Section 8.2

Integration by Parts

Learning Goals :

Learning Goal	Homework Problems
8.2.1 Evaluate an indefinite or definite integral using integration by parts once	8.2: 1,6,11,12,35,51,59
8.2.2 Evaluate an indefinite or definite integral by first using substitution and then integration by parts	8.2: 25,45
8.2.3 Evaluate an indefinite or definite integral using twice integration by parts	8.2: 3,23
8.2.4 Find reduction formula using integration by parts	8.2: 67,69,71,73,77,78,79,81
8.2.5 Evaluate an indefinite or definite integral with inverse trigonometric function by integration by parts	8.2: 11,12
8.2.6 Recognize when integration by parts is not needed	8.2: 37

Conceptual introduction :

u - substitution : backwards chain rule.

Integration by parts (IBP) : backwards product rule.

Recall the product rule: $\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

We can write this in differential form: $d(uv) = vdu + udv$.

If we integrate on both sides, we obtain:

$$\int d(uv) = \int vdu + \int udv$$
$$uv = \int vdu + \int udv$$

Isolating $\int udv$:

$$\boxed{\begin{array}{l} \int udv = uv - \int vdu \\ \int_a^b udv = [uv]_a^b - \int_a^b vdu \end{array}}$$

IBP for antiderivatives

IBP for definite integrals

Strategy to use in practice :

Choose u so that du is simpler.

Choose dv so that v is easy to compute.

In practice, we can often choose u in order of decreasing priority with the LIATE guideline :

Logarithmic, Inverse trig, Algebraic, Trig, Exp.

Examples: 1) $\int \underbrace{x}_{u} \underbrace{\cos(2x)dx}_{dv}$

IBP table

$u = x$	$dv = \cos(2x)dx$
$du = dx$	$v = \frac{1}{2}\sin(2x)$

$$= uv - \int v\,du$$

$$= \frac{1}{2}x\sin(2x) - \underbrace{\int \frac{1}{2}\sin(2x)dx}_{\text{easier to compute than the original integral}}$$

$$= \boxed{\frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C}$$

Remark: choosing $u = \cos(2x)$ and $dv = x\,dx$ instead would give:
$du = -2\sin(2x)dx$ $\quad v = \frac{x^2}{2}$

$$\int x\cos(2x)dx = \cos(2x)\frac{x^2}{2} - \underbrace{\int \frac{x^2}{2}(-2\sin(2x))dx}_{\text{harder than the original integral}}$$

2) $\int \underbrace{x^2}_{u} \underbrace{e^{3x-5}dx}_{dv}$

IBP table

$u = x^2$	$dv = e^{3x-5}dx$
$du = 2x\,dx$	$v = \frac{1}{3}e^{3x-5}$

Tip: it is often a good idea to choose u to be a polynomial

$$= x^2\frac{1}{3}e^{3x-5} - \int \frac{1}{3}e^{3x-5}2x\,dx$$

$$= \frac{1}{3}x^2e^{3x-5} - \frac{2}{3}\int xe^{3x-5}dx$$

IBP again for this one with

$u = x$	$dv = e^{3x-5}dx$
$du = dx$	$v = \frac{1}{3}e^{3x-5}$

$$= \frac{1}{3}x^2e^{3x-5} - \frac{2}{3}\left(\frac{1}{3}xe^{3x-5} - \frac{1}{3}\int e^{3x-5}dx\right)$$

$$= \frac{1}{3}x^2e^{3x-5} - \frac{2}{9}xe^{3x-5} + \frac{2}{9}\cdot\frac{1}{3}e^{3x-5}$$

$$= \boxed{\frac{1}{3}e^{3x-5}\left(x^2 - \frac{2}{3}x + \frac{2}{9}\right) + C}.$$

3) $\int_0^1 \tan^{-1}(x)\,dx$

IBP

$u = \tan^{-1}(x) \qquad dv = dx$

$du = \dfrac{dx}{1+x^2} \qquad v = x$

Sometimes, $dv = dx$ is a good choice

$$= \underbrace{\Big[\tan^{-1}(x)\,x\Big]_0^1}_{=\frac{\pi}{4}} - \underbrace{\int_0^1 \frac{x}{x^2+1}\,dx}$$

we can compute this integral
with substitution $t = x^2+1$, $dt = 2x\,dx$

$x = 0 \Rightarrow t = 1$

$x = 1 \Rightarrow t = 2$

$$\int_0^1 \frac{x}{x^2+1}\,dx = \int_1^2 \frac{1}{2}\cdot\frac{dt}{t} = \frac{1}{2}\Big[\ln(|t|)\Big]_1^2 = \frac{\ln(2)}{2}.$$

So $\boxed{\int_0^1 \tan^{-1}(x)\,dx = \frac{\pi}{4} - \frac{\ln(2)}{2}}$

We can use the same technique to evaluate integrals involving $\sin^{-1}(x)$.

