Rutgers University
Math 152

Section 8.3: Trigonometric Integrals - Worksheet Solutions

1. Calculate the following integrals.

(a) $\int \sin(5x)^2 dx$

Solution. We use the double angle formula

$$\sin(\theta)^2 = \frac{1-\cos(2\theta)}{2},$$

which gives

$$\begin{aligned}\int \sin(5x)^2 dx &= \int \frac{1-\cos(10x)}{2} dx \\ &= \boxed{\frac{x}{2} - \frac{\sin(10x)}{20} + C}.\end{aligned}$$

(b) $\int \sec(2x)^4 \tan(2x)^6 dx$

Solution. The exponent of secant is even, so we can split off a factor $\sec(2x)^2$, rewrite the rest of the integrand in terms of $\tan(2x)$ using the Pythagorean identity $\sec(2x)^2 = \tan(2x)^2 + 1$, and use the substitution $u = \tan(2x)$, $du = 2\sec(2x)^2 dx$. This gives

$$\begin{aligned}\int \sec(2x)^4 \tan(2x)^6 dx &= \int \sec(2x)^2 \tan(2x)^6 \sec(2x)^2 dx \\ &= \int (\tan(2x)^2 + 1) \tan(2x)^6 \sec(2x)^2 dx \\ &= \int (u^2+1)u^6 \frac{du}{2} \\ &= \frac{1}{2}\int (u^8 + u^6) du \\ &= \frac{1}{2}\left(\frac{u^9}{9} + \frac{u^7}{7}\right) + C \\ &= \boxed{\frac{1}{2}\left(\frac{\tan(2x)^9}{9} + \frac{\tan(2x)^7}{7}\right) + C}.\end{aligned}$$

(c) $\int_0^{\pi/21} \tan(7\theta)^3 d\theta$

Solution. We can split off a factor $\tan(7\theta)^2$, replace it by $\sec(7\theta)^2 - 1$ and distribute. This gives

$$\begin{aligned}\int_0^{\pi/21} \tan(7\theta)^3 d\theta &= \int_0^{\pi/21} \tan(7\theta)\tan(7\theta)^2 d\theta \\ &= \int_0^{\pi/21} \tan(7\theta)\left(\sec(7\theta)^2 - 1\right) d\theta \\ &= \int_0^{\pi/21} \tan(7\theta)\sec(7\theta)^2 d\theta - \int_0^{\pi/21} \tan(7\theta) d\theta.\end{aligned}$$

The first integral can be evaluated using the substitution $u = \tan(7\theta)$, which gives $du = 7\sec(7\theta)^2 d\theta$. The bounds become

$$\begin{aligned}\theta = 0 &\Rightarrow u = \tan(0) = 0, \\ \theta = \frac{\pi}{21} &\Rightarrow u = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}.\end{aligned}$$

So we get

$$\begin{aligned}\int_0^{\pi/21} \tan(7\theta)^3 d\theta &= \int_0^{\sqrt{3}} \frac{u}{7} du - \int_0^{\pi/21} \tan(7\theta) d\theta \\ &= \left[\frac{u^2}{21}\right]_0^{\sqrt{3}} - \left[\frac{\ln|\sec(7\theta)|}{7}\right]_0^{\pi/21} \\ &= \frac{\sqrt{3}^2}{21} - \frac{1}{7}\left(\ln\left(\sec\left(\frac{\pi}{3}\right)\right) - \ln(\sec(0))\right) \\ &= \boxed{\frac{1 - \ln(2)}{7}}.\end{aligned}$$

(d) $\displaystyle\int \sec(3x)^2 \ln(\sec(3x)) dx$

Solution. We can start with an IBP with parts

$$\begin{aligned}u = \ln(\sec(3x)) &\Rightarrow du = \frac{3\sec(3x)\tan(3x)}{\sec(3x)} dx = 3\tan(3x) dx, \\ dv = \sec(3x)^2 dx &\Rightarrow v = \frac{1}{3}\tan(3x).\end{aligned}$$

