Section 8.4

Trigonometric Substitution

Learning Goals

Learning Goal	*Homework Problems*
8.4.1 Evaluate an indefinite and a definite integral containing radical using trigonometric substitution	8.4: 5,7,9,11,17,19,20,21,23,25,33,53
8.4.2 Evaluate an indefinite and a definite integral containing radical by first completing the square and then trigonometric substitution	8.4: 49, 51
8.4.3 Evaluate an indefinite and a definite integral not containing radicals using trigonometric substitution	8.4: 16
8.4.4 Evaluate an indefinite integral by first substitution and then trigonometric substitution	8.4: 29,35,37,43,44,61,63

Goal: integrate expressions involving powers of

$$\sqrt{a^2-b^2x^2},\ \sqrt{a^2+b^2x^2} \text{ or } \sqrt{b^2x^2-a^2} \quad (a,b>0 \text{ constants}).$$

We try to choose a suitable substitution making the inside of the square root a perfect square.

① Expressions involving $\sqrt{a^2-b^2x^2}$: sine substitution.

We will want to replace

$$a^2 - b^2x^2 = a^2 - a^2\sin(\theta)^2 \overset{\text{PYTHAGOREAN IDENTITY}}{=} a^2\cos(\theta)^2 \quad \text{(perfect square)}$$

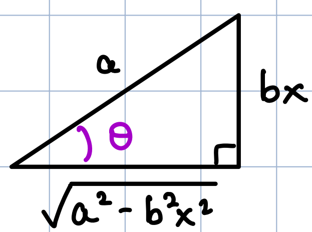
For this, we need $b^2x^2 = a^2\sin(\theta)^2$, so:

- Substitute $x = \frac{a}{b}\sin(\theta)$ with $\underbrace{-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}}$.

$$\Rightarrow dx = \frac{a}{b}\cos(\theta)\,d\theta$$

- Then

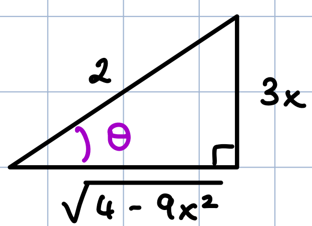
$$\sqrt{a^2-b^2x^2} = \sqrt{a^2-a^2\sin(\theta)^2} = \sqrt{a^2\cos(\theta)^2} = a|\cos(\theta)| \underset{\cos(\theta)>0}{=} a\cos(\theta)$$



Examples: 1) $\displaystyle\int \frac{dx}{(4-9x^2)^{3/2}}$

We will want $4-9x^2 = 4-4\sin(\theta)^2 \Rightarrow 9x^2 = 4\sin(\theta)^2$

$\Rightarrow$ substitute $x = \frac{2}{3}\sin(\theta)$, $dx = \frac{2}{3}\cos(\theta)\,d\theta$



Then $\sqrt{4-9x^2} = \sqrt{4-4\sin(\theta)^2} = 2\cos(\theta)$.

The integral becomes

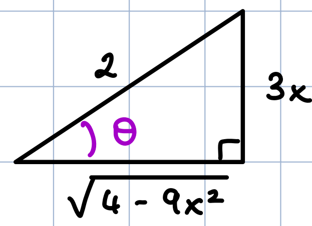
$$\int \frac{dx}{(4-9x^2)^{3/2}} = \int \frac{dx}{\left(\sqrt{4-9x^2}\right)^3} = \int \frac{\frac{2}{3}\cos(\theta)\,d\theta}{(2\cos(\theta))^3} = \frac{1}{12}\int \frac{d\theta}{\cos(\theta)^2}$$

$$= \frac{1}{12}\int \sec(\theta)^2\, d\theta = \frac{1}{12}\tan(\theta) + C$$

USE THE RIGHT TRIANGLE TO EXPRESS IN TERMS OF x.

$$= \frac{1}{12}\cdot\frac{3x}{\sqrt{4-9x^2}} + C$$

$$= \boxed{\frac{x}{4\sqrt{4-9x^2}} + C}.$$

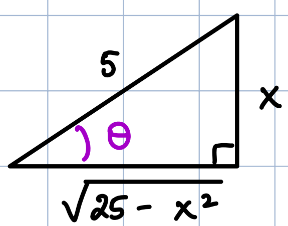


$\tan(\theta) = \frac{\text{opp}}{\text{adj}} = \frac{3x}{\sqrt{4-9x^2}}$

2) $\int \sqrt{25-x^2}\, dx$

We want $25 - x^2 = 25 - 25\sin(\theta)^2$

$\Rightarrow$ substitute $x = 5\sin(\theta)$, $dx = 5\cos(\theta)\, d\theta$



Then $\sqrt{25-x^2} = \sqrt{25 - 25\sin(\theta)^2} = 5\cos(\theta)$.

