Rutgers University
Math 152

Section 8.8: Improper Integrals - Worksheet Solutions

1. Calculate the following integrals or determine if they diverge.

(a) $\displaystyle\int_0^{\infty} e^{-5x}dx$

Solution.

$$\begin{aligned}\int_0^{\infty} e^{-5x}dx &= \lim_{b\to\infty}\int_0^b e^{-5x}dx\\ &= \lim_{b\to\infty}\left[-\frac{1}{5}e^{-5x}\right]_0^b\\ &= \lim_{b\to\infty}\left(-\frac{1}{5}e^{-5b}+\frac{1}{5}e^0\right)\\ &= \left(-\frac{1}{5}\cdot 0+\frac{1}{5}\right)\\ &= \boxed{\frac{1}{5}}\end{aligned}$$

(b) $\displaystyle\int_0^{\pi/4}\csc(x)dx$

Solution. This is a type II improper integral due to the vertical asymptote of $y=\csc(x)$ at $x=0$.

$$\begin{aligned}\int_0^{\pi/4}\csc(x)dx &= \lim_{a\to 0^+}\int_0^{\pi/4}\csc(x)dx\\ &= \lim_{a\to 0^+}\left[-\ln|\csc(x)+\cot(x)|\right]_a^{\pi/4}\\ &= \lim_{a\to 0^+}\Big(-\ln(\sqrt{2}+1)+\ln|\csc(a)+\cot(a)|\Big)\\ &= \infty\end{aligned}$$

since $\cot(a),\csc(a)\to\infty$ when $a\to 0^+$, so $\ln|\csc(a)+\cot(a)|\to\infty$ when $a\to 0^+$. Therefore $\boxed{\displaystyle\int_0^{\pi/4}\csc(x)dx \text{ diverges}}$.

(c) $\displaystyle\int_{-\infty}^{0} xe^{3x}dx$

Solution. We can start by finding an antiderivative using integration by parts. We use the parts

$$u = x \;\Rightarrow\; du = dx$$
$$dv = e^{3x}dx \;\Rightarrow\; v = \frac{e^{3x}}{3}.$$

We obtain

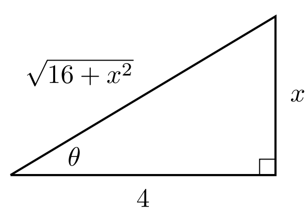
$$\begin{aligned}\int xe^{3x}dx &= \frac{xe^{3x}}{3} - \int \frac{e^{3x}}{3}dx \\ &= \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + C \\ &= \frac{(3x-1)e^{3x}}{9} + C.\end{aligned}$$

We can now compute the improper integral.

$$\begin{aligned}\int_{-\infty}^{0} xe^{3x}dx &= \lim_{a\to-\infty}\int_a^0 xe^{3x}dx \\ &= \lim_{a\to-\infty}\left[\frac{(3x-1)e^{3x}}{9}\right]_a^0 \\ &= \lim_{a\to-\infty}\left(-\frac{1}{9} - \frac{(3a-1)e^{3a}}{9}\right) \\ &= -\frac{1}{9} - \lim_{a\to-\infty}\frac{(3a-1)}{9e^{-3a}} \\ &\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} -\frac{1}{9} - \lim_{a\to-\infty}\frac{3}{-27e^{-3a}} \\ &= -\frac{1}{9} - 0 \\ &= \boxed{-\frac{1}{9}}.\end{aligned}$$

(d) $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{(16+x^2)^{3/2}}$

Solution. We can start by finding an antiderivative of the integrand. For this, we can use the trigonometric substitution $x = 4\tan(\theta)$, $dx = 4\sec(\theta)^2 d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\tan(\theta) = \frac{x}{4}$ as shown below.



