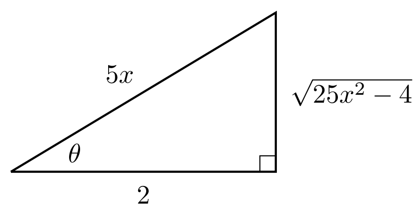
Rutgers University
Math 152

Section 8.4: Trigonometric Substitution - Worksheet Solutions

1. Calculate the following integrals.

(a) $\displaystyle\int \frac{\sqrt{25x^2-4}}{x}dx$ for $x > \frac{2}{5}$.

Solution. We want $25x^2 - 4 = 4\sec(\theta)^2 - 4$, so we substitute $x = \frac{2}{5}\sec(\theta)$ and $dx = \dfrac{2}{5}\sec(\theta)\tan(\theta)d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\sec(\theta) = \frac{5x}{2}$ as shown below.



We get $\sqrt{25x^2-4} = \sqrt{4\sec(\theta)^2 - 4} = 2\tan(\theta)$ and the integral becomes

$$\begin{aligned}\int \frac{\sqrt{25x^2-4}}{x}dx &= \int \frac{2\tan(\theta)}{\frac{2}{5}\sec(\theta)} \cdot \frac{2}{5}\tan(\theta)\sec(\theta)d\theta \\ &= 2\int \tan(\theta)^2 d\theta \\ &= 2\int \left(\sec(\theta)^2 - 1\right) d\theta \\ &= 2\left(\tan(\theta) - \theta\right) + C.\end{aligned}$$

We need to express this result in terms of x. Using the right triangle above, we see that $\tan(\theta) = \frac{\sqrt{25x^2-4}}{2}$ and $\theta = \sec^{-1}\left(\frac{5x}{2}\right)$. Thus

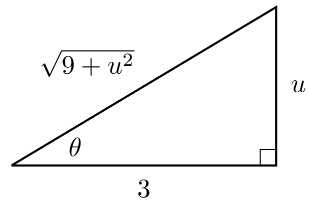
$$\boxed{\int \frac{\sqrt{25x^2-4}}{x}dx = \sqrt{25x^2-4} - 2\sec^{-1}\left(\frac{5x}{2}\right) + C}.$$

(b) $\displaystyle\int \frac{dt}{t\sqrt{9+\ln(t)^2}}$.

Solution. We stat by using the substitution $u = \ln(t)$, which gives $du = \frac{dt}{t}$ and

$$\int \frac{dt}{t\sqrt{9+\ln(t)^2}} = \int \frac{du}{\sqrt{9+u^2}}.$$

We compute this last integral using a trigonometric substitution. We want $9+u^2 = 9+9\tan(\theta)^2$, so we substitute $u = 3\tan(\theta)$ and $du = 3\sec(\theta)^2 d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\tan(\theta) = \frac{u}{3}$ as shown below.



We get $\sqrt{9+u^2} = \sqrt{9+9\tan(\theta)^2} = 3\sec(\theta)$ and the integral becomes

$$\begin{aligned}\int \frac{dt}{t\sqrt{9+\ln(t)^2}} &= \int \frac{du}{\sqrt{9+u^2}} \\ &= \int \frac{3\sec(\theta)^2 d\theta}{3\sec(\theta)} \\ &= \int \sec(\theta) d\theta \\ &= \ln|\tan(\theta)+\sec(\theta)| + C.\end{aligned}$$

We express this result in terms of u using the right triangle above, from which we see that

$$\tan(\theta) = \frac{u}{3}, \quad \sec(\theta) = \frac{\sqrt{9+u^2}}{3}.$$

We get

$$\begin{aligned}\int \frac{dt}{t\sqrt{9+\ln(t)^2}} &= \ln\left|\frac{u}{3} + \frac{\sqrt{9+u^2}}{3}\right| + C \\ &= \ln\left|u + \sqrt{9+u^2}\right| + C.\end{aligned}$$

We now finish by replacing u by $\ln(t)$ and we obtain

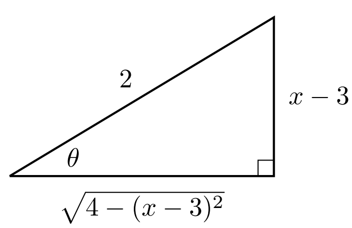
$$\boxed{\int \frac{dt}{t\sqrt{9+\ln(t)^2}} = \ln\left|\ln(t) + \sqrt{9+\ln(t)^2}\right| + C}.$$

(c) $\displaystyle\int \frac{dx}{(6x-x^2-5)^{5/2}}$.

