Rutgers University
Math 152

Section 8.2: Integration by Parts - Worksheet Solutions

1. Evaluate the following antiderivatives.

   (a) $\int x^3 \cos(5x)dx$

   Solution. We will evaluate this integral with three consecutive IBPs, taking u to be the power of x each time. For the first IBP, we take

$$u = x^3 \;\Rightarrow\; du = 3x^2 dx,$$
$$dv = \cos(5x)dx \;\Rightarrow\; v = \frac{1}{5}\sin(5x).$$

   This gives

$$\begin{aligned}\int x^3 \cos(5x)dx &= \frac{x^3 \sin(5x)}{5} - \int 3x^2 \frac{\sin(5x)}{5} dx \\ &= \frac{x^3 \sin(5x)}{5} - \frac{3}{5}\int x^2 \sin(5x)dx.\end{aligned}$$

   For the second IBP, we take

$$u = x^2 \;\Rightarrow\; du = 2x dx,$$
$$dv = \sin(5x)dx \;\Rightarrow\; v = -\frac{1}{5}\cos(5x).$$

   This gives

$$\begin{aligned}\int x^3 \cos(5x)dx &= \frac{x^3 \sin(5x)}{5} - \frac{3}{5}\left(-\frac{x^2\cos(5x)}{5} - \int -2x\frac{\cos(5x)}{5}dx\right) \\ &= \frac{x^3 \sin(5x)}{5} + \frac{3x^2\cos(5x)}{25} - \frac{6}{25}\int x\cos(5x)dx.\end{aligned}$$

   Finally, for the third and final IBP, we take

$$u = x \;\Rightarrow\; du = dx,$$
$$dv = \cos(5x)dx \;\Rightarrow\; v = \frac{1}{5}\sin(5x),$$

   and we obtain

$$\begin{aligned}\int x^3 \cos(5x)dx &= \frac{x^3 \sin(5x)}{5} + \frac{3x^2\cos(5x)}{25} - \frac{6}{25}\left(\frac{x\sin(5x)}{5} - \int \frac{\sin(5x)}{5}dx\right) \\ &= \frac{x^3 \sin(5x)}{5} + \frac{3x^2\cos(5x)}{25} - \frac{6x\sin(5x)}{125} + \frac{6}{125}\int \sin(5x)dx \\ &= \boxed{\frac{x^3 \sin(5x)}{5} + \frac{3x^2\cos(5x)}{25} - \frac{6x\sin(5x)}{125} - \frac{6\cos(5x)}{625} + C}.\end{aligned}$$

(b) $\int x^2 \sin^{-1}(x)dx$

Solution. We start with an IBP taking

$$u = \sin^{-1}(x) \;\Rightarrow\; du = \frac{dx}{\sqrt{1-x^2}},$$
$$dv = x^2 dx \;\Rightarrow\; v = \frac{x^3}{3},$$

which gives

$$\int x^2 \sin^{-1}(x)dx = \frac{x^3 \sin^{-1}(x)}{3} - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}dx.$$

For this last integral, we perform the substitution $u = 1 - x^2$, $du = -2xdx$. The numerator is x^2xdx, and we will replace $x^2 = 1 - u$ and $xdx = -\frac{du}{2}$. We obtain

$$\begin{aligned}
\int \frac{x^2xdx}{\sqrt{1-x^2}} &= \int -\frac{1-u}{2\sqrt{u}}du \\
&= \frac{1}{2}\int \left(u^{1/2} - u^{-1/2}\right) du \\
&= \frac{1}{2}\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right) \\
&= \frac{(1-x^2)^{3/2}}{3} - (1-x^2)^{1/2}.
\end{aligned}$$

Plugging back in the above equation gives

$$\begin{aligned}
\int x^2 \sin^{-1}(x)dx &= \frac{x^3 \sin^{-1}(x)}{3} - \frac{1}{3}\left(\frac{(1-x^2)^{3/2}}{3} - (1-x^2)^{1/2}\right) \\
&= \boxed{\frac{x^3 \sin^{-1}(x)}{3} - \frac{(1-x^2)^{3/2}}{9} + \frac{(1-x^2)^{1/2}}{3} + C}.
\end{aligned}$$