We get

$$\begin{aligned}\int \sec(3x)^2 \ln(\sec(3x)) dx &= \frac{\tan(3x)\ln(\sec(3x))}{3} - \int 3\tan(3x)\frac{\tan(3x)}{3} dx \\ &= \frac{\tan(3x)\ln(\sec(3x))}{3} - \int \tan(3x)^2 dx.\end{aligned}$$

This last integral can be computed using the Pythagorean identity $\tan(3x)^2 = \sec(3x)^2 - 1$, which gives

$$\begin{aligned}\int \sec(3x)^2 \ln(\sec(3x)) dx &= \frac{\tan(3x)\ln(\sec(3x))}{3} - \int \left(\sec(3x)^2 - 1\right) dx \\ &= \boxed{\frac{\tan(3x)\ln(\sec(3x))}{3} - \frac{\tan(3x)}{3} - x + C}.\end{aligned}$$

(e) $\displaystyle\int_{\pi}^{3\pi/2} \cos(z)^5 \sin(z)^8 dz$

Solution. Since the power of cos is odd, we can compute this integral by splitting off a factor $\cos(z)$, rewriting the remaining factors in terms of $\sin(z)$ using the Pythagorean identity $\cos(z)^2 = 1-\sin(z)^2$ and substituting $u = \sin(z)$, $du = \cos(z)dz$. The bounds will change to

$$\begin{aligned} z = \pi &\Rightarrow u = \sin(\pi) = 0, \\ z = \frac{3\pi}{2} &\Rightarrow u = \sin\left(\frac{3\pi}{2}\right) = -1. \end{aligned}$$

This yields

$$\begin{aligned} \int_{\pi}^{3\pi/2} \cos(z)^5 \sin(z)^8 dz &= \int_{\pi}^{3\pi/2} \cos(z)^4 \sin(z)^8 \cos(z) dz \\ &= \int_{\pi}^{3\pi/2} \left(1-\sin(z)^2\right)^2 \sin(z)^8 \cos(z) dz \\ &= \int_{0}^{-1} \left(1-u^2\right)^2 u^8 du \\ &= \int_{0}^{-1} \left(u^8 - u^{10}\right) du \\ &= \left[\frac{u^9}{9} - \frac{u^{11}}{11}\right]_0^{-1} \\ &= \frac{(-1)^9}{9} - \frac{(-1)^{11}}{11} \\ &= \boxed{-\frac{2}{99}}. \end{aligned}$$

(f) $\displaystyle\int_{\pi/3}^{\pi/2} \sqrt{\frac{1+\cos(t)}{1-\cos(t)}} dt$

Solution. We can rewrite the inside of the square root as a perfect square using the double angle formulas. From

$$\cos(\theta)^2 = \frac{1+\cos(2\theta)}{2}, \quad \sin(\theta)^2 = \frac{1-\cos(2\theta)}{2},$$

we obtain

$$1+\cos(t) = 2\cos\left(\frac{t}{2}\right)^2, \quad 1-\cos(t) = 2\sin\left(\frac{t}{2}\right)^2.$$

Using this for the integral gives

$$\begin{aligned} \int_{\pi/3}^{\pi/2} \sqrt{\frac{1+\cos(t)}{1-\cos(t)}} dt &= \int_{\pi/3}^{\pi/2} \sqrt{\frac{2\cos\left(\frac{t}{2}\right)^2}{2\sin\left(\frac{t}{2}\right)^2}} dt \\ &= \int_{\pi/3}^{\pi/2} \sqrt{\cot\left(\frac{t}{2}\right)^2} dt \\ &= \int_{\pi/3}^{\pi/2} \left|\cot\left(\frac{t}{2}\right)\right| dt \end{aligned}$$

$$
\begin{aligned}
&= \int_{\pi/3}^{\pi/2} \cot\left(\frac{t}{2}\right) dt \\
&= \left[2\ln\left|\sin\left(\frac{t}{2}\right)\right|\right]_{\pi/3}^{\pi/2} \\
&= 2\left(\ln\left(\sin\left(\frac{\pi}{4}\right)\right) - \ln\left(\sin\left(\frac{\pi}{6}\right)\right)\right) \\
&= 2\left(\ln\left(\frac{\sqrt{2}}{2}\right) - \ln\left(\frac{1}{2}\right)\right) \\
&= \boxed{\ln(2)}.
\end{aligned}
$$