Solution. We start by completing the square in the denominator:

$$6x - x^2 - 5 = -(x^2-6x) - 5 = -(x^2-6x+9) + 9 - 5 = 4 - (x-3)^2.$$

We can now use a trigonometric substitution. We want $4-(x-3)^2 = 4 - 4\sin(\theta)^2$, so we substitute $x - 3 = 2\sin(\theta)$ or $x = 3 + 2\sin(\theta)$. This gives $dx = 2\cos(\theta)d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\sin(\theta) = \frac{x-3}{2}$ as shown below.



We get $\left(4-(x+3)^2\right)^{5/2} = \left(4-4\sin(\theta)^2\right)^{5/2} = (4\cos(\theta)^2)^{5/2} = 32\cos(\theta)^5$. The integral becomes

$$\begin{aligned}\int \frac{dx}{(6x-x^2-5)^{5/2}} &= \int \frac{dx}{\left(4-(x+3)^2\right)^{5/2}} \\ &= \int \frac{2\cos(\theta)d\theta}{32\cos(\theta)^5} \\ &= \frac{1}{16}\int \frac{d\theta}{\cos(\theta)^4} \\ &= \frac{1}{16}\int \sec(\theta)^4 d\theta.\end{aligned}$$

Since the exponent of sec is even, we can split off a factor $\sec(\theta)^2$, rewrite the remaining factors using the Pythagorean identity $\sec(\theta)^2 = \tan(\theta)^2 + 1$ and then use the substitution $u = \tan(\theta)$, $du = \sec(\theta)^2 d\theta$. This gives

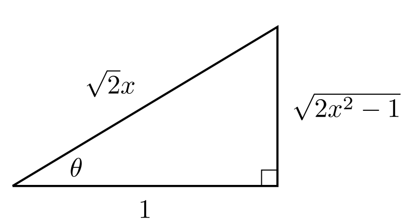
$$\begin{aligned}\int \frac{dx}{(6x-x^2-5)^{5/2}} &= \frac{1}{16}\int \sec(\theta)^2\sec(\theta)^2 d\theta \\ &= \frac{1}{16}\int \left(\tan(\theta)^2+1\right)\sec(\theta)^2 d\theta \\ &= \frac{1}{16}\int (u^2+1)du \\ &= \frac{1}{16}\left(\frac{u^3}{3}+u\right)+C \\ &= \frac{1}{16}\left(\frac{\tan(\theta)^3}{3}+\tan(\theta)\right)+C \\ &= \frac{\tan(\theta)}{16}\left(\frac{\tan(\theta)^2}{3}+1\right)+C.\end{aligned}$$

To express this antiderivative in terms of x, we use the right triangle above, from which we see that $\tan(\theta) = \frac{x-3}{\sqrt{4-(x-3)^2}}$. So we get

$$\boxed{\int \frac{dx}{(6x-x^2-5)^{3/2}} = \frac{x-3}{16\sqrt{4-(x-3)^2}}\left(\frac{(x-3)^2}{3(4-(x-3)^2)}+1\right)+C}.$$

(d) $\displaystyle\int_1^{\sqrt{2}} \frac{dx}{x(2x^2-1)^{3/2}}$.

Solution. We want $2x^2-1 = \sec(\theta)^2-1$, so we substitute $x = \frac{\sec(\theta)}{\sqrt{2}}$ and $dx = \frac{\sec(\theta)\tan(\theta)}{\sqrt{2}}d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\sec(\theta) = \sqrt{2}x$ as shown below.