We get $(16+u^2)^{3/2} = (16+16\tan(\theta)^2)^{3/2} = (16\sec(\theta)^2)^{3/2} = 64\sec(\theta)^3$, and the integral becomes

$$\begin{aligned}\int\frac{dx}{(16+x^2)^{3/2}} &= \int\frac{4\sec(\theta)^2 d\theta}{64\sec(\theta)^3} \\ &= \frac{1}{16}\int\frac{d\theta}{\sec(\theta)} \\ &= \frac{1}{16}\int\cos(\theta)d\theta \\ &= \frac{1}{16}\sin(\theta) + C.\end{aligned}$$

In this antiderivative, we can express $\sin(\theta)$ in terms of x using the right triangle above, in which we see that $\sin(\theta) = \frac{x}{\sqrt{16+x^2}}$. Thus

$$\int \frac{dx}{(16+x^2)^{3/2}} = \frac{x}{16\sqrt{16+x^2}} + C.$$

We can now compute the improper integral. Observe that the integrand is even, so the integral on $(-\infty, \infty)$ is equal to two times the integral on $[0, \infty)$.

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx}{(16+x^2)^{3/2}} &= 2\int_0^{\infty} \frac{dx}{(16+x^2)^{3/2}} \\
&= 2 \lim_{b\to\infty} \int_0^b \frac{dx}{(16+x^2)^{3/2}} \\
&= 2 \lim_{b\to\infty} \left[\frac{x}{16\sqrt{16+x^2}}\right]_0^b \\
&= 2 \lim_{b\to\infty} \frac{b}{16\sqrt{16+b^2}} \cdot \frac{\frac{1}{b}}{\frac{1}{b}} \\
&= 2 \lim_{b\to\infty} \frac{1}{16\sqrt{\frac{16+}{b^2}1}} \\
&= 2\frac{1}{16} \\
&= \boxed{\frac{1}{8}}.
\end{aligned}$$

(e) $\displaystyle\int_0^1 \ln(x)dx$

Solution. This is a type II improper integral due to the vertical asymptote of $y = \ln(x)$ at $x = 0$. First we compute an antiderivative using integration by parts with parts

$$\begin{aligned}
u &= \ln(x) \;\Rightarrow\; du = \frac{dx}{x} \\
dv &= dx \;\Rightarrow\; v = x.
\end{aligned}$$

We obtain

$$\begin{aligned}
\int \ln(x)dx &= x\ln(x) - \int x\frac{1}{x}dx \\
&= x\ln(x) - \int dx \\
&= x\ln(x) - x + C.
\end{aligned}$$

Next we compute the improper integral.

$$\begin{aligned}
\int_0^1 \ln(x)dx &= \lim_{a\to 0^+} \int_a^1 \ln(x)dx \\
&= \lim_{a\to 0^+} \left[x\ln(x) - x\right]_a^1 \\
&= \lim_{a\to 0^+} \left(-1 - a\ln(a) + a\right) \\
&= -1 - \lim_{a\to 0^+} a\ln(a) + 0 \\
&= -1 - \lim_{a\to 0^+} \frac{\ln(a)}{\frac{1}{a}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} -1 - \lim_{a\to 0^+} \frac{\frac{1}{a}}{-\frac{1}{a^2}} \\
&= -1 - \lim_{a\to 0^+} (-a) \\
&= -1 - 0 \\
&= \boxed{-1}.
\end{aligned}$$

(f) $\displaystyle\int_{-2}^{1} \frac{dx}{\sqrt[3]{3x-2}}$

Solution. This is a type II improper integral due to the vertical asymptote of $y = \frac{1}{\sqrt[3]{3x-2}}$ at $x = \frac{2}{3}$. Because the vertical asymptote is in the interior of the interval of integration, we need to break-up the integral into a sum of two integrals and compute each of them as a limit. We get