(c) $\int \frac{\ln(x)}{x^5}dx$

Solution. We use an IBP with

$$u = \ln(x) \;\Rightarrow\; du = \frac{dx}{x},$$
$$dv = \frac{dx}{x^5}, \;\Rightarrow\; v = -\frac{1}{4x^4}.$$

This gives

$$\begin{aligned}
\int \frac{\ln(x)}{x^5}dx &= -\frac{\ln(x)}{4x^4} + \frac{1}{4}\int \frac{1}{x^5}dx \\
&= \boxed{-\frac{\ln(x)}{4x^4} - \frac{1}{16x^4} + C}.
\end{aligned}$$

(d) $\int x^3 e^{-x^2}\,dx$

Solution. We can start with the substitution $t = -x^2$, $dt = -2x dx$, which gives

$$\int x^3 e^{-x^2} dx = \int x^2 e^{-x^2} x dx = \int (-t) e^t \frac{dt}{-2} = \frac{1}{2}\int t e^t dt$$

. We compute this new integral with an IBP taking

$$u = t \;\Rightarrow\; du = dt,$$
$$dv = e^t dt \;\Rightarrow\; v = e^t,$$

which gives

$$\int t e^t dt = t e^t - \int e^t dt = t e^t - e^t = e^t(t-1).$$

We plug back and replace t by $-x^2$:

$$\begin{aligned}\int x^3 e^{-x^2} dx &= \frac{e^t}{2}(t-1)\\ &= \boxed{\frac{e^{-x^2}}{2}\left(-x^2-1\right)+C}.\end{aligned}$$

(e) $\int e^{-2x}\sin(3x)dx$

Solution. We perform an IBP twice and then solve for the unknown integral. For the first IBP, the parts are

$$u = \sin(3x) \;\Rightarrow\; du = 3\cos(3x)dx,$$
$$dv = e^{-2x}dx \;\Rightarrow\; v = -\frac{e^{-2x}}{2}.$$

This gives

$$\begin{aligned}\int e^{-2x}\sin(3x)dx &= -\frac{e^{-2x}\sin(3x)}{2} - \int -\frac{e^{-2x}}{2} 3\cos(3x)dx\\ &= -\frac{e^{-2x}\sin(3x)}{2} + \frac{3}{2}\int e^{-2x}\cos(3x)dx.\end{aligned}$$

For the second IBP, the parts are

$$u = \cos(3x) \;\Rightarrow\; du = -3\sin(3x)dx,$$
$$dv = e^{-2x}dx \;\Rightarrow\; v = -\frac{e^{-2x}}{2}.$$

This gives

$$\begin{aligned}\int e^{-2x}\sin(3x)dx &= -\frac{e^{-2x}\sin(3x)}{2} + \frac{3}{2}\left(-\frac{e^{-2x}\cos(3x)}{2} - \int -\frac{e^{-2x}}{2}(-3\sin(3x))dx\right)\\ &= -\frac{e^{-2x}\sin(3x)}{2} - \frac{3e^{-2x}\cos(3x)}{4} - \frac{9}{4}\int e^{-2x}\sin(3x)dx.\end{aligned}$$

We can now solve the relation for the unknown integral. This gives

$$\begin{aligned}
\left(1+\frac{9}{4}\right)\int e^{-2x}\sin(3x)dx &= -\frac{e^{-2x}\sin(3x)}{2}-\frac{3e^{-2x}\cos(3x)}{4}\\
\frac{13}{4}\int e^{-2x}\sin(3x)dx &= -\frac{e^{-2x}\sin(3x)}{2}-\frac{3e^{-2x}\cos(3x)}{4}\\
\int e^{-2x}\sin(3x)dx &= \boxed{-\frac{2e^{-2x}\sin(3x)}{13}-\frac{3e^{-2x}\cos(3x)}{13}+C}
\end{aligned}$$

