the curve as a function of y as $x = \frac{y^2-5}{3}$, $1 \leqslant y \leqslant 2$. The area for a surface of revolution about the x-axis is given by

$$\begin{aligned} A &= \int_{\text{curve}} 2\pi y ds \\ &= \int_1^2 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \\ &= \int_1^2 2\pi y \sqrt{1 + \left(\frac{2y}{3}\right)^2} dy \\ &= 2\pi \int_1^2 y \sqrt{1 + \frac{4y^2}{9}} dy. \end{aligned}$$

We can finish evaluating this integral using the substitution $u = 1 + \frac{4y^2}{9}$, which gives $du = \frac{8ydy}{9}$. The bounds change as follows

$$\begin{aligned} y = 1 &\Rightarrow u = 1 + \frac{4}{9} = \frac{13}{9}, \\ y = 2 &\Rightarrow u = 1 + \frac{16}{9} = \frac{25}{9}. \end{aligned}$$

So the integral becomes

$$\begin{aligned} A &= 2\pi \int_{13/9}^{25/9} \frac{9}{8} \sqrt{u} du \\ &= \frac{9\pi}{4} \left[\frac{2}{3} u^{3/2}\right]_{13/9}^{25/9} \\ &= \frac{9\pi}{4} \cdot \frac{2}{3} \left(\left(\frac{25}{9}\right)^{3/2} - \left(\frac{13}{9}\right)^{3/2} \right) \\ &= \boxed{\frac{\pi\left(125 - 13^{3/2}\right)}{18} \text{ square units}}. \end{aligned}$$

(b) The curve $x = \sqrt{16y - y^2}$, $0 \leqslant y \leqslant 8$, revolved about the y-axis.

Solution. We use integration with respect to y. The area for a surface of revolution about the y-axis is given by

$$\begin{aligned} A &= \int_{\text{curve}} 2\pi x ds \\ &= \int_0^8 2\pi x(y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \\ &= \int_0^8 2\pi \sqrt{16y - y^2} \sqrt{1 + \left(\frac{16 - 2y}{2\sqrt{16y - y^2}}\right)^2} dy \end{aligned}$$

$$
\begin{aligned}
&= 2\pi \int_0^8 \sqrt{(16y - y^2)\left(1 + \frac{(8-y)^2}{16y - y^2}\right)} dy \\
&= 2\pi \int_0^8 \sqrt{16y - y^2 + (8-y)^2} dy \\
&= 2\pi \int_0^8 \sqrt{16y - y^2 + 64 - 16y + y^2} dy \\
&= 2\pi \int_0^8 8 dy \\
&= \boxed{128\pi \text{ square units}}.
\end{aligned}
$$

(c) The curve $x = 2\sqrt[3]{y}$, $0 \leqslant y \leqslant 1$, revolved about the x-axis.

Solution. Method 1: we use an x-integral. The curve can be expressed as a function of x as $y = \frac{x^3}{8}$, $0 \leqslant x \leqslant 2$. The area for a surface of revolution about the x-axis is given by

$$
\begin{aligned}
A &= \int_{\text{curve}} 2\pi y ds \\
&= \int_0^2 2\pi y(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
&= \int_0^2 2\pi \frac{x^3}{8} \sqrt{1 + \left(\frac{3x^2}{8}\right)^2} dx \\
&= \frac{\pi}{4} \int_0^2 x^3 \sqrt{1 + \frac{9x^4}{64}} dx.
\end{aligned}
$$

We can finish evaluating this integral using the substitution $u = 1 + \frac{9x^4}{64}$, which gives $du = \frac{9x^3 dx}{16}$. The bounds change as follows

$$
\begin{aligned}
x = 0 \;&\Rightarrow\; u = 1 + \frac{9 \cdot 0}{64} = 1, \\
x = 2 \;&\Rightarrow\; u = 1 + \frac{9 \cdot 2^4}{64} = \frac{13}{4}.
\end{aligned}
$$

So the integral becomes

$$
\begin{aligned}
A &= \frac{\pi}{4} \int_1^{13/4} \frac{16}{9} \sqrt{u} du \\
&= \frac{4\pi}{9} \left[\frac{2}{3} u^{3/2}\right]_1^{13/4} \\
&= \frac{4\pi}{9} \cdot \frac{2}{3} \left(\left(\frac{13}{4}\right)^{3/2} - 1\right) \\
&= \boxed{\frac{\pi\left(13^{3/2} - 8\right)}{27} \text{ square units}}.
\end{aligned}
$$

Method 2: we use a y-integral. The area for a surface of revolution about the x-axis is given by

$$\begin{aligned}
A &= \int_{\text{curve}} 2\pi y ds \\
&= \int_0^1 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy \\
&= \int_0^1 2\pi y\sqrt{1+\left(\frac{2}{3y^{2/3}}\right)^2}dy \\
&= 2\pi\int_0^1 y\sqrt{1+\frac{4}{9y^{4/3}}}dy \\
&= 2\pi\int_0^1 y\frac{\sqrt{y^{4/3}+\frac{4}{9}}}{y^{2/3}}dy \\
&= 2\pi\int_0^1 y^{1/3}\sqrt{y^{4/3}+\frac{4}{9}}dy.
\end{aligned}$$