$$\begin{aligned}
\int_{-2}^{1} \frac{dx}{\sqrt[3]{3x-2}} &= \int_{-2}^{2/3} \frac{dx}{\sqrt[3]{3x-2}} + \int_{2/3}^{1} \frac{dx}{\sqrt[3]{3x-2}} \\
&= \lim_{b\to\frac{2}{3}^-} \int_{-2}^{b} \frac{dx}{\sqrt[3]{3x-2}} + \lim_{a\to\frac{2}{3}^+} \int_{a}^{1} \frac{dx}{\sqrt[3]{3x-2}} \\
&= \lim_{b\to\frac{2}{3}^-} \left[\frac{(3x-2)^{2/3}}{2}\right]_{-2}^{b} + \lim_{a\to\frac{2}{3}^+} \left[\frac{(3x-2)^{2/3}}{2}\right]_{a}^{1} \\
&= \lim_{b\to\frac{2}{3}^-} \left(\frac{(3b-2)^{2/3}}{2} - 2\right) + \lim_{a\to\frac{2}{3}^+} \left(1 - \frac{(3a-2)^{2/3}}{2}\right) \\
&= (0-2) + (1-0) \\
&= \boxed{-1}.
\end{aligned}$$

(g) $\displaystyle\int_{0}^{3/2} \frac{dx}{\sqrt{9-4x^2}}$

Solution. This is a type II improper integral due to the vertical asymptote of $y = \frac{1}{\sqrt{9-4x^2}}$ at $x = \frac{3}{2}$.

$$\begin{aligned}
\int_0^{3/2} \frac{dx}{\sqrt{9-4x^2}} &= \lim_{b\to\frac{3}{2}^-} \int_0^b \frac{dx}{\sqrt{9-4x^2}} \\
&= \lim_{b\to\frac{3}{2}^-} \left[\frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right)\right]_0^b \\
&= \lim_{b\to\frac{3}{2}^-} \left(\frac{1}{2}\sin^{-1}\left(\frac{2b}{3}\right) - 0\right) \\
&= \frac{1}{2}\sin^{-1}(1) \\
&= \boxed{\frac{\pi}{4}}.
\end{aligned}$$

(h) $\displaystyle\int_e^\infty \frac{dx}{x\ln(x)}$

Solution. We use the substitution $u = \ln(x)$, $du = \frac{dx}{x}$ to compute the antiderivative.

$$\begin{aligned}
\int \frac{dx}{x\ln(x)} &= \int \frac{du}{u} \\
&= \ln|u| + C \\
&= \ln|\ln(x)| + C.
\end{aligned}$$

We can now use this antiderivative to compute the improper integral.

$$\begin{aligned}
\int_e^\infty \frac{dx}{x\ln(x)} &= \lim_{b\to\infty} \int_e^b \frac{dx}{x\ln(x)} \\
&= \lim_{b\to\infty} [\ln|\ln(x)|]_e^b \\
&= \lim_{b\to\infty} (\ln|\ln(b)| - \ln|\ln(e)|) \\
&= \infty
\end{aligned}$$

since $\ln(b) \to \infty$ when $b \to \infty$. Therefore $\boxed{\displaystyle\int_e^\infty \frac{dx}{x\ln(x)} \text{ diverges}}$.

(i) $\displaystyle\int_0^\infty e^{-x}\sin(x)dx$

Solution. We start by computing an antiderivative, using two successive integration by parts. For the first IBP, the parts are

$$\begin{aligned}
u &= \sin(x) \;\Rightarrow\; du = \cos(x)dx, \\
dv &= e^{-x}dx \;\Rightarrow\; v = -e^{-x}.
\end{aligned}$$

This gives

$$\begin{aligned}
\int e^{-x}\sin(x)dx &= -e^{-x}\sin(x) - \int -e^{-x}\cos(x)dx \\
&= -e^{-x}\sin(x) + \int e^{-x}\cos(x)dx.
\end{aligned}$$

The second IBP uses the parts

$$u = \cos(x) \;\Rightarrow\; du = -\sin(x)dx,$$
$$dv = e^{-x}dx \;\Rightarrow\; v = -e^{-x}.$$

We get

$$\int e^{-x}\sin(x)dx = -e^{-x}\sin(x) - e^{-x}\cos(x) - \int(-e^{-x})(-\sin(x))dx \quad = -e^{-x}\sin(x) - e^{-x}\cos(x) - \int e^{-x}\sin(x)dx.$$