So $\int \sqrt{25-x^2}\, dx = \int 5\cos(\theta)\cdot 5\cos(\theta)\, d\theta$

USE DOUBLE-ANGLE FORMULA

$$= 25\int \cos(\theta)^2\, d\theta \qquad \cos(\theta)^2 = \frac{1+\cos(2\theta)}{2}$$

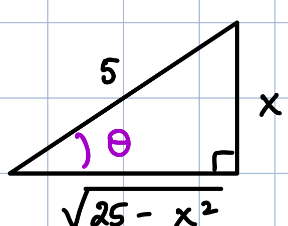
$$= 25\int \frac{1+\cos(2\theta)}{2}\, d\theta$$

$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$$= \frac{25}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right) + C$$

$$= \frac{25}{2}\left(\theta + \sin(\theta)\cos(\theta)\right) + C$$

USE THE RIGHT TRIANGLE TO EXPRESS IN TERMS OF x.



$\sin(\theta) = \frac{x}{5}$, $\cos(\theta) = \frac{\sqrt{25-x^2}}{5}$

$$= \frac{25}{2}\left(\sin^{-1}\left(\frac{x}{5}\right) + \frac{x}{5}\cdot\frac{\sqrt{25-x^2}}{5}\right) + C$$

$$= \boxed{\frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right) + \frac{x\sqrt{25-x^2}}{2} + C}.$$

② Expressions involving $\sqrt{a^2+b^2x^2}$: tangent substitution

We will want to replace

PYTHAGOREAN IDENTITY

$$a^2 + b^2x^2 = a^2 + a^2\tan(\theta)^2 = a^2\sec(\theta)^2 \quad \text{(perfect square)}$$

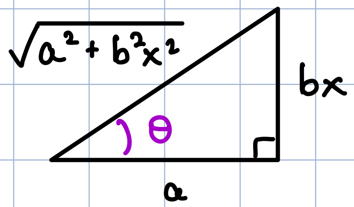
For this, we need $b^2x^2 = a^2\tan(\theta)^2$, so :

- Substitute $x = \frac{a}{b}\tan(\theta)$ with $\underbrace{-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}}$.

  $\Rightarrow dx = \frac{a}{b}\sec(\theta)^2 d\theta$

- Then

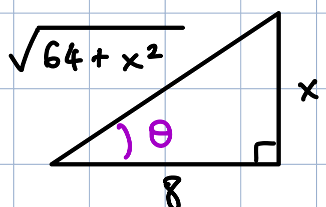
$$\sqrt{a^2+b^2x^2} = \sqrt{a^2+a^2\tan(\theta)^2}$$
$$= \sqrt{a^2\sec(\theta)^2}$$
$$= a|\sec(\theta)| \qquad \sec(\theta) > 0$$
$$= a\sec(\theta)$$



Examples : 1) $\displaystyle\int_0^8 \frac{dx}{\sqrt{64+x^2}} = \int_0^8 \frac{dx}{\sqrt{8^2+x^2}}$

We want $64 + x^2 = 64 + 64\tan(\theta)^2$

So we substitute $x = 8\tan(\theta)$, $dx = 8\sec(\theta)^2 d\theta$



Then $\sqrt{64+x^2} = \sqrt{64+64\tan(\theta)^2} = 8\sec(\theta)$

Because this is a definite integral, we also need to write the bounds in term of θ:

$x = 0 \Rightarrow 0 = 8\tan(\theta) \Rightarrow \theta = \tan^{-1}(0) = 0$

$x = 8 \Rightarrow 8 = 8\tan(\theta) \Rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}$.

So $\displaystyle\int_0^8 \frac{dx}{\sqrt{64+x^2}} = \int_0^{\pi/4} \frac{8\sec(\theta)^2 d\theta}{8\sec(\theta)} = \int_0^{\pi/4} \sec(\theta)\, d\theta$

$$= \Big[\ln|\sec(\theta) + \tan(\theta)|\Big]_0^{\pi/4} = \boxed{\ln(\sqrt{2}+1)}.$$

<u>Remark</u>: if we computed the indefinite integral, we would have

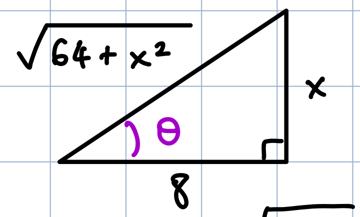
$$\int \frac{dx}{\sqrt{64+x^2}} = \int \sec(\theta)\,d\theta$$

$$= \ln|\sec(\theta) + \tan(\theta)| + C$$

USE THE RIGHT TRIANGLE TO EXPRESS IN TERMS OF x.