We get $\left(2x^2-1\right)^{3/2} = \left(\sec(\theta)^2-1\right)^{3/2} = (\tan(\theta)^2)^{3/2} = \tan(\theta)^3$. The bounds change as follows:

$$x = 1 \;\Rightarrow\; \sec(\theta) = \sqrt{2}\cdot 1 = \sqrt{2} \;\Rightarrow\; \theta = \sec^{-1}(\sqrt{2}) = \frac{\pi}{4},$$
$$x = \sqrt{2} \;\Rightarrow\; \sec(\theta) = \sqrt{2}\cdot\sqrt{2} = 2 \;\Rightarrow\; \theta = \sec^{-1}(2) = \frac{\pi}{3}.$$

The integral becomes

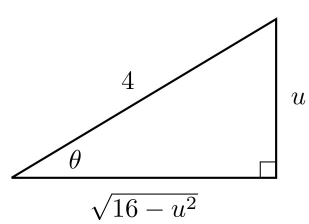
$$\begin{aligned}
\int_1^{\sqrt{2}} \frac{dx}{x(2x^2-1)^{3/2}} &= \int_{\pi/4}^{\pi/3} \frac{\frac{\sec(\theta)\tan(\theta)}{\sqrt{2}}d\theta}{\frac{\sec(\theta)}{\sqrt{2}}\tan(\theta)^3} \\
&= \int_{\pi/4}^{\pi/3} \frac{d\theta}{\tan(\theta)^2} \\
&= \int_{\pi/4}^{\pi/3} \cot(\theta)^2 d\theta \\
&= \int_{\pi/4}^{\pi/3} \left(\csc(\theta)^2 - 1\right) d\theta \\
&= \left[-\cot(\theta) - \theta\right]_{\pi/4}^{\pi/3} \\
&= -\cot\left(\frac{\pi}{3}\right) - \frac{\pi}{3} + \cot\left(\frac{\pi}{4}\right) + \frac{\pi}{4} \\
&= \boxed{1 - \frac{1}{\sqrt{3}} - \frac{\pi}{12}}.
\end{aligned}$$

(e) $\displaystyle\int \frac{e^{6x}}{\sqrt{16-e^{4x}}}dx.$

Solution. We start with the substitution $u = e^{2x}$, so that $du = 2e^{2x}dx$. The extraneous factor e^{4x} in the numerator can be expressed as $e^{4x} = (e^{2x})^2 = u^2$. So the integral becomes

$$\int \frac{e^{6x}}{\sqrt{16-e^{4x}}}dx = \int \frac{e^{4x}}{\sqrt{16-e^{4x}}}e^{2x}dx = \int \frac{u^2}{2\sqrt{16-u^2}}du.$$

We can now use a trigonometric substitution. We want $16 - u^2 = 16 - 16\sin(\theta)^2$, so we substitute $u = 4\sin(\theta)$ and $du = 4\cos(\theta)d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\sin(\theta) = \frac{u}{4}$ as shown below.



We get $\sqrt{16-u^2} = \sqrt{16-16\sin(\theta)^2} = \sqrt{16\cos(\theta)^2} = 4\cos(\theta)$. The integral becomes

$$\begin{aligned}
\int \frac{e^{6x}}{\sqrt{16-e^{4x}}}dx &= \int \frac{u^2}{2\sqrt{16-u^2}}du \\
&= \int \frac{(4\sin(\theta))^2}{2(4\cos(\theta))}4\cos(\theta)d\theta
\end{aligned}$$

$$= 8\int \sin(\theta)^2 d\theta.$$

We can compute this integral using the double angle formulas $\sin(\theta)^2 = \dfrac{1-\cos(2\theta)}{2}$. We get

$$\begin{aligned}\int \frac{e^{6x}}{\sqrt{16-e^{4x}}}dx &= 8\int \frac{1-\cos(2\theta)}{2} 2d\theta \\ &= 4\left(\theta - \frac{\sin(2\theta)}{2}\right) + C \\ &= 4\left(\theta - \cos(\theta)\sin(\theta)\right) + C\end{aligned}$$

where we have used the trigonometric identity $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$ in the last step. We can use the right triangle above to express this result in terms of u, observing that

$$\theta = \sin^{-1}\left(\frac{u}{4}\right), \quad \cos(\theta) = \frac{\sqrt{16-u^2}}{4}, \quad \sin(\theta) = \frac{u}{4}.$$