We can solve this relation for the unknown integral by moving the term $-\int e^{-x}\sin(x)dx$ to the left-hand side and we get

$$2\int e^{-x}\sin(x)dx = -e^{-x}\sin(x) - e^{-x}\cos(x)$$
$$\Rightarrow \quad \int e^{-x}\sin(x)dx = -\frac{e^{-x}(\sin(x)+\cos(x))}{2} + C$$

We can now compute the improper integral.

$$\begin{aligned}\int_0^\infty e^{-x}\sin(x)dx &= \lim_{b\to\infty}\int_0^b e^{-x}\sin(x)dx\\ &= \lim_{b\to\infty}\left[-\frac{e^{-x}(\sin(x)+\cos(x))}{2}\right]_0^b\\ &= \lim_{b\to\infty}\left(-\frac{e^{-b}(\sin(b)+\cos(b))}{2}+\frac{1}{2}\right)\\ &= \frac{1}{2} - \lim_{b\to\infty}\frac{\sin(b)+\cos(b)}{e^b}.\end{aligned}$$

This last limit can be computed using the Sandwich Theorem. We have the inequalities

$$-2 \leqslant \sin(b)+\cos(b) \leqslant 2$$
$$\Rightarrow \quad -\frac{2}{e^b} \leqslant \frac{\sin(b)+\cos(b)}{e^b} \leqslant \frac{2}{e^b}$$

Since $\lim_{b\to\infty}\frac{2}{e^b} = 0 = \lim_{b\to\infty} -\frac{2}{e^b}$, it follows that $\lim_{b\to\infty}\frac{\sin(b)+\cos(b)}{e^b} = 0$. Hence

$$\boxed{\int_0^\infty e^{-x}\sin(x)dx = \frac{1}{2}}.$$

2. Use a convergence test to determine if the following improper integrals converge or diverge.

   (a) $\displaystyle\int_3^\infty \frac{dx}{xe^x}$

Solution. We use the DCT. Observe that for x in $[3, \infty)$, we have

$$\begin{aligned} & 0 \leqslant e^x \leqslant xe^x \\ \Rightarrow \quad & 0 \leqslant \frac{1}{xe^x} \leqslant \frac{1}{e^x}. \end{aligned}$$

Furthermore, $\displaystyle\int_3^\infty \frac{dx}{e^x}$ converges since

$$\begin{aligned} \int_3^\infty \frac{dx}{e^x} &= \lim_{b\to\infty} \int_3^b e^{-x} dx \\ &= \lim_{b\to\infty} \left(-e^{-b} + e^3\right) \\ &= e^3. \end{aligned}$$

Thus, $\boxed{\displaystyle\int_3^\infty \frac{dx}{xe^x} \text{ converges}}$ as well.

Remark: the inequality

$$0 \leqslant \frac{1}{xe^x} \leqslant \frac{1}{x}$$

is also true, but it does not help establish the convergence of $\displaystyle\int_3^\infty \frac{dx}{xe^x}$ since the integral of the "bigger function" $\displaystyle\int_3^\infty \frac{dx}{x}$ diverges (type I p-integral with $p = 1$).

(b) $\displaystyle\int_1^\infty \frac{dx}{x^2 + 3x + 1}$

Solution. We use the DCT. Observe that for x in $[1, \infty)$ we have the inequalities

$$\begin{aligned} & 0 \leqslant x^2 \leqslant x^2 + 3x + 1 \\ \Rightarrow \quad & 0 \leqslant \frac{1}{x^2 + 3x + 1} \leqslant \frac{1}{x^2}. \end{aligned}$$

Furthermore, the integral $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges since it is a type I p-integral with $p = 2 > 1$. Therefore, $\boxed{\displaystyle\int_1^\infty \frac{dx}{x^2 + 3x + 1} \text{ converges}}$ as well.

Remark. We can also use the LCT, observing that

$$\lim_{x\to\infty} \frac{\frac{1}{x^2+3x+1}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{1 + 3/x + 1/x^2} = 1 > 0$$

and the integral $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges since it is a type I p-integral with $p = 2 > 1$.