4) $\int \ln(x)\,dx$

IBP

$u = \ln(x) \qquad dv = dx$

$du = \dfrac{dx}{x} \qquad v = x$

$$= x\ln(x) - \int x\,\frac{dx}{x}$$

$$= x\ln(x) - \int dx = \boxed{x\ln(x) - x + C}$$

The same technique works for any integral of the form $\int \ln(x)x^a\,dx$. For example:

$$\int \underbrace{\ln(x)}_{u}\underbrace{\sqrt{x}\,dx}_{dv} = \frac{2}{3}\ln(x)x^{3/2} - \frac{2}{3}\int \frac{1}{x}x^{3/2} = \frac{2}{3}\ln(x)x^{3/2} - \frac{2}{3}\int x^{1/2}\,dx$$

$$= \boxed{\frac{2}{3}\ln(x)x^{3/2} - \frac{4}{9}x^{3/2} + C}.$$

5) $\int \sin(\sqrt{x})\,dx$

First, we use the substitution $t = \sqrt{x}$, $dt = \frac{dx}{2\sqrt{x}} \Rightarrow dx = 2t\,dt$

IBP

$u = 2t$	$dv = \sin(t)dt$
$du = 2dt$	$v = -\cos(t)$

$$\int \sin(\sqrt{x})\,dx = \int \underbrace{2t}_{u}\,\underbrace{\sin(t)dt}_{v}$$

$$= -2t\cos(t) - \int 2(-\cos(t))dt$$

$$= -2t\cos(t) + 2\sin(t)$$

$$= \boxed{-2\sqrt{x}\cos(\sqrt{x}) + 2\sin(\sqrt{x}) + C}$$

Sometimes, IBP does not give us a direct result. Instead, we get a <u>relation that we solve for the unknown integral.</u>

<u>Typical example:</u> (Exponential)$\cdot$(Trigonometric)

Calculate $\int e^{4x}\cos(x)\,dx$.

We do a first IBP with $u = e^{4x}$, $du = 4e^{4x}dx$, $dv = \cos(x)dx$, $v = \sin(x)$

$$\int e^{4x}\cos(x)dx = e^{4x}\sin(x) - 4\underbrace{\int e^{4x}\sin(x)dx}$$

IBP with $u = e^{4x}$, $du = 4e^{4x}dx$, $dv = \sin(x)dx$, $v = -\cos(x)$

$$\int e^{4x}\cos(x)dx = e^{4x}\sin(x) - 4\left[e^{4x}(-\cos(x)) - \int(-\cos(x))4e^{4x}dx\right]$$

$$\int e^{4x}\cos(x)dx = e^{4x}\sin(x) + 4e^{4x}\cos(x) - 16\underbrace{\int e^{4x}\cos(x)dx}$$

we recover the unknown integral

we solve this relation for $\int e^{4x}\cos(x)dx$

$$17\int e^{4x}\cos(x)\,dx = e^{4x}\sin(x) + 4e^{4x}\cos(x)$$

$$\boxed{\int e^{4x}\cos(x)\,dx = \frac{1}{17}e^{4x}\sin(x) + \frac{4}{17}e^{4x}\cos(x) + C}$$

We can use a similar technique to obtain <u>reduction formulas</u>. These are formulas expressing an integral of a power in terms of an integral of a lower power and explicit quantities.

<u>Examples</u> : 1) Express $\int \cos(x)^7 dx$ in terms of $\int \cos(x)^5 dx$.

$$\int \cos(x)^7 dx = \int \underbrace{\cos(x)^6}_{u} \underbrace{\cos(x)\,dx}_{dv}$$