We can then replace $u = e^{2x}$ and we get

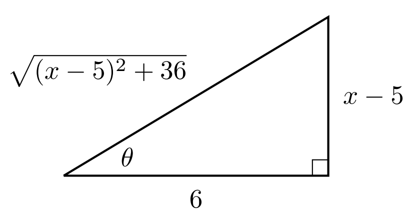
$$\begin{aligned}\int \frac{e^{6x}}{\sqrt{16-e^{4x}}}dx &= 4\left(\sin^{-1}\left(\frac{u}{4}\right) - \frac{\sqrt{16-u^2}}{4}\frac{u}{4}\right) + C \\ &= \boxed{\frac{1}{4}\sin^{-1}\left(\frac{e^{2x}}{4}\right) - \frac{e^{2x}\sqrt{16-e^{4x}}}{4} + C}.\end{aligned}$$

(f) $\displaystyle\int_5^{11} \frac{dx}{(x^2-10x+61)^{5/2}}$.

Solution. We start by completing the square in the denominator:

$$x^2 - 10x + 61 = (x^2 - 10x + 25) - 25 + 61 = (x-5)^2 + 36.$$

We can now use a trigonometric substitution. We want $(x-5)^2 + 36 = 36\tan(\theta)^2 + 36$, so we substitute $x - 5 = 6\tan(\theta)$, or $x = 5 + 6\tan(\theta)$. This gives $dx = 6\sec(\theta)^2 d\theta$ and the following right triangle with base angle θ such that $\tan(\theta) = \frac{x-5}{6}$.



Then $((x-5)^2+36)^{5/2} = (36\tan(\theta)^2+36)^{5/2} = (36\sec(\theta)^2)^{5/2} = 6^5\sec(\theta)^5$. The bounds change as follows:

$$\begin{aligned}x = 5 &\Rightarrow \tan(\theta) = \frac{5-5}{6} = 0 \Rightarrow \theta = \tan^{-1}(0) = 0, \\ x = 11 &\Rightarrow \tan(\theta) = \frac{11-5}{6} = 1 \Rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}.\end{aligned}$$

The integral becomes

$$\begin{aligned}
\int_5^{11} \frac{dx}{(x^2-10x+61)^{5/2}} &= \int_5^{11} \frac{dx}{((x-5)^2+36)^{5/2}} \\
&= \int_0^{\pi/4} \frac{6\sec(\theta)^2 d\theta}{6^5 \sec(\theta)^5} \\
&= \frac{1}{1296}\int_0^{\pi/4} \frac{d\theta}{\sec(\theta)^3} \\
&= \frac{1}{1296}\int_0^{\pi/4} \cos(\theta)^3 d\theta.
\end{aligned}$$

Since the exponent of cos is odd, we can compute this integral by splitting off a factor $\cos(\theta)$, rewriting the remaining factors with the trigonometric identity $\cos(\theta)^2 = 1-\sin(\theta)^2$ and using the substitution $u=\sin(\theta)$, $du=\cos(\theta)d\theta$. The bounds will change as follows

$$\begin{aligned}
\theta = 0 &\Rightarrow u = \sin(0) = 0, \\
\theta = \frac{\pi}{4} &\Rightarrow u = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.
\end{aligned}$$