(f) $\int x\sec(5x)^2dx$

*Solution.*We use an IBP with the parts

$$\begin{aligned}
&u = x \;\Rightarrow\; du = dx,\\
&dv = \sec(5x)^2dx \;\Rightarrow\; v = \frac{\tan(5x)}{5},
\end{aligned}$$

which gives

$$\begin{aligned}
\int x\sec(5x)^2dx &= \frac{x\tan(5x)}{5}-\frac{1}{5}\int \tan(5x)dx\\
&= \boxed{\frac{x\tan(5x)}{5}-\frac{\ln|\sec(5x)|}{25}+C}.
\end{aligned}$$

2. Calculate the volume of the solid obtained by revolving the given region about the given axis using (i) the method of disks/washers and (ii) the method of cylindrical shells.

   (a) The region between the graph of $y=\sqrt{\tan^{-1}(x)}$ and the x-axis for $0\leqslant x\leqslant 1$ revolved about the x-axis.

   Solution. (i) Revolving the vertical strip at x in the region about the x-axis forms a disk of radius $r(x)=\sqrt{\tan^{-1}(x)}$. So the volume is

$$\begin{aligned}
V &= \int_0^1 \pi r(x)^2dx\\
&= \int_0^1 \pi\sqrt{\tan^{-1}(x)}^{\,2}\,dx\\
&= \pi\int_0^1 \tan^{-1}(x)dx.
\end{aligned}$$

   We calculate this integral with an IBP using the parts

$$\begin{aligned}
&u = \tan^{-1}(x) \;\Rightarrow\; du = \frac{dx}{x^2+1},\\
&dv = dx \;\Rightarrow\; v = x.
\end{aligned}$$

   We obtain

$$V = \pi\left(\left[x\tan^{-1}(x)\right]_0^1 - \int_0^1 \frac{x}{x^2+1}dx\right)$$

$$= \pi \left(\frac{\pi}{4} - \int_0^1 \frac{x}{x^2+1} dx \right).$$

This remaining integral can be evaluated with the substitution $u = x^2 + 1$, so that $du = 2xdx$. The bounds are $u = 1$ when $x = 0$ and $u = 2$ when $x = 1$. We get

$$\begin{aligned} V &= \pi \left(\frac{\pi}{4} - \int_1^2 \frac{du}{2u} \right) \\ &= \pi \left(\frac{\pi}{4} - \frac{1}{2} \left[\ln|u| \right]_1^2 \right) \\ &= \boxed{\pi \left(\frac{\pi}{4} - \frac{\ln(2)}{2} \right) \text{ cubic units}}. \end{aligned}$$

(ii) We will need to express the curve as a function of y:

$$y = \sqrt{\tan^{-1}(x)} \Rightarrow \tan^{-1}(x) = y^2 \Rightarrow x = \tan\left(y^2\right).$$

Then the region is bounded between the curves $x = \tan\left(y^2\right)$ and $x = 1$ for $0 \leqslant y \leqslant \sqrt{\frac{\pi}{4}}$. Revolving the horizontal strip at y in the region about the x-axis forms a cylindrical shell of radius $r(y) = y$ and height $h(y) = 1 - \tan(y^2)$. So the volume is

$$\begin{aligned} V &= \int_0^{\sqrt{pi/4}} 2\pi r(y) h(y) dy \\ &= \int_0^{\sqrt{pi/4}} 2\pi y \left(1 - \tan\left(y^2\right)\right) dy. \end{aligned}$$

We can calculate this integral with the substituion $u = y^2$, which gives $du = 2ydy$. The new bounds are $u = 0$ to $u = \frac{\pi}{4}$. We get

$$\begin{aligned} V &= \int_0^{\pi/4} \pi \left(1 - \tan(u)\right) du \\ &= \pi \left[u - \ln|\sec(u)| \right]_0^{\pi/4} \\ &= \pi \left(\frac{\pi}{4} - \ln\left(\sec\left(\frac{\pi}{4} \right) \right) + \ln\left(\sec(0)\right) \right) \\ &= \pi \left(\frac{\pi}{4} - \ln\left(\sqrt{2}\right) + \ln(1) \right) \\ &= \boxed{\pi \left(\frac{\pi}{4} - \frac{\ln(2)}{2} \right) \text{ cubic units}}. \end{aligned}$$

(b) The region bounded by the y-axis, the graph of $y = \sin(x)$ and the line $y = 1$ revolved about the y-axis.