IBP

$u = \cos(x)^6$	$dv = \cos(x)\,dx$
$du = -6\cos(x)^5 \sin(x)$	$v = \sin(x)$

$$\int \cos(x)^7 dx = \cos(x)^6 \sin(x) + 6 \int \cos(x)^5 \sin(x)^2 dx$$

replace by $1 - \cos(x)^2$

$$\int \cos(x)^7 dx = \cos(x)^6 \sin(x) + 6 \int \cos(x)^5 (1 - \cos(x)^2)\, dx$$

$$\int \cos(x)^7 dx = \cos(x)^6 \sin(x) + 6 \int \cos(x)^5 dx - 6 \underbrace{\int \cos(x)^7 dx}$$

we recover the original integral

we solve this relation for $\int \cos(x)^7 dx$

$$7 \int \cos(x)^7 dx = \cos(x)^6 \sin(x) + 6 \int \cos(x)^5 dx$$

$$\boxed{\int \cos(x)^7 dx = \frac{1}{7} \cos(x)^6 \sin(x) + \frac{6}{7} \int \cos(x)^5 dx}$$

lower power.

this is an example of a reduction formula.

2) Express $\int \sec(4x)^{11} dx$ in terms of $\int \sec(4x)^9 dx$.

$$\int \sec(4x)^{11} dx = \int \underbrace{\sec(4x)^9}_{u} \underbrace{\sec(4x)^2 dx}_{dv}$$

IBP

$u = \sec(4x)^9$

$du = 36 \sec(4x)^8 \sec(4x) \tan(4x)\, dx$

$\quad = 36 \sec(4x)^9 \tan(4x)\, dx$

$dv = \sec(4x)^2 dx$

$v = \frac{1}{4} \tan(4x)$

$$= \frac{1}{4} \sec(4x)^9 \tan(4x) - \frac{36}{4} \int \sec(4x)^9 \tan(4x)^2 dx$$

replace by $\sec(4x)^2 - 1$

$$= \frac{1}{4}\sec(4x)^9\tan(4x) - 9\int \sec(4x)^9\left(\sec(4x)^2 - 1\right)dx$$

$$= \frac{1}{4}\sec(4x)^9\tan(4x) - 9\underbrace{\int \sec(4x)^{11}dx}_{\text{original integral}} + 9\int \sec(4x)^9 dx$$

Solve for $\int \sec(4x)^{11}dx$:

$$10\int \sec(4x)^{11}dx = \frac{1}{4}\sec(4x)^9\tan(4x) + 9\int \sec(4x)^9 dx$$

$$\boxed{\int \sec(4x)^{11}dx = \frac{1}{40}\sec(4x)^9\tan(4x) + \frac{9}{10}\int \sec(4x)^9 dx}$$

3) Find a reduction formula for $I_n = \int_1^e \ln(x)^n dx$.

IBP

$u = \ln(x)^n$	$dv = dx$
$du = n\ln(x)^{n-1}\frac{1}{x}dx$	$v = x$

$$I_n = \int_1^e \ln(x)^n dx$$

$$= \left[x\ln(x)^n\right]_1^e - \int_1^e n\ln(x)^{n-1}\frac{1}{x}x\,dx$$

$$= e - n\int_1^e \ln(x)^{n-1}dx = e - nI_{n-1}.$$

So we obtain the reduction formula $\boxed{I_n = e - nI_{n-1}}$

We can use this to compute values of I_n explicitly.

$I_0 = \int_1^e \ln(x)^0 dx = e - 1$

$I_1 = e - 1\cdot I_0 = e - (e-1) = 1$

$I_2 = e - 2\cdot I_1 = e - 2$

$I_3 = e - 3\cdot I_2 = e - 3(e-2) = 6 - 2e$

$I_4 = e - 4\cdot I_3 = e - 4(6-2e) = 9e - 24$

etc.

Practice: evaluate the following integrals.

1) $\int \sin^{-1}(4x)\,dx$

2) $\int_1^e \frac{\ln(x)^2}{x^4}\,dx$

3) $\int \cos(3x)\cos(7x)\,dx$

4) $\int x\csc^2(6x)\,dx$

Solutions:

1) $\int \underbrace{\sin^{-1}(4x)}_{u}\,\underbrace{dx}_{dv}$

IBP

$u = \sin^{-1}(4x)$	$dv = dx$
$du = \frac{4dx}{\sqrt{1-16x^2}}$	$v = x$

$$= x\sin^{-1}(4x) - \int \frac{4x}{\sqrt{1-16x^2}}\,dx$$

substitution: $t = 1-16x^2$
$dt = -32x\,dx$
$\Rightarrow x\,dx = -\frac{dt}{32}$

$$= x\sin^{-1}(4x) - \int \frac{4}{\sqrt{t}} \cdot \frac{-dt}{32}$$

$$= x\sin^{-1}(4x) + \frac{1}{8}\int \frac{dt}{\sqrt{t}}$$

$$= x\sin^{-1}(4x) + \frac{1}{8}\cdot 2\sqrt{t} + C$$

$$= \boxed{x\sin^{-1}(4x) + \frac{\sqrt{1-16x^2}}{4} + C}.$$

2) $\int_1^e \frac{\ln(x)^2}{x^4}\,dx$

IBP

$u = \ln(x)^2$	$dv = \frac{dx}{x^4}$
$du = \frac{2\ln(x)}{x}\,dx$	$v = -\frac{1}{3x^3}$

