Section 8.3

Trigonometric Integrals

Learning Goals:

Learning Goal	*Homework Problems*
8.3.1 Evaluate an indefinite or definite integral of product of trigonometric functions	8.3: 3,22,28,35,38
8.3.2 Evaluate an indefinite or definite integral of a unique trigonometric function to a power of a constant, includes sec^3	8.3: 7,9,13,17,39,41,44,47,71,75
8.3.3 Evaluate an indefinite or definite integral of quotient of trigonometric functions	8.3: 64,65
8.3.4 Find reduction formula for integrals containing trigonometric functions to a power of n	8.3: Not in textbook - lecture should cover sin^n x and/or cos^n x

Goal: compute integrals of the form:

$$\int f(x)^n g(x)^m \, dx$$

where f, g are trigonometric functions.

① $\int \cos(x)^n \sin(x)^m \, dx$

Key identities:

$$\boxed{\begin{aligned} &\cos(x)^2 + \sin(x)^2 = 1 \\ &\cos(x)^2 = \frac{1+\cos(2x)}{2} \\ &\sin(x)^2 = \frac{1-\cos(2x)}{2} \end{aligned}}$$

$$\boxed{\begin{aligned} &\frac{d}{dx}\cos(x) = -\sin(x) \\ &\frac{d}{dx}\sin(x) = \cos(x) \end{aligned}}$$

Strategies:

- If at least one power is odd:
  → split-off one factor from the odd power and rewrite the remaining factors using $\cos(x)^2 + \sin(x)^2 = 1$.
  → do a u-substitution.

- If powers of cos, sin are both even: decrease powers using

$$\rightarrow \begin{cases} \cos(x)^2 = \dfrac{1+\cos(2x)}{2} \\ \sin(x)^2 = \dfrac{1-\cos(2x)}{2} \end{cases}$$

  double angle identities.

- If we have a single sin or cos to a power, we can also decrease the power using IBP as seen in 8.2. (This would give us a reduction formula.)

Examples:

1) $\int \underbrace{\sin(x)^5}\cos(x)^{18}\,dx$ power of sin is odd: split off one factor

$= \int \sin(x)^4 \cos(x)^{18} \sin(x)\,dx$

$= \int (\sin(x)^2)^2 \cos(x)^{18} \sin(x)\,dx$

rewrite remaining powers of sin using $\sin(x)^2 = 1-\cos(x)^2$

$= \int (1-\cos(x)^2)^2 \cos(x)^{18}\sin(x)\,dx$

u-sub

$u = \cos(x)$, $du = -\sin(x)dx$

$= \int (1-u^2)^2 u^{18}(-du)$

$= \int -(1+u^4-2u^2)u^{18}\,du$

$= \int (-u^{18}-u^{22}+2u^{20})\,du$

$= -\frac{u^{19}}{19} - \frac{u^{23}}{23} + \frac{2u^{21}}{21} + C$

$= \boxed{-\frac{\cos(x)^{19}}{19} - \frac{\cos(x)^{23}}{23} + \frac{2\cos(x)^{21}}{21} + C}$

2) $\int \sin(x)^4\,dx$ both powers even (0 and 4).

Method 1: double-angle formula

$\int \sin(x)^4\,dx = \int (\sin(x)^2)^2\,dx$

$= \int \left(\frac{1-\cos(2x)}{2}\right)^2 dx$

$$= \frac{1}{4}\int \left(1 - 2\cos(2x) + \cos(2x)^2\right)dx$$

$$= \frac{1}{4}\left(x - \sin(2x) + \underline{\int \cos(2x)^2 dx}\right)$$

we use a double-angle formula again for the remaining integral

$$= \frac{1}{4}\left(x - \sin(2x) + \int \frac{1+\cos(4x)}{2}dx\right)$$

$$= \frac{1}{4}\left(x - \sin(2x) + \frac{x}{2} + \frac{\sin(4x)}{8}\right) + C$$

$$= \boxed{\frac{3x}{8} - \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} + C}$$

Method 2: IBP

$$\int \sin(x)^4 dx = \int \underbrace{\sin(x)^3}_{u}\underbrace{\sin(x)\,dx}_{dv}$$