(c) $\displaystyle\int_4^\infty \frac{\cos(x) + 5}{x^{3/5}} dx$

Solution. We use the DCT, observing that for x in $[4, \infty)$ we have

$$\begin{aligned} & -1 \leqslant \cos(x) \\ \Rightarrow\quad & 0 \leqslant 4 \leqslant \cos(x) + 5 \\ \Rightarrow\quad & 0 \leqslant \frac{4}{x^{3/5}} \leqslant \frac{\cos(x)+5}{x^{3/5}} \end{aligned}$$

Furthermore, the integral $\int_4^\infty \frac{4}{x^{3/5}} dx$ diverges since it is a type I p-integral with $p = \frac{3}{5} \leqslant 1$. It follows that $\boxed{\int_4^\infty \frac{\cos(x)+5}{x^{3/5}} dx \text{ diverges}}$ as well.

Remark: we would not be able to use the LCT to compare with the divergent p-integral integral $\int_4^\infty \frac{dx}{x^{3/5}}$ since

$$\lim_{x\to\infty} \frac{\frac{\cos(x)+5}{x^{3/5}}}{\frac{1}{x^{3/5}}} = \lim_{x\to\infty} (\cos(x)+5) \quad \text{does not exist.}$$

(d) $\int_0^1 \frac{dx}{\sqrt{x} + x^2}$

Solution. We use the DCT, observing that for x in $(0, 1]$ we have

$$\begin{aligned} & 0 \leqslant \sqrt{x} \leqslant \sqrt{x} + x^2 \\ \Rightarrow\quad & 0 \leqslant \frac{1}{x^2 + \sqrt{x}} \leqslant \frac{1}{\sqrt{x}} \end{aligned}$$

Furthermore, the integral $\int_0^1 \frac{dx}{\sqrt{x}}$ converges since it is a type II p-integral with $p = \frac{1}{2} < 1$. It follows that $\boxed{\int_0^1 \frac{dx}{x^2+\sqrt{x}} \text{ converges}}$ as well.

Remark 1. The inequality

$$0 \leqslant \frac{1}{x^2 + \sqrt{x}} \leqslant \frac{1}{x^2}$$

is also true, but it does not help establish the convergence of $\int_0^1 \frac{dx}{x^2+\sqrt{x}}$ since the integral of the "bigger function" $\int_0^1 \frac{dx}{x^2}$ diverges (type II p-integral with $p = 2 \geqslant 1$).

Remark 2. We could have also used the LCT to compare with the convergent type II p-integral

$\int_0^1 \frac{dx}{\sqrt{x}}$, remarking that

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\frac{1}{x^2+\sqrt{x}}}{\frac{1}{\sqrt{x}}} &= \lim_{x\to 0^+} \frac{\sqrt{x}}{x^2+\sqrt{x}} \\
&= \lim_{x\to 0^+} \frac{\sqrt{x}}{x^2+\sqrt{x}} \cdot \frac{\frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x}}} \\
&= \lim_{x\to 0^+} \frac{1}{x^{3/2}+1} \\
&= 1 > 0.
\end{aligned}$$

(e) $\int_5^\infty \frac{xdx}{x^4-1}$

*Solution.*We use the LCT, comparing with $\frac{1}{x^3}$. We have

$$\begin{aligned}
\lim_{x\to\infty} \frac{\frac{x}{x^4-1}}{\frac{1}{x^3}} &= \lim_{x\to\infty} \frac{x^4}{x^4-1} \\
&= \lim_{x\to\infty} \frac{x^4}{x^4-1} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}} \\
&= \lim_{x\to\infty} \frac{1}{1-\frac{1}{x^4}} \\
&= 1 > 0.
\end{aligned}$$

Furthermore, the integral $\int_5^\infty \frac{dx}{x^3}$ converges since it is a type I p-integral with $p = 3 > 1$. Hence, $\boxed{\int_5^\infty \frac{xdx}{x^4-1} \text{ converges}}$ as well.