2. Express $\int \sin(3x)^n dx$ in terms of $\int \sin(3x)^{n-2} dx$.

Solution. We split off a factor $\sin(3x)$ and use IBP with parts

$$
\begin{aligned}
u &= \sin(3x)^{n-1} \;\Rightarrow\; du = 3(n-1)\sin(3x)^{n-2}\cos(3x)dx, \\
dv &= \sin(3x)dx \;\Rightarrow\; v = -\frac{\cos(3x)}{3}.
\end{aligned}
$$

This gives

$$
\begin{aligned}
\int \sin(3x)^n dx &= \int \sin(3x)^{n-1}\sin(3x)dx \\
\int \sin(3x)^n dx &= -\frac{\sin(3x)^{n-1}\cos(3x)}{3} - \int 3(n-1)\sin(3x)^{n-2}\cos(3x)\frac{-\cos(3x)}{3}dx \\
\int \sin(3x)^n dx &= -\frac{\sin(3x)^{n-1}\cos(3x)}{3} + (n-1)\int \sin(3x)^{n-2}\cos(3x)^2 dx.
\end{aligned}
$$

In this last integral, we use the Phytagorean identity $\cos(3x)^2 = 1 - \sin(3x)^2$ to obtain

$$
\begin{aligned}
\int \sin(3x)^n dx &= -\frac{\sin(3x)^{n-1}\cos(3x)}{3} + (n-1)\int \sin(3x)^{n-2}\left(1 - \sin(3x)^2\right) dx \\
\int \sin(3x)^n dx &= -\frac{\sin(3x)^{n-1}\cos(3x)}{3} + (n-1)\int \sin(3x)^{n-2}dx - (n-1)\int \sin(3x)^n dx
\end{aligned}
$$

We can now solve for the original integral by moving the term $-(n-1)\int \sin(3x)^n dx$ to the left-hand side.

$$
\begin{aligned}
\int \sin(3x)^n dx + (n-1)\int \sin(3x)^n dx &= -\frac{\sin(3x)^{n-1}\cos(3x)}{3} + (n-1)\int \sin(3x)^{n-2}dx \\
n\int \sin(3x)^n dx &= -\frac{\sin(3x)^{n-1}\cos(3x)}{3} + (n-1)\int \sin(3x)^{n-2}dx
\end{aligned}
$$

We now divide by n to obtain the reduction formula

$$
\boxed{\int \sin(3x)^n dx = -\frac{\sin(3x)^{n-1}\cos(3x)}{3n} + \frac{(n-1)}{n}\int \sin(3x)^{n-2}dx}.
$$

3. Consider the region bounded by the x-axis, the graph of $y = \sec(x)^2 \tan(x)$ and the lines $x = 0$, $x = \frac{\pi}{4}$. Calculate the volume of the solid obtained by revolving $\mathcal{R}$ about (a) the x-axis, (b) the y-axis.

Solution. (a) We use the disk method. Revolving the vertical strip at x about the x-axis forms a disk with radius $r(x) = \sec(x)^2 \tan(x)$. Therefore the volume is

$$\begin{aligned} V &= \int_0^{\pi/4} \pi r(x)^2 dx \\ &= \pi \int_0^{\pi/4} \sec(x)^4 \tan(x)^2 dx. \end{aligned}$$

The exponent of secant is even, so we can split off a factor $\sec(x)^2$, rewrite the rest of the integrand in terms of $\tan(x)$ using the Pythagorean identity $\sec(x)^2 = \tan(x)^2 + 1$, and use the substitution $u = \tan(x)$, $du = \sec(x)^2 dx$. The bounds become

$$\begin{aligned} x = 0 &\;\Rightarrow\; u = \tan(0) = 0, \\ x = \frac{\pi}{4} &\;\Rightarrow\; u = \tan\left(\frac{\pi}{4}\right) = 1. \end{aligned}$$

