Rutgers University
Math 152

Math 152 All Worksheet Solutions
Laurent Vera
(Last updated: 7/29/23)

Contents

23 Sections 11.5: Areas and Lengths in Polar Coordinates 180

Rutgers University
Math 152

Calculus 1 Review Worksheet - Solutions

1. Simplify the following expressions. Your answer should not involve any trigonometric or inverse trigonometric functions.

(a) $\cos^{-1}\left(\frac{1}{2}\right)$

Solution. Recall that $\cos^{-1}(x)$ is the angle θ in $[0, \pi]$ such that $\cos(\theta) = x$. Therefore

$$\boxed{\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}}.$$

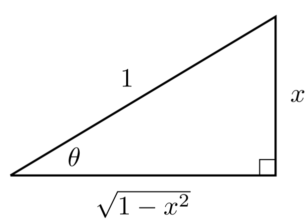
(b) $\sin^{-1}\left(\sin\left(\frac{7\pi}{4}\right)\right)$

Solution. Recall that $\sin^{-1}(x)$ is the angle θ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin(\theta) = x$. Since $\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, we deduce that

$$\boxed{\sin^{-1}\left(\sin\left(\frac{7\pi}{4}\right)\right) = -\frac{\pi}{4}}.$$

(c) $\cos(\sin^{-1}(x))$

Solution. Consider a right triangle with acute angle $\theta = \sin^{-1}(x)$. Then $\sin(\theta) = x = \frac{\text{opp}}{\text{hyp}}$. So we can choose the sides to be $\text{opp} = x$ and $\text{hyp} = 1$ as shown below.



The Pythagoran identity then gives $\text{adj} = \sqrt{\text{hyp}^2 - \text{opp}^2} = \sqrt{1 - x^2}$. Therefore

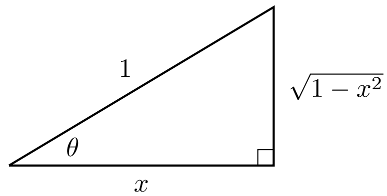
$$\cos(\sin^{-1}(x)) = \frac{\text{adj}}{\text{hyp}} = \boxed{\sqrt{1 - x^2}}.$$

(d) $\sin(2\cos^{-1}(x))$

Solution. First, using the trigonometric identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, we have

$$\sin(2\cos^{-1}(x)) = 2\sin(\sin^{-1}(x))\sin(\cos^{-1}(x)) = 2x\sin(\cos^{-1}(x)).$$

To simplify $\sin(\cos^{-1}(x))$, we consider a right triangle with acute angle $\theta = \cos^{-1}(x)$. Then $\cos(\theta) = x = \frac{\text{adj}}{\text{hyp}}$. So we can choose the sides to be adj $= x$ and hyp $= 1$ as shown below.



The Pythagoran identity then gives opp $= \sqrt{\text{hyp}^2 - \text{adj}^2} = \sqrt{1-x^2}$. Therefore

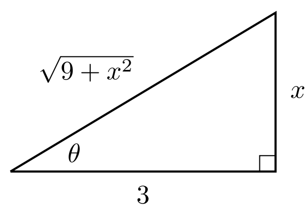
$$\sin(\cos^{-1}(x)) = \frac{\text{opp}}{\text{hyp}} = \sqrt{1-x^2}.$$

We conclude that

$$\boxed{\sin(2\cos^{-1}(x)) == 2x\sqrt{1-x^2}}.$$

(e) $\sec\left(\tan^{-1}\left(\frac{x}{3}\right)\right)$

Solution. Consider a right triangle with acute angle $\theta = \tan^{-1}\left(\frac{x}{3}\right)$. Then $\tan(\theta) = \frac{x}{3} = \frac{\text{opp}}{\text{adj}}$. So we can choose the sides to be opp $= x$ and adj $= 3$ as shown below.