The integral becomes

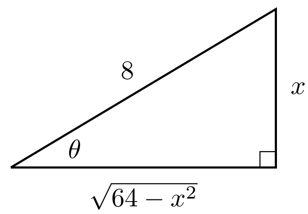
$$\begin{aligned}
\int_5^{11} \frac{dx}{(x^2-10x+61)^{5/2}} &= \frac{1}{1296}\int_0^{\pi/4} \cos(\theta)^2\cos(\theta)d\theta \\
&= \frac{1}{1296}\int_0^{\pi/4} \left(1-\sin(\theta)^2\right)\cos(\theta)d\theta \\
&= \frac{1}{1296}\int_0^{\sqrt{2}/2} \left(1-u^2\right)du \\
&= \frac{1}{1296}\left[u-\frac{u^3}{3}\right]_0^{\sqrt{2}/2} \\
&= \frac{1}{1296}\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}^3}{2^3\cdot 3}\right) \\
&= \boxed{\frac{5\sqrt{2}}{15552}}.
\end{aligned}$$

2. Calculate the average value of the function $f(x) = \dfrac{1}{x\sqrt{64-x^2}}$ on the interval $[4, 4\sqrt{2}]$.

Solution. The average value on the interval $[4, 4\sqrt{2}]$ is given by

$$\text{av}(f) = \frac{1}{4\sqrt{2}-4}\int_4^{4\sqrt{2}} \frac{dx}{x\sqrt{64-x^2}}.$$

We compute this integral using the substitution $x = 8\sin(\theta)$ and $dx = 8\cos(\theta)d\theta$. The right triangle for this trigonometric substitution has base angle θ such that $\sin(\theta) = \frac{x}{8}$ as shown below.



Then $\sqrt{64-x^2} = \sqrt{64-64\sin(\theta)^2} = \sqrt{64\cos(\theta)^2} = 8\cos(\theta)$. The bounds change as follows:

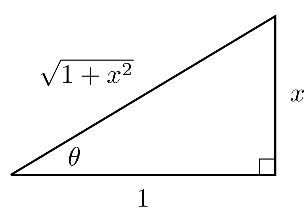
$$x = 4 \;\Rightarrow\; \sin(\theta) = \frac{4}{8} = \frac{1}{2} \;\Rightarrow\; \theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6},$$
$$x = 4\sqrt{2} \;\Rightarrow\; \sin(\theta) = \frac{4\sqrt{2}}{8} = \frac{\sqrt{2}}{2} \;\Rightarrow\; \theta = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.$$

Therefore

$$\begin{aligned}
\text{av}(f) &= \frac{1}{4(\sqrt{2}-1)}\int_{\pi/6}^{\pi/4} \frac{8\cos(\theta)d\theta}{8\sin(\theta)8\cos(\theta)} \\
&= \frac{1}{32(\sqrt{2}-1)}\int_{\pi/6}^{\pi/4} \frac{d\theta}{\sin(\theta)} \\
&= \frac{1}{32(\sqrt{2}-1)}\int_{\pi/6}^{\pi/4} \csc(\theta)d\theta \\
&= \frac{1}{32(\sqrt{2}-1)}\left[\ln|\csc(\theta)-\cot(\theta)|\right]_{\pi/6}^{\pi/4} \\
&= \frac{1}{32(\sqrt{2}-1)}\left(\ln\left|\csc\left(\frac{\pi}{4}\right)-\cot\left(\frac{\pi}{4}\right)\right| - \ln\left|\csc\left(\frac{\pi}{6}\right)-\cot\left(\frac{\pi}{6}\right)\right|\right) \\
&= \boxed{\frac{1}{32(\sqrt{2}-1)}\left(\ln\left(\sqrt{2}-1\right) - \ln(2-\sqrt{(3)})\right)}.
\end{aligned}$$

3. (a) Evaluate $\int \sqrt{1+x^2}dx$.

Solution. We want $1 + x^2 = 1 + \tan(\theta)^2$, so we substitute $x = \tan(\theta)$, $dx = \sec(\theta)^2 d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\tan(\theta) = x$, as shown below.