This gives

$$\begin{aligned} V &= \pi \int_0^{\pi/4} \sec(x)^2 \tan(x)^2 \sec(x)^2 dx \\ &= \pi \int_0^{\pi/4} \sec(x)^2 \tan(x)^2 \sec(x)^2 dx \\ &= \pi \int_0^{\pi/4} \left(\tan(x)^2 + 1\right) \tan(x)^2 \sec(x)^2 dx \\ &= \pi \int_0^1 \left(u^2 + 1\right) u^2 du \\ &= \pi \int_0^1 \left(u^4 + u^2\right) du \\ &= \pi \left[\frac{u^5}{5} + \frac{u^3}{3}\right]_0^1 \\ &= \pi \left(\frac{1}{5} + \frac{1}{3}\right) \\ &= \boxed{\frac{8\pi}{15} \text{ cubic units}}. \end{aligned}$$

(b) We use the method of cylindrical shells. Revolving the vertical strip at x about the y-axis forms a shell with radius $r(x) = x$ and height $h(x) = \sec(x)^2 \tan(x)$. So the volume is

$$\begin{aligned} V &= \int_0^{\pi/4} 2\pi r(x) h(x) \\ &= 2\pi \int_0^{\pi/4} x \sec(x)^2 \tan(x) dx. \end{aligned}$$

We can compute this integral with an IBP taking the parts

$$u = x \;\Rightarrow\; du = dx,$$

$$dv = \sec(x)^2 \tan(x) dx \;\Rightarrow\; v = \frac{\tan(x)^2}{2}.$$

We get

$$\begin{aligned} V &= 2\pi \left(\left[\frac{x\tan(x)^2}{2} \right]_0^{\pi/4} - \int_0^{\pi/4} \frac{\tan(x)^2}{2} dx \right) \\ &= 2\pi \left(\frac{\pi}{8} - \frac{1}{2} \int_0^{\pi/4} \left(\sec(x)^2 - 1 \right) dx \right) \\ &= 2\pi \left(\frac{\pi}{8} - \frac{1}{2} \left[\tan(x) - x \right]_0^{\pi/4} \right) \\ &= 2\pi \left(\frac{\pi}{8} - \frac{1}{2} \left(1 - \frac{\pi}{4} \right) \right) \\ &= \boxed{\frac{\pi(\pi - 2)}{2} \text{ cubic units}}. \end{aligned}$$

4. Evaluate $\int \sec(\theta)^3 d\theta$ and $\int \sec(\theta) \tan(\theta)^2 d\theta$.

Solution. We can evaluate $\int \sec(\theta)^3 d\theta$ with an IBP and solving for the unknown integral when it reappears on the right-hand side. For the IBP we use the parts

$$\begin{aligned} u &= \sec(\theta) \;\Rightarrow\; du = \sec(\theta)\tan(\theta) d\theta, \\ dv &= \sec(\theta)^2 d\theta \;\Rightarrow\; v = \tan(\theta). \end{aligned}$$

We get

$$\begin{aligned} \int \sec(\theta)^3 d\theta &= \int \sec(\theta)^2 \sec(\theta) d\theta \\ \int \sec(\theta)^3 d\theta &= \tan(\theta)\sec(\theta) - \int \tan(\theta)\sec(\theta)\tan(\theta) d\theta \\ \int \sec(\theta)^3 d\theta &= \tan(\theta)\sec(\theta) - \int \tan(\theta)^2 \sec(\theta) d\theta \end{aligned}$$

We will use the Pythagorean identity $\tan(\theta)^2 = \sec(\theta)^2 - 1$ to see the original integral reappear on the right-hand side.

$$\begin{aligned} \int \sec(\theta)^3 d\theta &= \tan(\theta)\sec(\theta) - \int \left(\sec(\theta)^2 - 1 \right) \sec(\theta) d\theta \\ \int \sec(\theta)^3 d\theta &= \tan(\theta)\sec(\theta) - \int \sec(\theta)^3 d\theta + \int \sec(\theta) d\theta \\ \int \sec(\theta)^3 d\theta &= \tan(\theta)\sec(\theta) - \int \sec(\theta)^3 d\theta + \ln\left|\sec(\theta) + \tan(\theta)\right| \end{aligned}$$