Solution. (i) The region can be described as the region between the y-axis and $x = \sin^{-1}(y)$ for $0 \leqslant y \leqslant 1$. Revolving the horizontal strip at y in the region will form a disk of radius $r(y) = \sin^{-1}(y)$. So the volume is

$$V = \int_0^1 \pi r(y)^2 dy$$

$$= \int_0^1 \pi \sin^{-1}(y)^2 dy.$$

We can calculate this integral with two successive IBPs. The first one uses the parts

$$u = \sin^{-1}(y)^2 \;\Rightarrow\; du = \frac{2\sin^{-1} dy}{\sqrt{1-y^2}},$$
$$dv = dy \;\Rightarrow\; v = y.$$

This gives

$$V = \pi\left(\left[y\sin^{-1}(y)^2\right]_0^1 - \int_0^1 \frac{2y\sin^{-1}(y)}{\sqrt{1-y^2}}dy\right)$$
$$= \pi\left(\frac{\pi^2}{4} - 2\int_0^1 \frac{y\sin^{-1}(y)}{\sqrt{1-y^2}}dy\right).$$

In this last integral, we use an IBP with parts

$$u = \sin^{-1}(y) \;\Rightarrow\; du = \frac{dy}{\sqrt{1-y^2}},$$
$$dv = \frac{ydy}{\sqrt{1-y^2}} \;\Rightarrow\; v = -\sqrt{1-y^2}.$$

We get

$$V = \pi\left(\frac{\pi^2}{4} - 2\left(\left[-\sin^{-1}(y)\sqrt{1-y^2}\right]_0^1 - \int_0^1 \frac{-\sqrt{1-y^2}}{\sqrt{1-y^2}}dy\right)\right)$$
$$= \pi\left(\frac{\pi^2}{4} - 2\left(0 + \int_0^1 dy\right)\right)$$
$$= \boxed{\pi\left(\frac{\pi^2}{4} - 2\right) \text{ cubic units}}.$$

(ii) Revolving the vertical strip at x in the region about the y-axis creates a cylindrical shell of radius $r(x) = x$ and height $h(x) = 1 - \sin(x)$. Therefore the volume is

$$V = \int_0^{\pi/2} 2\pi r(x)h(x)dx$$
$$= 2\pi\int_0^{\pi/2} x\left(1 - \sin(x)\right)dx.$$

We can compute this integral with an IBP, taking the parts

$$u = x \;\Rightarrow\; du = dx,$$
$$dv = 1 - \sin(x) \;\Rightarrow\; v = x + \cos(x).$$

We get

$$V = 2\pi\left(\left[x(x+\cos(x))\right]_0^{\pi/2} - \int_0^{\pi/2}(x+\cos(x))dx\right)$$

$$
\begin{aligned}
&= 2\pi\left(\frac{\pi}{2}\left(\frac{\pi}{2}+0\right)-0-\left[\frac{x^2}{2}+\sin(x)\right]_0^{\pi/2}\right)\\
&= 2\pi\left(\frac{\pi^2}{4}-\frac{\pi^2}{8}-1\right)\\
&= \boxed{\pi\left(\frac{\pi^2}{4}-2\right)\text{ cubic units}}.
\end{aligned}
$$

(c) The region between the graph of $y = \ln(x)$ and the x-axis for $1 \leqslant x \leqslant e$ revolved about the line $x = -2$.