Then $\sqrt{1+x^2} = \sqrt{1+\tan(\theta)^2} = \sqrt{\sec(\theta)^2} = \sec(\theta)$. The integral becomes

$$\int \sqrt{1+x^2}dx = \int \sec(\theta)\sec(\theta)^2 d\theta$$

$$= \int \sec(\theta)^3 d\theta.$$

We can evaluate $\int \sec(\theta)^3 d\theta$ with an IBP and solving for the unknown integral when it reappears on the right-hand side. For the IBP we use the parts

$$\begin{aligned} u &= \sec(\theta) \ \Rightarrow \ du = \sec(\theta)\tan(\theta)d\theta, \\ dv &= \sec(\theta)^2 d\theta \ \Rightarrow \ v = \tan(\theta). \end{aligned}$$

We get

$$\begin{aligned} \int \sec(\theta)^3 d\theta &= \int \sec(\theta)^2 \sec(\theta) d\theta \\ \int \sec(\theta)^3 d\theta &= \tan(\theta)\sec(\theta) - \int \tan(\theta)\sec(\theta)\tan(\theta) d\theta \\ \int \sec(\theta)^3 d\theta &= \tan(\theta)\sec(\theta) - \int \tan(\theta)^2 \sec(\theta) d\theta \end{aligned}$$

We will use the Pythagorean identity $\tan(\theta)^2 = \sec(\theta)^2 - 1$ to see the original integral reappear on the right-hand side.

$$\begin{aligned} \int \sec(\theta)^3 d\theta &= \tan(\theta)\sec(\theta) - \int \left(\sec(\theta)^2 - 1\right)\sec(\theta) d\theta \\ \int \sec(\theta)^3 d\theta &= \tan(\theta)\sec(\theta) - \int \sec(\theta)^3 d\theta + \int \sec(\theta) d\theta \\ \int \sec(\theta)^3 d\theta &= \tan(\theta)\sec(\theta) - \int \sec(\theta)^3 d\theta + \ln|\sec(\theta) + \tan(\theta)| \end{aligned}$$

We can now move the term $-\int \sec(\theta)^3 d\theta$ to the left hand side and finish solving

$$\begin{aligned} 2\int \sec(\theta)^3 d\theta &= \tan(\theta)\sec(\theta) + \ln|\sec(\theta) + \tan(\theta)| \\ \Rightarrow \int \sec(\theta)^3 d\theta &= \frac{1}{2}\left(\tan(\theta)\sec(\theta) + \ln|\sec(\theta) + \tan(\theta)|\right) + C. \end{aligned}$$

Using the right triangle above, we can express this result in terms of x, observing that $\tan(\theta) = x$ and $\sec(\theta) = \sqrt{x^2+1}$. We get

$$\boxed{\int \sqrt{1+x^2}dx = \frac{1}{2}\left(x\sqrt{x^2+1} + \ln\left|x + \sqrt{x^2+1}\right|\right) + C}.$$

(b) Use your result from part (a) for the following applications.

(i) Calculate the length of the curve $y = x^2$, $0 \leqslant x \leqslant 1$.

Solution. The arc length is given by

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$
\begin{aligned}
&= \int_0^1 \sqrt{1+(2x)^2}\,dx \\
&= \frac{1}{2}\int_0^2 \sqrt{1+u^2}\,du \quad (u = 2x) \\
&= \frac{1}{2}\left[\frac{1}{2}\left(u\sqrt{u^2+1} + \ln\left|u+\sqrt{u^2+1}\right|\right)\right]_0^2 \\
&= \boxed{\frac{1}{4}\left(2\sqrt{5} + \ln\left(2+\sqrt{5}\right)\right)\ \text{units}}.
\end{aligned}
$$

(ii) Calculate the area of the surface obtained by revolving the curve $y = e^x$, $0 \leqslant x \leqslant \ln(2)$, about the x-axis.