We can now move the term $-\int \sec(\theta)^3 d\theta$ to the left hand side and finish solving

$$2\int \sec(\theta)^3 d\theta = \tan(\theta)\sec(\theta) + \ln\left|\sec(\theta) + \tan(\theta)\right|$$

$$\Rightarrow \int \sec(\theta)^3 d\theta = \boxed{\frac{1}{2}\left(\tan(\theta)\sec(\theta) + \ln|\sec(\theta) + \tan(\theta)|\right) + C}.$$

For the other integral, we can use the Pythagorean identity and the integral of $\sec^3$ that we just computed as follows:

$$\begin{aligned}
\int \tan(\theta)^2 \sec(\theta) d\theta &= \int \left(\sec(\theta)^2 - 1\right)\sec(\theta) d\theta \\
&= \int \sec(\theta)^3 d\theta - \int \sec(\theta) d\theta \\
&= \frac{1}{2}\left(\tan(\theta)\sec(\theta) + \ln|\sec(\theta) + \tan(\theta)|\right) - \ln|\sec(\theta) + \tan(\theta)| \\
&= \boxed{\frac{1}{2}\left(\tan(\theta)\sec(\theta) - \ln|\sec(\theta) + \tan(\theta)|\right) + C}.
\end{aligned}$$

5. Calculate the arc length of the curve $y = x + \cos(x)\sin(x) - \frac{1}{8}\tan(x)$, $0 \leqslant x \leqslant \frac{\pi}{4}$.

Solution. We have

$$\begin{aligned}
\frac{dy}{dx} &= 1 - \sin(x)^2 + \cos(x)^2 - \frac{1}{8}\sec(x)^2 \\
&= \cos(x)^2 + \cos(x)^2 - \frac{1}{8}\sec(x)^2 \\
&= 2\cos(x)^2 - \frac{1}{8}\sec(x)^2.
\end{aligned}$$

So

$$\begin{aligned}
1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \left(2\cos(x)^2 - \frac{1}{8}\sec(x)^2\right)^2 \\
&= 1 + 4\cos(x)^4 + \frac{1}{64}\sec(x)^4 - 2\cdot 2\cos(x)^2 \cdot \frac{1}{8}\sec(x)^2 \\
&= 1 + 4\cos(x)^4 + \frac{1}{64}\sec(x)^4 - \frac{1}{2} \\
&= 4\cos(x)^4 + \frac{1}{64}\sec(x)^4 + \frac{1}{2} \\
&= 4\cos(x)^4 + \frac{1}{64}\sec(x)^4 + 2\cdot 2\cos(x)^2 \cdot \frac{1}{8}\sec(x)^2 \\
&= \left(2\cos(x)^2 + \frac{1}{8}\sec(x)^2\right)^2.
\end{aligned}$$

Therefore the arc length is given by

$$\begin{aligned}
L &= \int_0^{\pi/4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
&= \int_0^{\pi/4} \sqrt{\left(2\cos(x)^2 + \frac{1}{8}\sec(x)^2\right)^2} dx \\
&= \int_0^{\pi/4} \left|2\cos(x)^2 + \frac{1}{8}\sec(x)^2\right| dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\pi/4} \left(2\cos(x)^2 + \frac{1}{8}\sec(x)^2\right) dx \\
&= \int_0^{\pi/4} \left(2\frac{1+\cos(2x)}{2} + \frac{1}{8}\sec(x)^2\right) dx \\
&= \int_0^{\pi/4} \left(1+\cos(2x) + \frac{1}{8}\sec(x)^2\right) dx \\
&= \left[x + \frac{1}{2}\sin(2x) + \frac{1}{8}\tan(x)\right]_0^{\pi/4} \\
&= \left(\frac{\pi}{4} + \frac{1}{2}\sin\left(\frac{\pi}{2}\right) + \frac{1}{8}\tan\left(\frac{\pi}{4}\right)\right) - 0 \\
&= \boxed{\frac{\pi}{4} + \frac{5}{8}\ \text{units}}.
\end{aligned}
$$