Remark. The DCT cannot be used to compare with the convergent type *I* p-integral $\int_5^\infty \frac{dx}{x^3}$ since we have the inequalities

$$\begin{aligned}
& 0 \leqslant x^4 - 1 \leqslant x^4 \\
\Rightarrow\ & 0 \leqslant \frac{1}{x^4} \leqslant \frac{1}{x^4-1} \\
\Rightarrow\ & 0 \leqslant \frac{1}{x^3} \leqslant \frac{x}{x^4-1}
\end{aligned}$$

and knowing that the integral of the "smaller function" converges does not say anything about the integral of the "bigger function".

(f) $\int_1^\infty \frac{x^3+5x^2+1}{\sqrt{x^7+4x+2}}dx$

Solution. We use the LCT. To find a good function to compare to, we keep the terms of the numerator and denominator that are dominant when $x \to \infty$:

$$\frac{x^3+5x^2+1}{\sqrt{x^7+4x+2}} \sim \frac{x^3}{\sqrt{x^7}} = \frac{x^3}{x^{7/2}} = \frac{1}{x^{1/2}}.$$

Now that we have found our reference function, we properly establish the limit comparison.

$$\begin{aligned}
\lim_{x\to\infty} \frac{\frac{x^3+5x^2+1}{\sqrt{x^7+4x+2}}}{\frac{1}{x^{1/2}}} &= \lim_{x\to\infty} \frac{x^{7/2}+5x^{5/2}+x^{1/2}}{\sqrt{x^7+4x+2}} \\
&= \lim_{x\to\infty} \frac{x^{7/2}+5x^{5/2}+x^{1/2}}{\sqrt{x^7+4x+2}} \cdot \frac{\frac{1}{x^{7/2}}}{\frac{1}{x^{7/2}}} \\
&= \lim_{x\to\infty} \frac{1+\frac{5}{x^2}+\frac{1}{x^3}}{\sqrt{1+\frac{4}{x^6}+\frac{2}{x^7}}} \\
&= 1 > 0.
\end{aligned}$$

We also know that the integral $\displaystyle\int_1^\infty \frac{dx}{x^{1/2}}$ diverges since it is a type I p-integral with $p = \frac{1}{2} \leqslant 1$.

Therefore, $\boxed{\displaystyle\int_1^\infty \frac{x^3+5x^2+1}{\sqrt{x^7+4x+2}}dx \text{ diverges}}$ as well.

3. Consider the unbounded region $\mathcal{R}$ between the graph of $y = \dfrac{\ln(x)}{x}$ and the x-axis for $x \geqslant 1$.

   (a) Find the area of the region $\mathcal{R}$ or determine if $\mathcal{R}$ has infinite area.

Solution. The area of $\mathcal{R}$ is given by

$$A = \int_1^\infty \frac{\ln(x)}{x}dx.$$

The antiderivative of the integrand can be found with the substitution $u = \ln(x)$, $du = \frac{dx}{x}$, which gives

$$\int \frac{\ln(x)}{x}dx = \int u du = \frac{u^2}{2} + C = \frac{\ln(x)^2}{2} + C.$$

We can use this to compute the area, as follows

$$\begin{aligned}
A &= \int_1^\infty \frac{\ln(x)}{x}dx \\
&= \lim_{b\to\infty} \int_1^b \frac{\ln(x)}{x}dx \\
&= \lim_{b\to\infty} \left[\frac{\ln(x)^2}{2}\right]_1^b \\
&= \lim_{b\to\infty} \frac{\ln(b)^2}{2} \\
&= \infty.
\end{aligned}$$

So $\boxed{\mathcal{R} \text{ has infinite area}}$.

(b) We now revolve the region $\mathcal{R}$ about the x-axis to form a solid of revolution. Calculate the volume of the solid or determine if the solid has infinite volume.