Solution. The area of a surface of revolution about the x-axis is given by

$$
\begin{aligned}
L &= \int_0^{\ln(2)} 2\pi y(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\
&= \int_0^{\ln(2)} 2\pi e^x \sqrt{1+e^{2x}}\,dx \\
&= 2\pi \int_1^2 \sqrt{1+u^2}\,du \quad (u = e^x) \\
&= 2\pi \left[\frac{1}{2}\left(u\sqrt{u^2+1} + \ln\left|u+\sqrt{u^2+1}\right|\right)\right]_1^2 \\
&= \boxed{\pi\left(2\sqrt{5} + \ln(2+\sqrt{5}) - \sqrt{2} - \ln(1+\sqrt{2})\right)\ \text{square units}}.
\end{aligned}
$$

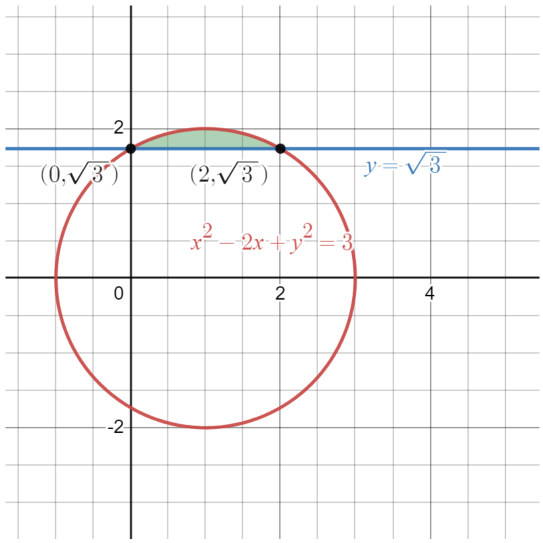
(iii) Calculate the area of the surface obtained by revolving the curve $y = \sin^{-1}(x)$, $0 \leqslant x \leqslant 1$ about the y-axis.

Solution. Note that the curve can be expressed as a function of y as $x = \sin(y)$, $0 \leqslant y \leqslant \frac{\pi}{2}$. The area of a surface of revolution about the y-axis is given by

$$
\begin{aligned}
L &= \int_0^{\pi/2} 2\pi x(y)\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \\
&= \int_0^{\pi/2} 2\pi \sin(y)\sqrt{1+\cos(y)^2}\,dy \\
&= 2\pi \int_1^0 -\sqrt{1+u^2}\,du \quad (u = \cos(y)) \\
&= 2\pi \int_0^1 \sqrt{1+u^2}\,du \\
&= 2\pi \left[\frac{1}{2}\left(u\sqrt{u^2+1} + \ln\left|u+\sqrt{u^2+1}\right|\right)\right]_0^1 \\
&= \boxed{\pi\left(\sqrt{2} + \ln(1+\sqrt{2})\right)\ \text{square units}}.
\end{aligned}
$$

4. Calculate the area of the region inside the circle of equation $x^2 - 2x + y^2 = 3$ and above the line $y = \sqrt{3}$.

Solution. The region is sketched below.



Note that the upper half semi-circle can be expressed as a function of x as $y = \sqrt{3 + 2x - x^2}$. We will compute the area using vertical strips. The vertical strip at x in the region has length $\ell(x) = \sqrt{3 + 2x - x^2} - \sqrt{3}$. Therefore, the area is given by

$$A = \int_0^2 \ell(x)dx = \int_0^2 \left(\sqrt{3 + 2x - x^2} - \sqrt{3}\right) dx = \int_0^2 \sqrt{3 + 2x - x^2}dx - 2\sqrt{3}.$$

To compute the remaining integral, we start by completing the square in the square root:

$$3 + 2x - x^2 = 3 - (x^2 - 2x) = 3 - (x^2 - 2x + 1) + 1 = 4 - (x - 1)^2.$$

We can then use a trigonometric substitution. We want $4 - (x - 1)^2 = 4 - 4\sin(\theta)^2$, so we substitute $x - 1 = 2\sin(\theta)$ or $x = 1 + 2\sin(\theta)$. This gives $dx = 2\cos(\theta)d\theta$ and $\sqrt{4 - (x - 1)^2} = \sqrt{4 - 4\sin(\theta)^2} = \sqrt{4\cos(\theta)^2} = 2\cos(\theta)$. The bounds of the integral become