$$= \left[-\frac{\ln(x)^2}{3x^3}\right]_1^e - \int_1^e \frac{2\ln(x)}{x} \cdot \frac{-1}{3x^3}\,dx$$

$$= -\frac{1}{3e^3} + \frac{2}{3}\int_1^e \frac{\ln(x)}{x^4}\,dx$$

IBP

$u = \ln(x)$	$dv = \frac{dx}{x^4}$
$du = \frac{dx}{x}$	$v = -\frac{1}{3x^3}$

$$= -\frac{1}{3e^3} + \frac{2}{3}\left(\left[-\frac{\ln(x)}{3x^3}\right]_1^e - \int_1^e \frac{1}{x}\cdot\frac{-1}{3x^3}\,dx\right)$$

$$= -\frac{1}{3e^3} + \frac{2}{3}\left(-\frac{1}{3e^3} + \frac{1}{3}\int_1^e \frac{dx}{x^4}\right)$$

$$= -\frac{1}{3e^3} - \frac{2}{9e^3} + \frac{2}{9}\left[-\frac{1}{3x^3}\right]_1^e$$

$$= -\frac{5}{9e^3} + \frac{2}{9}\left(-\frac{1}{3e^3} + \frac{1}{3}\right)$$

$$= \boxed{\frac{2e^3 - 17}{27e^3}}$$

3) $\int \cos(3x)\cos(7x)\,dx$

IBP:

$u = \cos(3x)$	$dv = \cos(7x)dx$
$du = -3\sin(3x)dx$	$v = \frac{1}{7}\sin(7x)$

$$= \frac{\cos(3x)\sin(7x)}{7} - \int -3\sin(3x)\cdot\frac{1}{7}\sin(7x)\,dx$$

$$= \frac{\cos(3x)\sin(7x)}{7} + \frac{3}{7}\int \sin(3x)\sin(7x)\,dx$$

$\hookrightarrow$ IBP: $u = \sin(3x)$, $du = 3\cos(3x)dx$, $dv = \sin(7x)dx$, $v = -\frac{1}{7}\cos(7x)dx$

$$= \frac{\cos(3x)\sin(7x)}{7} + \frac{3}{7}\left(-\frac{\sin(3x)\cos(7x)}{7} - \int 3\cos(3x)\cdot\frac{-1}{7}\cos(7x)\,dx\right)$$

$$= \frac{\cos(3x)\sin(7x)}{7} - \frac{3\sin(3x)\cos(7x)}{49} + \frac{9}{49}\int \cos(3x)\cos(7x)\,dx$$

we get the original integral back $\Rightarrow$ solve for it.

$$\left(1 - \frac{9}{49}\right)\int \cos(3x)\cos(7x)\,dx = \frac{\cos(3x)\sin(7x)}{7} - \frac{3\sin(3x)\cos(7x)}{49}$$

$$\frac{40}{49}\int \cos(3x)\cos(7x)\,dx = \frac{\cos(3x)\sin(7x)}{7} - \frac{3\sin(3x)\cos(7x)}{49}$$

$$\boxed{\int \cos(3x)\cos(7x)\,dx = \frac{7\cos(3x)\sin(7x)}{40} - \frac{3\sin(3x)\cos(7x)}{40} + C}$$

4) $\int x\csc^2(6x)\,dx$

IBP : $\boxed{\begin{array}{ll} u = x & dv = \csc^2(6x)\,dx \\ du = dx & v = -\frac{1}{6}\cot(6x) \end{array}}$

$$= -\frac{x\cot(6x)}{6} - \int -\frac{\cot(6x)}{6}\,dx$$

$$= -\frac{x\cot(6x)}{6} + \frac{1}{6}\cdot\frac{1}{6}\ln|\sin(6x)| + C$$

$$= \boxed{-\frac{x\cot(6x)}{6} + \frac{\ln|\sin(x)|}{36} + C}.$$