(i) The region can be described as the region between $x = e^y$ and $x = e$ for $0 \leqslant y \leqslant 1$. Revolving the horizontal strip at y in the region about the line $x = -2$ forms a washer with inner radius $r_{\text{in}}(y) = e^y - (-2) = e^y + 2$ and outer radius $r_{\text{out}}(y) = e - (-2) = e + 2$. So the volume is

$$
\begin{aligned}
V &= \int_0^1 \pi\left(r_{\text{out}}(y)^2 - r_{\text{in}}(y)^2\right)dy\\
&= \int_0^1 \pi\left((e+2)^2-(e^y+2)^2\right)dy\\
&= \pi\int_0^1 \left((e+2)^2-4-e^{2y}-4e^y\right)dy\\
&= \pi\left[((e+2)^2-4)y-\frac{e^{2y}}{2}-4e^y\right]_0^1\\
&= \pi\left((e+2)^2-4-\frac{e^2}{2}-4e+\frac{1}{2}+4\right)\\
&= \boxed{\frac{\pi(e^2+9)}{9}\text{ cubic units}}.
\end{aligned}
$$

(ii) Revolving the vertical strip at x about the line $x = -2$ forms a cylindrical shell with radius $r(x) = x - (-2) = x + 2$ and height $h(x) = \ln(x)$. So the volume is

$$
\begin{aligned}
V &= \int_1^e 2\pi r(x)h(x)dx\\
&= 2\pi\int_1^e (x+2)\ln(x)dx.
\end{aligned}
$$

We compute this integral with an IBP, taking the parts

$$
\begin{aligned}
u = \ln(x) &\;\Rightarrow\; du = \frac{dx}{x},\\
dv = (x+2)dx &\;\Rightarrow\; v = \frac{x^2}{2}+2x.
\end{aligned}
$$

This gives

$$
\begin{aligned}
V &= 2\pi\left(\left[\left(\frac{x^2}{2}+2x\right)\ln(x)\right]_1^e - \int_1^e\left(\frac{x^2}{2}+2x\right)\frac{1}{x}dx\right)\\
&= 2\pi\left(\frac{e^2}{2}+2e-\int_1^e\left(\frac{x}{2}+2\right)dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&= 2\pi\left(\frac{e^2}{2} + 2e - \left[\frac{x^2}{4} + 2x\right]_1^e\right) \\
&= 2\pi\left(\frac{e^2}{2} + 2e - \frac{e^2}{4} - 2e + \frac{1}{4} + 2\right) \\
&= \boxed{\frac{\pi(e^2+9)}{9} \text{ cubic units}}.
\end{aligned}
$$

3. Find reduction formulas for the following integrals.

   (a) $\displaystyle\int \cos(3x)^n dx$

   Solution. We split off a factor $\cos(3x)$ and use IBP with parts

$$
\begin{aligned}
u &= \cos(3x)^{n-1} \;\Rightarrow\; du = -3(n-1)\cos(3x)^{n-2}\sin(3x)dx, \\
dv &= \cos(3x)dx \;\Rightarrow\; v = \frac{\sin(3x)}{3}.
\end{aligned}
$$

   This gives

$$
\begin{aligned}
\int \cos(3x)^n dx &= \int \cos(3x)^{n-1}\cos(3x)dx \\
\int \cos(3x)^n dx &= \frac{\cos(3x)^{n-1}\sin(3x)}{3} - \int -3(n-1)\cos(3x)^{n-2}\sin(3x)\frac{\sin(3x)}{3}dx \\
\int \cos(3x)^n dx &= \frac{\cos(3x)^{n-1}\sin(3x)}{3} + (n-1)\int \cos(3x)^{n-2}\sin(3x)^2 dx.
\end{aligned}
$$

   In this last integral, we use the Phytagorean identity $\sin(3x)^2 = 1 - \cos(3x)^2$ to obtain

$$
\begin{aligned}
\int \cos(3x)^n dx &= \frac{\cos(3x)^{n-1}\sin(3x)}{3} + (n-1)\int \cos(3x)^{n-2}\left(1 - \cos(3x)^2\right)dx \\
\int \cos(3x)^n dx &= \frac{\cos(3x)^{n-1}\sin(3x)}{3} + (n-1)\int \cos(3x)^{n-2}dx - (n-1)\int \cos(3x)^n dx
\end{aligned}
$$