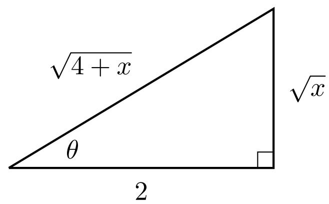


The Pythagoran identity then gives hyp $= \sqrt{\text{adj}^2 + \text{opp}^2} = \sqrt{9+x^2}$. Therefore

$$\sec\left(\tan^{-1}\left(\frac{x}{3}\right)\right) = \frac{\text{hyp}}{\text{adj}} = \boxed{\frac{\sqrt{9+x^2}}{3}}.$$

(f) $\sin\left(\cot^{-1}\left(\frac{2}{\sqrt{x}}\right)\right)$

Solution. Consider a right triangle with acute angle $\theta = \cot^{-1}\left(\frac{2}{\sqrt{x}}\right)$. Then $\cot(\theta) = \frac{2}{\sqrt{x}} = \frac{\text{adj}}{\text{opp}}$. So we can choose the sides to be adj $= 2$ and opp $= \sqrt{x}$ as shown below.



The Pythagorean identity then gives $\text{hyp} = \sqrt{\text{adj}^2 + \text{opp}^2} = \sqrt{4+x}$. Therefore

$$\sin\left(\cot^{-1}\left(\frac{2}{\sqrt{x}}\right)\right) = \frac{\text{opp}}{\text{hyp}} = \boxed{\frac{\sqrt{x}}{\sqrt{4+x}}}.$$

2. Calculate the following limits.

(a) $\displaystyle\lim_{x\to\infty} \frac{\ln(x)^2}{\sqrt{x}}$

Solution. We can compute this limit using L'Hôpital's Rule twice with the indeterminate form $\frac{\infty}{\infty}$. This gives

$$\begin{aligned}
\lim_{x\to\infty} \frac{\ln(x)^2}{\sqrt{x}} &\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{2\ln(x)\frac{1}{x}}{\frac{1}{2\sqrt{x}}} \\
&= \lim_{x\to\infty} \frac{4\ln(x)}{\sqrt{x}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{\frac{4}{x}}{\frac{1}{2\sqrt{x}}} \\
&= \lim_{x\to\infty} \frac{8}{\sqrt{x}} \\
&= \boxed{0}.
\end{aligned}$$

(b) $\displaystyle\lim_{x\to 0} \frac{5^x - 3^x}{\sin(2x)}$

Solution. This limit is an indeterminate form $\frac{0}{0}$. We can resolve the indeterminate form using L'Hôpital's Rule, remembering that for a positive constant a, we have

$$\frac{d}{dx}a^x = \ln(a)a^x.$$

We obtain

$$\begin{aligned}
\lim_{x\to 0} \frac{5^x - 3^x}{\sin(2x)} &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to 0} \frac{\ln(5)5^x - \ln(3)3^x}{2\cos(2x)} \\
&= \frac{\ln(5)5^0 - \ln(3)^0}{2\cos(2\cdot 0)} \\
&= \boxed{\frac{\ln(5) - \ln(3)}{2}}.
\end{aligned}$$

(c) $\lim_{x\to\infty} \tan^{-1}(x^2 - x^3)$

Solution. We start by investigating the behavior of the "inside" $x^2 - x^3$ as $x \to \infty$. Since $\lim_{x\to\infty} \left(x^2 - x^3\right)$ is an indeterminate form $\infty - \infty$, we will need a bit of algebra to be able to compute the limit. We have

$$\lim_{x\to\infty} \left(x^2 - x^3\right) = \lim_{x\to\infty} x^3 \left(\frac{1}{x} - 1\right) = \text{“}\infty(0-1)\text{”} = -\infty.$$

Therefore, letting $u = x^2 - x^3$, we have $u \to -\infty$ as $x \to \infty$, so

$$\lim_{x\to\infty} \tan^{-1}(x^2 - x^3) = \lim_{u\to-\infty} \tan^{-1}(u) = \boxed{-\frac{\pi}{2}}.$$