$$\begin{aligned} x = 0 &\Rightarrow \sin(\theta) = \frac{0 - 1}{2} = -\frac{1}{2} \Rightarrow \theta = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}, \\ x = 2 &\Rightarrow \sin(\theta) = \frac{2 - 1}{2} = \frac{1}{2} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}. \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^2 \sqrt{3 + 2x - x^2}dx &= \int_0^2 \sqrt{4 - (x - 1)^2}dx \\ &= \int_{-\pi/6}^{\pi/6} 2\cos(\theta)2\cos(\theta)d\theta \\ &= 4\int_{-\pi/6}^{\pi/6} \cos(\theta)^2 d\theta \\ &= 8\int_0^{\pi/6} \cos(\theta)^2 d\theta. \end{aligned}$$

where we have used the fact that the integrand is even in the last step. We can now compute this integral with the double angle formula as follows:

$$\begin{aligned}
\int_0^2 \sqrt{3+2x-x^2}dx &= 8\int_0^{\pi/6} \frac{1+\cos(2\theta)}{2} 2d\theta \\
&= 4\left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{\pi/6} \\
&= 4\left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right) \\
&= \frac{2\pi}{3} + \sqrt{3}.
\end{aligned}$$

In conclusion, the area is

$$\begin{aligned}
A &= \int_0^2 \sqrt{3+2x-x^2}dx - 2\sqrt{3} \\
&= \frac{2\pi}{3} + \sqrt{3} - 2\sqrt{3} \\
&= \boxed{\frac{2\pi}{3} - \sqrt{3} \text{ square units}}.
\end{aligned}$$

5. Consider the region $\mathcal{R}$ bounded between the graph of $y = \dfrac{1}{16-x^2}$ and the x-axis for $0 \leqslant x \leqslant 2$. Find the volume of the solid obtained by revolving $\mathcal{R}$ about the line $x = -3$.

Solution. We use the shell method. Revolving the vertical strip at x about the line $x = -3$ forms a cylindrical shell of radius $r(x) = x + 3$ and height $h(x) = \frac{1}{16-x^2}$. Therefore

$$\begin{aligned}
V &= \int_0^2 2\pi r(x)h(x)dx \\
&= 2\pi\int_0^2 \frac{x+3}{16-x^2}dx.
\end{aligned}$$

We can evaluate this integral with a trigonometric substitution. We substitute $x = 4\sin(\theta)$, so $dx = 4\cos(\theta)d\theta$ and $16 - x^2 = 16 - 16\sin(\theta)^2 = 16\cos(\theta)^2$. The bounds become

$$\begin{aligned}
x = 0 &\Rightarrow \sin(\theta) = \frac{0}{4} = 0 \Rightarrow \theta = \sin^{-1}(0) = 0, \\
x = 2 &\Rightarrow \sin(\theta) = \frac{2}{4} = \frac{1}{2} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}.
\end{aligned}$$

So

$$\begin{aligned}
V &= 2\pi\int_0^{\pi/6} \frac{4\sin(\theta)+3}{16\cos(\theta)^2} 4\cos(\theta)d\theta \\
&= \frac{\pi}{2}\int_0^{\pi/6} \frac{4\sin(\theta)+3}{\cos(\theta)}d\theta \\
&= \frac{\pi}{2}\int_0^{\pi/6} \left(4\tan(\theta) + 3\sec(\theta)\right)d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\pi}{2}\left[4\ln|\sec(\theta)| + 3\ln|\sec(\theta)+\tan(\theta)|\right]_0^{\pi/6} \\
&= \frac{\pi}{2}\left(4\ln\left(\frac{2}{\sqrt{3}} + 3\ln\left(\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)\right)\right) \\
&= \boxed{\frac{\pi}{2}\left(4\ln(2) - \frac{1}{2}\ln(3)\right) \text{ cubic units}}.
\end{aligned}
$$