We can finish evaluating this integral using the substitution $u = y^{4/3} + \frac{4}{9}$, which gives $du = \frac{4y^{1/3}dy}{3}$. The bounds change as follows

$$\begin{aligned}
y = 0 &\Rightarrow u = 0^{4/3} + \frac{4}{9} = \frac{4}{9}, \\
y = 1 &\Rightarrow u = 1^{4/3} + \frac{4}{9} = \frac{13}{9}.
\end{aligned}$$

So the integral becomes

$$\begin{aligned}
A &= 2\pi\int_{4/9}^{13/9} \frac{3}{4}\sqrt{u}du \\
&= \frac{3\pi}{2}\left[\frac{2}{3}u^{3/2}\right]_{4/9}^{13/9} \\
&= \frac{3\pi}{2}\cdot\frac{2}{3}\left(\left(\frac{13}{9}\right)^{3/2} - \left(\frac{4}{9}\right)^{3/2}\right) \\
&= \boxed{\frac{\pi\left(13^{3/2}-8\right)}{27}\text{ square units}}.
\end{aligned}$$

(d) The curve $x = \frac{3}{5}y^{5/3}$, $0 \leqslant y \leqslant 1$, revolved about the y-axis.

Solution. We use integration with respect to y. The area for a surface of revolution about the y-axis is given by

$$\begin{aligned}
A &= \int_{\text{curve}} 2\pi x ds \\
&= \int_0^1 2\pi x(y)\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 2\pi \frac{3}{5} y^{5/3} \sqrt{1 + \left(y^{2/3}\right)^2} dy \\
&= \frac{6\pi}{5} \int_0^1 y^{5/3} \sqrt{1 + y^{4/3}} dy.
\end{aligned}
$$

We finish evaluating this integral using the substitution $u = 1 + y^{4/3}$, which gives $du = \frac{4y^{1/3}dy}{3}$. The bounds change as follows

$$
\begin{aligned}
x = 0 &\Rightarrow u = 1 + 0^{4/3} = 1, \\
x = 1 &\Rightarrow u = 1 + 1^{4/3} = 2.
\end{aligned}
$$

There will be an extraneous factor $y^{4/3}$ in the integrand, which we can express in terms of u as $y^{4/3} = u - 1$. So the area becomes

$$
\begin{aligned}
A &= \frac{6\pi}{15} \int_0^1 y^{4/3} \sqrt{1 + y^{4/3}} y^{1/3} dy \\
&= \frac{6\pi}{15} \int_1^2 (u-1) \sqrt{u} \frac{3}{4} dy \\
&= \frac{3\pi}{10} \int_1^2 \left(u^{3/2} - \sqrt{u}\right) du \\
&= \frac{3\pi}{10} \left[\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right]_1^2 \\
&= \boxed{\frac{6\pi}{75} \left(\sqrt{2} + 1\right) \text{ units}}.
\end{aligned}
$$

(e) The curve $y = x^{3/2}$, $1 \leqslant x \leqslant 4$, revolved about the y-axis.

Solution. We use an x-integral. The area for a surface of revolution about the y-axis is given by

$$
\begin{aligned}
A &= \int_{\text{curve}} 2\pi x ds \\
&= \int_1^4 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
&= \int_1^4 2\pi x \sqrt{1 + \left(\frac{3x^{1/2}}{2}\right)^2} dx \\
&= 2\pi \int_1^4 x \sqrt{1 + \frac{9x}{4}} dx.
\end{aligned}
$$

We can finish evaluating this integral using the substitution $u = 1 + \frac{9x}{4}$, which gives $du = \frac{9dx}{4}$. The bounds change as follows

$$
\begin{aligned}
x = 1 &\Rightarrow u = 1 + \frac{9}{4} = \frac{13}{4}, \\
x = 4 &\Rightarrow u = 1 + \frac{9 \cdot 4}{4} = 10.
\end{aligned}
$$

We have an extraneous factor x in the integrand which we express in terms of u as

$$x = \frac{4}{9}(u - 1).$$

So the integral becomes

$$\begin{aligned} A &= 2\pi \int_{13/4}^{10} \frac{4}{9} \cdot \frac{4}{9}(u-1)\sqrt{u}du \\ &= \frac{32\pi}{81} \int_{13/4}^{10} \left(u^{3/2} - \sqrt{u}\right) du \\ &= \frac{32\pi}{81} \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_{13/4}^{10} \\ &= \boxed{\frac{32\pi}{81} \left(\frac{2}{5}\left(10^{5/2} - \left(\frac{25}{4}\right)^{5/2}\right) - \frac{2}{3}\left(10^{3/2} - \left(\frac{25}{4}\right)^{3/2}\right)\right) \text{ square units}}. \end{aligned}$$