   We can now solve for the original integral by moving the term $-(n-1)\displaystyle\int \cos(3x)^n dx$ to the left-hand side.

$$
\begin{aligned}
\int \cos(3x)^n dx + (n-1)\int \cos(3x)^n dx &= \frac{\cos(3x)^{n-1}\sin(3x)}{3} + (n-1)\int \cos(3x)^{n-2}dx \\
n\int \cos(3x)^n dx &= \frac{\cos(3x)^{n-1}\sin(3x)}{3} + (n-1)\int \cos(3x)^{n-2}dx
\end{aligned}
$$

   We now divide by n to obtain the reduction formula

$$
\boxed{\int \cos(3x)^n dx = \frac{\cos(3x)^{n-1}\sin(3x)}{3n} + \frac{(n-1)}{n}\int \cos(3x)^{n-2}dx}.
$$

(b) $\displaystyle\int \ln(x)^n dx$

Solution. We use IBP with parts

$$\begin{aligned} u &= \ln(x)^n \;\Rightarrow\; du = \frac{n\ln(x)^{n-1}dx}{x}, \\ dv &= dx \;\Rightarrow\; v = x. \end{aligned}$$

This gives

$$\int \ln(x)^n dx = x\ln(x)^n - \int x\frac{n\ln(x)^{n-1}}{x}dx$$

$$\boxed{\int \ln(x)^n dx = x\ln(x)^n - n\int \ln(x)^{n-1}dx}$$

(c) $\displaystyle\int \sec(5x)^n dx$

Solution. We separate a factor $\sec(5x)^2$ and use IBP with parts

$$\begin{aligned} u &= \sec(5x)^{n-2} \;\Rightarrow\; du = 5(n-2)\sec(5x)^{n-2}\tan(5x)dx, \\ dv &= \sec(5x)^2 dx \;\Rightarrow\; v = \frac{\tan(5x)}{5}. \end{aligned}$$

We get

$$\begin{aligned} \int \sec(5x)^n dx &= \int \sec(5x)^{n-2}\sec(5x)^2 dx \\ Int\sec(5x)^n dx &= \frac{\sec(5x)^{n-2}\tan(5x)}{5} - \int 5(n-2)\sec(5x)^{n-2}\tan(5x)\frac{\tan(5x)}{5}dx \\ \int \sec(5x)^n dx &= \frac{\sec(5x)^{n-2}\tan(5x)}{5} - (n-2)\int \sec(5x)^{n-2}\tan(5x)^2 dx \end{aligned}$$

In this last integral, we use the Pythagorean identity $\tan(5x)^2 = \sec(5x)^2 - 1$, which gives

$$\begin{aligned} \int \sec(5x)^n dx &= \frac{\sec(5x)^{n-2}\tan(5x)}{5} - (n-2)\int \sec(5x)^{n-2}\left(\sec(5x)^2 - 1\right)dx \\ \int \sec(5x)^n dx &= \frac{\sec(5x)^{n-2}\tan(5x)}{5} - (n-2)\int \sec(5x)^n dx + (n-2)\int \sec(5x)^{n-2}dx \end{aligned}$$

We can now solve for the original integral by moving the term $-(n-2)\displaystyle\int \sec(5x)^n dx$ to the left-hand side.

$$\begin{aligned} \int \sec(5x)^n dx + (n-2)\int \sec(5x)^n dx &= \frac{\sec(5x)^{n-2}\tan(5x)}{5} + (n-2)\int \sec(5x)^{n-2}dx \\ (n-1)\int \sec(5x)^n dx &= \frac{\sec(5x)^{n-2}\tan(5x)}{5} + (n-2)\int \sec(5x)^{n-2}dx \end{aligned}$$

Dividing by $n-1$ gives the reduction formula

$$\boxed{\int \sec(5x)^n dx = \frac{\sec(5x)^{n-2}\tan(5x)}{5(n-1)} + \frac{n-2}{n-1}\int \sec(5x)^{n-2} dx}.$$