$$= \ln\left|\frac{\sqrt{64+x^2}}{8} + \frac{x}{8}\right| + C$$

$$= \boxed{\ln\left|\sqrt{64+x^2} + x\right| + C}.$$



$$\sec(\theta) = \frac{\text{hyp.}}{\text{adj.}} = \frac{\sqrt{64+x^2}}{8}$$

$$\tan(\theta) = \frac{\text{opp.}}{\text{adj.}} = \frac{x}{8}$$

2) $\displaystyle\int \frac{dx}{(x^2-6x+11)^2}$

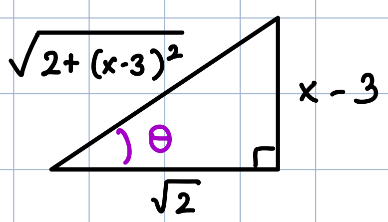
Complete the square first to see the form $a^2+b^2x^2$:

$$x^2 - 6x + 11 = (x^2-6x+9) - 9 + 11 = (x-3)^2 + 2$$

We want $2 + (x-3)^2 = 2 + 2\tan(\theta)^2$

$\Rightarrow$ Substitute $x-3 = \sqrt{2}\tan(\theta)$ or $x = 3 + \sqrt{2}\tan(\theta)$

$$dx = \sqrt{2}\sec(\theta)^2\,d\theta$$



Then $(x-3)^2 + 2 = 2\tan(\theta)^2 + 2 = 2\sec(\theta)^2$.

So $$\int \frac{dx}{(x^2-6x+11)^2} = \int \frac{dx}{((x-3)^2+2)^2} = \int \frac{\sqrt{2}\sec(\theta)^2\,d\theta}{(2\sec(\theta)^2)^4}$$

$$= \frac{\sqrt{2}}{4}\int \frac{d\theta}{\sec(\theta)^2} = \frac{\sqrt{2}}{4}\int \cos(\theta)^2\,d\theta$$

USE DOUBLE-ANGLE FORMULA

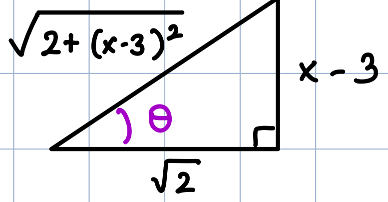
$$\cos(\theta)^2 = \frac{1+\cos(2\theta)}{2}$$

$$= \frac{\sqrt{2}}{4} \int \frac{1+\cos(2\theta)}{2}\, d\theta = \frac{\sqrt{2}}{8}\left(\theta + \frac{\sin(2\theta)}{2}\right) + C$$

USE THE RIGHT TRIANGLE TO EXPRESS IN TERMS OF x.

$$= \frac{\sqrt{2}}{8}\left(\theta + \sin(\theta)\cos(\theta)\right) + C$$

$$= \boxed{\frac{\sqrt{2}}{8}\left(\tan^{-1}\left(\frac{x-3}{\sqrt{2}}\right) + \frac{\sqrt{2}(x-3)}{(x-3)^2+2}\right) + C}$$



$$\sin(\theta) = \frac{\text{opp.}}{\text{hyp.}} = \frac{x-3}{\sqrt{2+(x-3)^2}}$$

$$\cos(\theta) = \frac{\text{adj.}}{\text{hyp.}} = \frac{\sqrt{2}}{\sqrt{2+(x-3)^2}}$$

③ Expressions involving $\sqrt{b^2x^2 - a^2}$ ($x \geq \frac{a}{b}$ only) : secant substitution.

We will want to replace

PYTHAGOREAN IDENTITY

$$b^2x^2 - a^2 = a^2\sec(\theta)^2 - a^2 = a^2\tan(\theta)^2 \quad \text{(perfect square)}$$

For this, we need $b^2x^2 = a^2\sec(\theta)^2$, so :

- Substitute $x = \frac{a}{b}\sec(\theta)$ with $0 \leq \theta \leq \frac{\pi}{2}$.