IBP

$u = \sin(x)^3$	$dv = \sin(x)dx$
$du = 3\sin(x)^2\cos(x)\,dx$	$v = -\cos(x)$

$$\int \sin(x)^4 dx = -\sin(x)^3\cos(x) + 3\int \sin(x)^2\cos(x)^2 dx$$

replace by $1 - \sin(x)^2$

$$\int \sin(x)^4 dx = -\sin(x)^3\cos(x) + 3\int \sin(x)^2 dx - 3\int \sin(x)^4 dx$$

we solve for $\int \sin(x)^4 dx$

$$4\int \sin(x)^4 dx = -\sin(x)^3\cos(x) + 3\int \sin(x)^2 dx$$

$$\int \sin(x)^4 dx = -\frac{1}{4}\sin(x)^3\cos(x) + \frac{3}{4}\int \sin(x)^2 dx$$

we repeat the pocess for this integral

$$\int \sin(x)^2 dx = \int \underbrace{\sin(x)}_{u}\underbrace{\sin(x)\,dx}_{dv}$$

$$\overset{IBP}{=} -\sin(x)\cos(x) + \int \cos(x)^2 dx$$

$1 - \sin(x)^2$

$$= -\sin(x)\cos(x) + x - \int \sin(x)^2 dx$$

solve for $\int \sin(x)^2 dx$

$$2\int \sin(x)^2\,dx = -\sin(x)\cos(x) + x$$

$$\int \sin(x)^2 = -\frac{1}{2}\sin(x)\cos(x) + \frac{x}{2}$$

We plug back:

$$\int \sin(x)^4 dx = -\frac{1}{4}\sin(x)^3\cos(x) + \frac{3}{4}\int \sin(x)^2 dx$$

$$= -\frac{1}{4}\sin(x)^3\cos(x) + \frac{3}{4}\left(-\frac{1}{2}\sin(x)\cos(x) + \frac{x}{2}\right) + c$$

$$= \boxed{\frac{3x}{8} - \frac{\sin(x)^3\cos(x)}{4} - \frac{3\sin(x)\cos(x)}{8} + C}$$

<u>Remark</u>: the results look different, but they are equal up to an additive constant.

3) $\displaystyle\int_0^{\pi/2} \sqrt{1+\cos(x)}\,dx$ double-angle formula: $1+\cos(x) = 2\cos\left(\frac{x}{2}\right)^2$

$$= \int_0^{\pi/2} \sqrt{2\cos\left(\frac{x}{2}\right)^2}\,dx = \sqrt{2}\int_0 \left|\cos\left(\frac{x}{2}\right)\right| dx = \sqrt{2}\int_0 \cos\left(\frac{x}{2}\right) dx$$

$\cos\left(\frac{x}{2}\right) > 0$ if $0 \leq x \leq \frac{\pi}{2}$

$$= \sqrt{2}\left[2\sin\left(\frac{x}{2}\right)\right]_0^{\pi/2} = \sqrt{2}\left(\sqrt{2} - 0\right) = \boxed{2}$$

4) $\displaystyle\int \frac{\cos(7x)^3}{\sin(7x)^6}\,dx$ power of cos odd: split off one factor

$$= \int \frac{\cos(7x)^2}{\sin(7x)^6}\cos(7x)\,dx$$

rewrite remaining factors in terms of sin

$$= \int \frac{1-\sin(7x)^2}{\sin(7x)}\cos(7x)\,dx$$

$$= \int \frac{1-u^2}{7u^6}\, du$$ u-sub: $u = \sin(7x)$, $du = 7\cos(7x)\,dx$

$$= \frac{1}{7}\int (u^{-6} - u^{-4})\, du = \frac{1}{7}\left(\frac{u^{-5}}{-5} - \frac{u^{-3}}{-3}\right) + C$$

$$= \boxed{\frac{1}{21\sin(7x)^3} - \frac{1}{35\sin(7x)^5} + C}$$

② $\int \tan(x)^n \sec(x)^m \, dx$

Key identities:

$$\boxed{\begin{array}{l} 1 + \tan(x)^2 = \sec(x)^2 \\ \frac{d}{dx}\tan(x) = \sec(x)^2, \quad \frac{d}{dx}\sec(x) = \sec(x)\tan(x) \\ \int \tan(x)\,dx = \ln|\sec(x)| + C, \quad \int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C \end{array}}$$

Strategies:

- If power of tan is odd:
  - → split off a factor $\sec(x)\tan(x)$ and rewrite the remaining factors in terms of sec using $\tan(x)^2 = \sec(x)^2 - 1$
  - → do the u-sub $u = \sec(x)$

- If power of sec is even and positive:
  - → split off a factor $\sec(x)^2$ and rewrite the remaining factors in terms of $\tan(x)$ using $\sec(x)^2 = \tan(x)^2 + 1$
  - → do the u-sub $u = \tan(x)$

- If power of tan is even and power of sec is odd or 0:
  - → rewrite integrand as sum of powers of sec
  - → use IBP with $dv = \sec(x)^2 dx$ and rewrite $\tan(x)^2$ as $\sec(x)^2 - 1$ in resulting integrals.