(d) $\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^x$

Solution. This limit is an indeterminate power 1^∞. **Warning:** limits of the form 1^∞ need not be equal to 1! We can resolve the indeterminate form by rewriting the power with an exponential using the formula

$$a^b = e^{b\ln(a)}$$

and applying L'Hôpital's Rule in the resulting exponent. This gives

$$\begin{aligned}
\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^x &= \lim_{x\to\infty} e^{x\ln\left(1+\frac{2}{x}\right)} \\
&= e^{\lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}}} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} e^{\lim_{x\to\infty} \frac{-\frac{2}{x^2}\cdot\frac{1}{1+\frac{2}{x}}}{-\frac{1}{x^2}}} \\
&= e^{\lim_{x\to\infty} \frac{2}{1+\frac{2}{x}}} \\
&= \boxed{e^2}.
\end{aligned}$$

(e) $\lim_{x\to-\infty} \frac{2x + 3\cos(x)}{5x}$

Solution. This limit is an indeterminate form $\frac{\infty}{\infty}$. However, we **cannot use L'Hôpital's Rule** here. This is because L'Hôpital's Rule only applies if the resulting limit exists or is infinite, but here, the resulting limit

$$\lim_{x\to-\infty} \frac{2 - 3\sin(x)}{5}$$

does not exist. The Squeeze (or Sandwich) Theorem will work for this limit. Since $-1 \leqslant \cos(x) \leqslant 1$ for all x, we have

$$\frac{2x-3}{5x} \leqslant \frac{2x + 3\cos(x)}{5x} \leqslant \frac{2x+3}{5x}$$

for any $x \neq 0$. Furthermore, we have

$$\lim_{x\to-\infty} \frac{2x-3}{5x} = \lim_{x\to\infty} \frac{2}{5} - \frac{3}{5x} = \frac{2}{5},$$
$$\lim_{x\to-\infty} \frac{2x+3}{5x} = \lim_{x\to\infty} \frac{2}{5} + \frac{3}{5x} = \frac{2}{5}.$$

Since the two limits are equal, we conclude that

$$\boxed{\lim_{x\to-\infty} \frac{2x+3\cos(x)}{5x} = \frac{2}{5}}.$$

(f) $\lim\limits_{x\to\infty} x^{1/x}$

Solution. This limit is an indeterminate power ∞^0. **Warning:** limits of the form ∞^0 need not be equal to 1! We can resolve the indeterminate form by rewriting the power with an exponential using the formula

$$a^b = e^{b\ln(a)}$$

and applying L'Hôpital's Rule in the resulting exponent. This gives

$$\begin{aligned}
\lim_{x\to\infty} x^{1/x} &= \lim_{x\to\infty} e^{\frac{\ln(x)}{x}} \\
&= e^{\lim\limits_{x\to\infty} \frac{\ln(x)}{x}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} e^{\lim\limits_{x\to\infty} \frac{1/x}{1}} \\
&= e^0 \\
&= \boxed{1}.
\end{aligned}$$

3. Find the horizontal asymptotes of the following functions.

(a) $f(x) = \dfrac{11x^3+2x-1}{2x^3-x^2+3}$

Solution. This function is rational. We will be able to compute its limits at infinity after dividing the numerator and denominator by the highest power of x appearing in the expression, here x^3. This gives

$$\lim_{x\to\infty} \frac{11x^3+2x-1}{2x^3-x^2+3} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{11+\frac{2}{x^2}-\frac{1}{x^3}}{2-\frac{1}{x}+\frac{3}{x^3}} = \frac{11}{2},$$
$$\lim_{x\to-\infty} \frac{11x^3+2x-1}{2x^3-x^2+3} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to-\infty} \frac{11+\frac{2}{x^2}-\frac{1}{x^3}}{2-\frac{1}{x}+\frac{3}{x^3}} = \frac{11}{2}.$$

Therefore, the only horizontal asymptote of f is $\boxed{y = \frac{11}{2}}$.