Solution. Revolving the vertical strip at x in the region about the x-axis forms a disk of radius $r(x) = \frac{\ln(x)}{x}$. So the volume is given by

$$V = \int_1^\infty \pi r(x)^2 dx = \pi \int_1^\infty \frac{\ln(x)^2}{x^2} dx.$$

To compute the antiderivative of the integrand, we use two successive IPBs. The first one will use the parts

$$\begin{aligned} u = \ln(x)^2 \ &\Rightarrow \ du = \frac{2\ln(x)dx}{x}, \\ dv = \frac{dx}{x^2} \ &\Rightarrow \ v = -\frac{1}{x}. \end{aligned}$$

This gives

$$\begin{aligned} \int \frac{\ln(x)^2}{x^2} dx &= -\frac{\ln(x)^2}{x} - \int \frac{2\ln(x)}{x} \left(-\frac{1}{x}\right) dx \\ &= -\frac{\ln(x)^2}{x} + 2\int \frac{\ln(x)}{x^2} dx. \end{aligned}$$

For the second IBP, we take

$$\begin{aligned} u = \ln(x) \ &\Rightarrow \ du = \frac{dx}{x}, \\ dv = \frac{dx}{x^2} \ &\Rightarrow \ v = -\frac{1}{x}. \end{aligned}$$

We obtain

$$\begin{aligned} \int \frac{\ln(x)^2}{x^2} dx &= -\frac{\ln(x)^2}{x} + 2\left(-\frac{\ln(x)}{x} - \int \frac{1}{x}\left(-\frac{1}{x}\right) dx\right) \\ &= -\frac{\ln(x)^2}{x} - 2\frac{\ln(x)}{x} + 2\int \frac{1}{x^2} dx \\ &= -\frac{\ln(x)^2}{x} - 2\frac{\ln(x)}{x} - \frac{2}{x} + C \\ &= -\frac{\ln(x)^2 + 2\ln(x) + 2}{x} + C. \end{aligned}$$

We can now use this to compute the volume.

$$\begin{aligned} V &= \pi \int_1^\infty \frac{\ln(x)^2}{x^2} dx \\ &= \pi \lim_{b\to\infty} \int_1^b \frac{\ln(x)^2}{x^2} dx \\ &= \pi \lim_{b\to\infty} \left[-\frac{\ln(x)^2 + 2\ln(x) + 2}{x}\right]_1^b \\ &= \pi \lim_{b\to\infty} \left(2 - \frac{\ln(b)^2 + 2\ln(b) + 2}{b}\right) \\ &= \pi \left(2 - \lim_{b\to\infty} \frac{\ln(b)^2 + 2\ln(b) + 2}{b}\right). \end{aligned}$$

To compute the remaining limit, we use L'Hôpital's Rule twice for the indeterminate form $\frac{\infty}{\infty}$.

$$\begin{aligned} V &\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \pi \left(2 - \lim_{b\to\infty} \frac{2\frac{\ln(b)}{b} + \frac{2}{b}}{1} \right) \\ &= \pi \left(2 - \lim_{b\to\infty} \frac{2\ln(b) + 2}{b} \right) \\ &\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \pi \left(2 - \lim_{b\to\infty} \frac{\frac{2}{b}}{1} \right) \\ &= \pi (2 - 0) \\ &= \boxed{2\pi \text{ cubic units}}. \end{aligned}$$

(c) We now revolve the region $\mathcal{R}$ about the y-axis to form a solid of revolution. Calculate the volume of the solid or determine if the solid has infinite volume.

Solution. Revolving the vertical strip at x about the y-axis forms a shell with radius $r(x) = x$ and height $h(x) = \frac{\ln(x)}{x}$. So the volume is

$$V = \int_1^\infty 2\pi r(x)h(x)dx = \int_1^\infty 2\pi x \frac{\ln(x)}{x} dx = 2\pi \int_1^\infty \ln(x)dx.$$

We have previously computed the antiderivative of the integrand using integration by parts and found that

$$\int \ln(x)dx = x(\ln(x) - 1) + C.$$

So the volume is

$$\begin{aligned} V &= 2\pi \lim_{b\to\infty} \int_1^b \ln(x)dx \\ &= 2\pi \lim_{b\to\infty} \left[x(\ln(x) - 1)\right]_1^b \\ &= 2\pi \lim_{b\to\infty} b(\ln(b) - 1) \\ &= \infty, \end{aligned}$$

so $\boxed{\text{the solid has infinite volume}}$.