  $\Rightarrow dx = \frac{a}{b}\sec(\theta)\tan(\theta)\,d\theta$

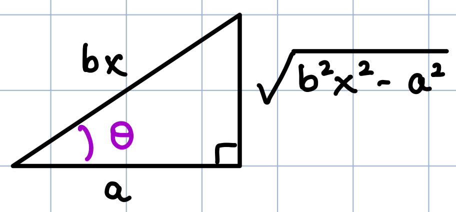
- Then $\sqrt{b^2x^2 - a^2} = \sqrt{a^2\sec(\theta)^2 - a^2}$

$$= \sqrt{a^2\tan(\theta)^2}$$

$$= a\,|\tan(\theta)|$$

$$= a\tan(\theta)$$

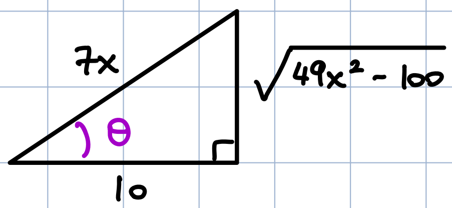
$\tan(\theta) > 0$



Examples : 1) $\displaystyle\int \frac{dx}{x^2\sqrt{49x^2 - 100}}$ for $x > \frac{10}{7}$

We want $49x^2 - 100 = 100\sec(\theta)^2 - 100$

$\Rightarrow$ Substitute $\quad x = \frac{10}{7}\sec(\theta)$, $\quad dx = \frac{10}{7}\sec(\theta)\tan(\theta)\,d\theta$



Then $\sqrt{49x^2-100} = \sqrt{100\sec(\theta)^2-100} = 10\tan(\theta)$

Then $$\int \frac{dx}{x^2\sqrt{49x^2-100}} = \int \frac{\frac{10}{7}\sec(\theta)\tan(\theta)\,d\theta}{\left(\frac{10}{7}\sec(\theta)\right)^2\left(10\tan(\theta)\right)} = \frac{7}{100}\int \frac{d\theta}{\sec(\theta)}$$

$$= \frac{7}{100}\int \cos(\theta)\,d\theta = \frac{7}{100}\sin(\theta) + C$$

USE THE RIGHT TRIANGLE TO EXPRESS IN TERMS OF x.

$$= \frac{7}{100}\cdot\frac{\sqrt{49x^2-100}}{7x} + C$$

$$= \boxed{\frac{\sqrt{49x^2-100}}{100x} + C}.$$

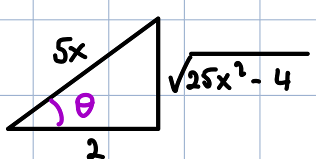
2) $$\int \frac{dx}{\sqrt{25x^2-4}} = \int \frac{dx}{\sqrt{(5x)^2-2^2}}$$ substitute $5x = 2\sec(\theta)$

$5dx = 2\sec(\theta)\tan(\theta)\,d\theta$

$\sqrt{(5x)^2-2^2} = 2\tan(\theta)$

$$= \int \frac{2\sec(\theta)\tan(\theta)\,d\theta}{5\cdot 2\tan(\theta)}$$

$$= \frac{1}{5}\int \sec(\theta)\,d\theta = \frac{1}{5}\ln\left|\sec(\theta)+\tan(\theta)\right| + C$$



$\sec(\theta) = \frac{5x}{2} \Rightarrow \tan(\theta) = \frac{\sqrt{25x^2-4}}{2}$

So $$\int \frac{dx}{\sqrt{25x^2-4}} = \frac{1}{5}\ln\left|\frac{5x}{2}+\frac{\sqrt{25x^2-4}}{2}\right| + C$$

$$= \boxed{\frac{1}{5}\ln\left|5x+\sqrt{25x^2-4}\right| + C}$$

Practice:

1) Show that $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)+C$, $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

using trigonometric substitution.

2) Calculate $\int \frac{dx}{x^2\sqrt{1-9x^2}}$.

3) Calculate $\int \frac{x^2}{\sqrt{x^2+4}}dx$ and $\int \frac{x^3}{\sqrt{x^2+4}}dx$.

Solutions:

1) $\int \frac{dx}{\sqrt{a^2-x^2}}$

substitute
$$x = a\sin(\theta)$$
$$dx = a\cos(\theta)\,d\theta$$
$$\sqrt{a^2-x^2} = a\cos(\theta)$$

$$= \int \frac{a\cos(\theta)\,d\theta}{a\cos(\theta)}$$

$$= \int d\theta = \theta + C$$

$\sin(\theta) = \frac{x}{a}$ so $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

$$= \boxed{\sin^{-1}\left(\frac{x}{a}\right) + C}$$

$\int \frac{dx}{a^2+x^2}$

substitute
$$x = a\tan(\theta)$$
$$dx = a\sec(\theta)^2 d\theta$$
$$a^2+x^2 = a^2\sec(\theta)^2$$

$$= \int \frac{a\sec(\theta)^2 d\theta}{a^2\sec(\theta)^2}$$

$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C$$

$\tan(\theta) = \frac{x}{a}$ so $\theta = \tan^{-1}\left(\frac{x}{a}\right)$.