(b) $f(x) = \dfrac{5x + \sqrt{16x^2+25}}{18x-7}$

Solution. Note that

$$\sqrt{16x^2+25} = \sqrt{x^2}\sqrt{16+\frac{25}{x^2}} = |x|\sqrt{16+\frac{25}{x^2}}.$$

When $x \to \infty$, we have $|x| = x$, so

$$\begin{aligned}
\lim_{x\to\infty} \frac{5x+\sqrt{16x^2+25}}{18x-7} &= \lim_{x\to\infty} \frac{5x+|x|\sqrt{16+\frac{25}{x^2}}}{18x-7} \\
&= \lim_{x\to\infty} \frac{5x+x\sqrt{16+\frac{25}{x^2}}}{18x-7} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{5+\sqrt{16+\frac{25}{x^2}}}{18-\frac{7}{x}} \\
&= \frac{5+\sqrt{16}}{18} \\
&= \frac{1}{2}.
\end{aligned}$$

When $x \to -\infty$, we have $|x| = -x$, so

$$\begin{aligned}
\lim_{x\to\infty} \frac{5x+\sqrt{16x^2+25}}{18x-7} &= \lim_{x\to\infty} \frac{5x+|x|\sqrt{16+\frac{25}{x^2}}}{18x-7} \\
&= \lim_{x\to\infty} \frac{5x-x\sqrt{16+\frac{25}{x^2}}}{18x-7} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{5-\sqrt{16+\frac{25}{x^2}}}{18-\frac{7}{x}} \\
&= \frac{5-\sqrt{16}}{18} \\
&= \frac{1}{18}.
\end{aligned}$$

Therefore the two horizontal asymptotes of f are $\boxed{y = \frac{1}{18},\ y = \frac{1}{2}}$.

(c) $f(x) = \dfrac{3e^{2x} - 2e^x + 4x^2}{x^2 - 6e^{2x}}$

Solution. When $x \to \infty$, the dominant term in the expression is e^{2x}, so we will divide numerator and denominator by e^{2x} to compute the limit when $x \to \infty$. We obtain

$$\begin{aligned}
\lim_{x\to\infty} \frac{3e^{2x} - 2e^x + 4x^2}{x^2 - 6e^{2x}} &= \lim_{x\to\infty} \frac{3e^{2x} - 2e^x + 4x^2}{x^2 - 6e^{2x}} \cdot \frac{\frac{1}{e^{2x}}}{\frac{1}{e^{2x}}} \\
&= \lim_{x\to\infty} \frac{3 - \frac{2}{e^x} + \frac{4x^2}{e^{2x}}}{\frac{x^2}{e^{2x}} - 6}.
\end{aligned}$$

Using L'Hôpital's Rule, we have

$$\lim_{x\to\infty}\frac{x^2}{e^{2x}} \overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty}\frac{2x}{2e^{2x}} \overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty}\frac{2}{4e^{2x}} = 0.$$

Therefore,

$$\lim_{x\to\infty}\frac{3e^{2x}-2e^x+4x^2}{x^2-6e^{2x}} = \lim_{x\to\infty}\frac{3-\frac{2}{e^x}+\frac{4x^2}{e^{2x}}}{\frac{x^2}{e^{2x}}-6} = \frac{3}{-6} = -\frac{1}{2}.$$

When $x \to -\infty$, the dominant term is x^2 as $e^x, e^{2x} \to 0$. So we will divide the numerator and denominator by x^2 to compute the limit when $x \to -\infty$. This gives

$$\begin{aligned}\lim_{x\to-\infty}\frac{3e^{2x}-2e^x+4x^2}{x^2-6e^{2x}} &= \lim_{x\to-\infty}\frac{3e^{2x}-2e^x+4x^2}{x^2-6e^{2x}}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\\ &= \lim_{x\to-\infty}\frac{\frac{3e^{2x}}{x^2}-\frac{2e^x}{x^2}+4}{1-\frac{6e^{2x}}{x^2}}\\ &= 4.\end{aligned}$$

Note that $\lim_{x\to-\infty}\frac{e^x}{x^2} = \lim_{x\to-\infty}\frac{e^{2x}}{x^2} = \text{“}\frac{0}{\infty}\text{”} = 0$. These are not indeterminate forms and we cannot apply L'Hôpital's Rule (nor would we need to).