→ solve for original integral and repeat for unknown integrals.

Examples:

1) $\int \tan(x)^3 \sec(x)^7 dx$ power of tan is odd: split off sec tan

$$= \int \tan(x)^2 \sec(x)^6 \tan(x)\sec(x)\,dx$$

rewrite remaining factors in terms of $\tan(x)$

$$= \int (\sec(x)^2 - 1)\sec(x)^6 \tan(x)\sec(x)\,dx$$

u-sub
$u = \sec(x)$, $du = \sec(x)\tan(x)dx$

$$= \int (u^2 - 1)u^6\,du$$

$$= \int (u^8 - u^6)\,du = \frac{u^9}{9} - \frac{u^7}{7} + C$$

$$= \boxed{\frac{\sec(x)^9}{9} - \frac{\sec(x)^7}{7} + C}$$

2) $\int \sec(x)^3 dx$ power of sec is odd, power of tan even
⇒ use IBP to get reduction formula

$$= \int \underbrace{\sec(x)}_{u}\,\underbrace{\sec(x)^2 dx}_{dv}$$

IBP

$u = \sec(x)$	$dv = \sec(x)^2 dx$
$du = \sec(x)\tan(x)\,dx$	$v = \tan(x)$

$$= \sec(x)\tan(x) - \int \sec(x)\underbrace{\tan(x)^2}_{\sec(x)^2-1}dx$$

$$= \sec(x)\tan(x) - \int \sec(x)^3 dx + \int \sec(x)\,dx$$

$$= \sec(x)\tan(x) - \int \sec(x)^3 dx + \ln|\tan(x) + \sec(x)|$$

we recover the original integral
⇒ we solve for $\int \sec(x)^3 dx$

$$2\int \sec(x)^3 dx = \sec(x)\tan(x) + \ln|\tan(x)+\sec(x)|$$

$$\boxed{\int \sec(x)^3 dx = \frac{1}{2}\Big(\sec(x)\tan(x) + \ln|\tan(x)+\sec(x)|\Big) + C}$$

3) $\int \tan(x)^4 dx$

Method 1 : $\int \tan(x)^4 dx = \int (\tan(x)^2)^2 dx = \int (\sec(x)^2-1)^2 dx$

$$= \int \left(\sec(x)^4 - 2\sec(x)^2 + 1\right) dx$$

$$= \int \sec(x)^4 dx - 2\tan(x) + x$$

compute this one with method for even power of sec.

$$\int \sec(x)^4 dx = \int \sec(x)^2 \sec(x)^2 dx$$

$$= \int (\tan(x)^2+1) \sec(x)^2 dx$$

u- sub

$u = \tan(x)$

$du = \sec(x)^2 dx$

$$= \int (u^2+1)\, du$$

$$= \frac{u^3}{3} + u = \frac{\tan(x)^3}{3} + \tan(x)$$

So $\int \tan(x)^4 dx = \int \sec(x)^4 dx - 2\tan(x) + x$

$$= \frac{\tan(x)^3}{3} + \tan(x) - 2\tan(x) + x$$

$$= \boxed{\frac{\tan(x)^3}{3} - \tan(x) + x + C}$$

Method 2 :

$$\int \tan(x)^4 dx = \int \tan(x)^2 \tan(x)^2 dx$$

$$= \int \tan(x)^2 \left(\sec(x)^2 - 1\right) dx$$

$$= \underbrace{\int \tan(x)^2 \sec(x)^2 dx}_{} - \int \tan(x)^2 dx$$

$u = \tan(x)$
$du = \sec(x)^2 dx$

$$= \int u^2 du - \int \left(\sec(x)^2 - 1\right) dx$$

$$= \frac{u^3}{3} - \tan(x) + x$$

$$= \boxed{\frac{\tan(x)^3}{3} - \tan(x) + x + C}$$

Practice:

1) Express $\int \cos(x)^n\,dx$ in terms of $\int \cos(x)^{n-2}\,dx$.
2) Express $\int \sec(5x)^n\,dx$ in terms of $\int \sec(5x)^{n-2}\,dx$
   Use this to compute $\int \sec(5x)^5\,dx$.
3) Calculate $\int \tan(x)^{11}\sec(x)^4\,dx$
4) Calculate $\int \cos(x)\ln(\cos(x))\,dx$

Solutions:

IBP

$u = \cos(x)^{n-1}$

$du = -(n-1)\cos(x)^{n-2}\sin(x)\,dx$

$dv = \cos(x)\,dx \Rightarrow v = \sin(x)$

1) $$\int \cos(x)^n\,dx = \int \underbrace{\cos(x)^{n-1}}_{u}\underbrace{\cos(x)\,dx}_{dv}$$

$$= \cos(x)^{n-1}\sin(x) + (n-1)\int \cos(x)^{n-2}\underbrace{\sin(x)^2}_{1-\cos(x)^2}\,dx$$

$$= \cos(x)^{n-1}\sin(x) + (n-1)\int \cos(x)^{n-2}\,dx - (n-1)\int \cos(x)^n\,dx$$

recover the original integral
$\Rightarrow$ solve for $\int \cos(x)^n\,dx$

$$n\int \cos(x)^n\,dx = \cos(x)^{n-1}\sin(x) + n-1\int \cos(x)^{n-2}\,dx$$

$$\boxed{\int \cos(x)^n\,dx = \frac{\cos(x)^{n-1}\sin(x)}{n} + \frac{n-1}{n}\int \cos(x)^{n-2}\,dx}$$

2) $\int \sec(5x)^n\,dx$

IBP

$u = \sec(5x)^{n-2}$

$du = 5(n-2)\sec(5x)^{n-2}\tan(5x)\,dx$

$dv = \sec(5x)^2 dx \Rightarrow v = \frac{1}{5}\tan(5x)$

$$= \int \underbrace{\sec(5x)^{n-2}}_{u}\underbrace{\sec(5x)^2\,dx}_{dv}$$

$$= \frac{\sec(5x)^{n-2}\tan(5x)}{5} - (n-2)\int \sec(5x)^{n-2}\underbrace{\tan(5x)^2}_{\sec(5x)^2-1}\,dx$$

$$= \frac{\sec(5x)^{n-2}\tan(5x)}{5} - (n-2)\int \sec(5x)^n\,dx + (n-2)\int \sec(5x)^{n-2}\,dx$$

recover the original integral
$\Rightarrow$ solve for $\int \sec(5x)^n dx$

$$(n-1)\int \sec(5x)^n\,dx = \frac{\sec(5x)^{n-2}\tan(5x)}{5} + (n-2)\int \sec(5x)^{n-2}\,dx$$

$$\boxed{\int \sec(5x)^n\,dx = \frac{\sec(5x)^{n-2}\tan(5x)}{5(n-1)} + \frac{n-2}{n-1}\int \sec(5x)^{n-2}\,dx}$$

We now compute $\int \sec(5x)^5 dx$ using the reduction formula we found for $n=5$:

$$\int \sec(5x)^5\,dx = \frac{\sec(5x)^3\tan(5x)}{20} + \frac{3}{4}\int \sec(5x)^3\,dx$$

use the reduction formula for $n=3$

$$= \frac{\sec(5x)^3\tan(5x)}{20} + \frac{3}{4}\left(\frac{\sec(5x)\tan(5x)}{10} + \frac{1}{2}\int \sec(5x)\,dx\right)$$

$$= \boxed{\frac{\sec(5x)^3\tan(5x)}{20} + \frac{3\sec(5x)\tan(5x)}{40} + \frac{3\ln|\sec(5x)+\tan(5x)|}{40} + C}$$

3) $\int \tan(x)^{11}\sec(x)^4\,dx$ power of sec is even: split off a $\sec^2$

$$= \int \tan(x)^{11}\sec(x)^2\sec(x)^2\,dx$$

rewrite remaining factors in terms of tan

$$= \int \tan(x)^{11}(\tan(x)^2+1)\sec(x)^2\,dx$$

u-sub
$u = \tan(x)$, $du = \sec(x)^2 dx$

$$= \int u^{11}(u^2+1)\,du$$

$$= \int (u^{13}+u^{11})\,du = \frac{u^{14}}{14} + \frac{u^{12}}{12} + C$$

$$= \boxed{\frac{\tan(x)^{14}}{14} + \frac{\tan(x)^{12}}{12} + C}$$

IBP

$u = \ln(\cos(x))$	$dv = \cos(x)\,dx$
$du = -\frac{\sin(x)}{\cos(x)}\,dx$	$v = \sin(x)$

4) $\int \cos(x)\ln(\cos(x))\,dx$

$$= \ln(\cos(x))\sin(x) + \int \frac{\sin(x)^2}{\cos(x)}\,dx$$

$$= \ln(\cos(x))\sin(x) + \int \frac{1-\cos(x)^2}{\cos(x)}\,dx$$

$$= \ln(\cos(x))\sin(x) + \int (\sec(x) - \cos(x))\,dx$$

$$= \boxed{\ln(\cos(x))\sin(x) + \ln|\sec(x)+\tan(x)| - \sin(x) + C}$$