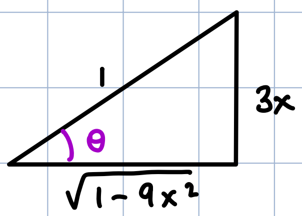
$$= \boxed{\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C}$$

2) $$\int \frac{dx}{x^2\sqrt{1-9x^2}} = \int \frac{\frac{1}{3}\cos(\theta)d\theta}{\left(\frac{1}{3}\sin(\theta)\right)^2\cos(\theta)} = 3\int \frac{d\theta}{\sin(\theta)^2}$$

Sub. $x = \frac{1}{3}\sin(\theta)$, $dx = \frac{1}{3}\cos(\theta)d\theta$

$\sqrt{1-9x^2} = \cos(\theta)$

$$= 3\int \csc(\theta)^2 d\theta$$



$$= -3\cot(\theta) + C$$

$$= \boxed{-\frac{\sqrt{1-9x^2}}{x} + C}$$

$$\cot(\theta) = \frac{\sqrt{1-9x^2}}{3x}$$

3) $$\int \frac{x^2}{\sqrt{x^2+4}}dx$$

substitute $x = 2\tan(\theta)$

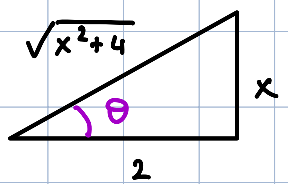
$dx = 2\sec(\theta)^2 d\theta$

$\sqrt{x^2+4} = 2\sec(\theta)$

$$= \int \frac{(2\tan(\theta))^2\, 2\sec(\theta)^2 d\theta}{2\sec(\theta)}$$

$$= 4\int \underbrace{\tan(\theta)^2}_{\sec(\theta)^2-1}\sec(\theta)\,d\theta = 4\underbrace{\int \sec(\theta)^3 d\theta}_{\frac{1}{2}\sec(\theta)\tan(\theta) + \frac{1}{2}\ln|\sec(\theta)+\tan(\theta)|\ \text{(previous section)}} - 4\overbrace{\int \sec(\theta)\,d\theta}^{\ln|\sec(\theta)+\tan(\theta)|}$$

$$= 2\sec(\theta)\tan(\theta) - 2\ln|\sec(\theta)+\tan(\theta)| + C$$



$$\tan(\theta) = \frac{x}{2} \Rightarrow \sec(\theta) = \frac{\sqrt{x^2+4}}{2}$$

So $$\int \frac{x^2}{\sqrt{x^2+4}}dx = \boxed{\frac{1}{2}x\sqrt{x^2+4} - 2\ln|x+\sqrt{x^2+4}| + C}$$

$$\int \frac{x^3}{\sqrt{x^2+4}}dx$$

substitute $x = 2\tan(\theta)$

$dx = 2\sec(\theta)^2 d\theta$

$\sqrt{x^2+4} = 2\sec(\theta)$

$$= \int \frac{(2\tan(\theta))^3\, 2\sec(\theta)^2 d\theta}{2\sec(\theta)}$$

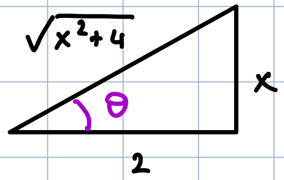
$$= 8 \int \tan(\theta)^3 \sec(\theta)\, d\theta = 8 \int \tan(\theta)^2 \left(\tan(\theta)\sec(\theta)\, d\theta\right)$$

$$= 8 \int \left(\sec(\theta)^2 - 1\right) \tan(\theta) \sec(\theta)\, d\theta$$

u - sub

$u = \sec(\theta)$, $du = \tan(\theta)\sec(\theta)\,d\theta$

$$= 8 \int (u^2 - 1)\, du = \frac{8u^3}{3} - 8u + C = \frac{8\sec(\theta)^3}{3} - 8\sec(\theta) + C$$



$$\tan(\theta) = \frac{x}{2} \Rightarrow \sec(\theta) = \frac{\sqrt{x^2+4}}{2}$$

So $$\boxed{\int \frac{x^3}{\sqrt{x^2+4}}\, dx = \frac{(x^2+4)^{3/2}}{3} - 4\sqrt{x^2+4} + C}$$