Therefore, the two horizontal asymptotes of the function f are $\boxed{y = -\frac{1}{2},\ y = 4}$.

4. Calculate the following indefinite or definite integrals.

(a) $\displaystyle\int (3x+1)\left(x^2-\frac{5}{x}\right)dx$

Solution. We fully distribute the integrand, then use the power rule. This gives

$$\begin{aligned}\int (3x+1)\left(x^2-\frac{5}{x}\right)dx &= \int\left(3x^3+x^2-15-\frac{5}{x}\right)dx\\ &= \boxed{\frac{3}{4}x^4+\frac{1}{3}x^3-15x-5\ln|x|+C}.\end{aligned}$$

(b) $\displaystyle\int x^3\sin(x^4+2)dx$

Solution. We use the substitution $u = x^4+2$, $du = 4x^3dx$. Therefore, $x^3dx = \frac{du}{4}$, and we obtain

$$\begin{aligned}\int x^3\sin(x^4+2)dx &= \int\frac{1}{4}\sin(u)du\\ &= -\frac{1}{4}\cos(u)+C\\ &= \boxed{-\frac{1}{4}\cos(x^4+2)+C}.\end{aligned}$$

(c) $\int_0^1 \frac{x^3}{\sqrt{3+x^2}}dx$

Solution 1. We use the substitution $u = 3 + x^2$. So $du = 2xdx$, that is $xdx = \frac{du}{2}$. The extraneous factor x^2 in the numerator can be expressed in terms of u using $x^2 = u - 3$. Finally, the bounds become

$$\begin{aligned} x = 0 &\Rightarrow u = 3 + 0^2 = 3, \\ x = 1 &\Rightarrow u = 3 + 1^2 = 4. \end{aligned}$$

So the integral becomes

$$\begin{aligned} \int_0^1 \frac{x^3}{\sqrt{3+x^2}}dx &= \int_0^1 \frac{x^2}{\sqrt{3+x^2}}xdx \\ &= \int_3^4 \frac{u-3}{2\sqrt{u}}du \\ &= \int_3^4 \left(\frac{1}{2}\sqrt{u} - \frac{3}{2\sqrt{u}}\right)du \\ &= \left[\frac{1}{2}\cdot\frac{2}{3}u^{3/2} - 3\sqrt{u}\right]_3^4 \\ &= \left(\frac{1}{3}4^{3/2} - 3\sqrt{4}\right) - \left(\frac{1}{3}3^{3/2} - 3\sqrt{3}\right) \\ &= \boxed{2\sqrt{3} - \frac{10}{3}}. \end{aligned}$$

Solution 2. We use the substitution $u = \sqrt{3+x^2}$. So $du = \frac{xdx}{\sqrt{3+x^2}}$. The extraneous factor x in the numerator can be expressed in terms of u using $x^2 = u^2 - 3$. Finally, the bounds become

$$\begin{aligned} x = 0 &\Rightarrow u = \sqrt{3+0^2} = \sqrt{3}, \\ x = 1 &\Rightarrow u = \sqrt{3+1^2} = 2. \end{aligned}$$

So the integral becomes

$$\begin{aligned} \int_0^1 \frac{x^3}{\sqrt{3+x^2}}dx &= \int_0^1 x^2\frac{xdx}{\sqrt{3+x^2}} \\ &= \int_{\sqrt{3}}^2 (u^2-3)du \\ &= \left[\frac{1}{3}u^3 - 3u\right]_{\sqrt{3}}^2 \\ &= \left(\frac{1}{3}2^3 - 3\cdot 2\right) - \left(\frac{1}{3}\sqrt{3}^3 - 3\sqrt{3}\right) \\ &= \boxed{2\sqrt{3} - \frac{10}{3}}. \end{aligned}$$

(d) $\displaystyle\int t\sec^2\left(3t^2\right)e^{7\tan\left(3t^2\right)}dt$

Solution. We use the substitution $u = 7\tan\left(3t^2\right)$. This gives

$$du = 7\sec^2(3t^2)\cdot 6t dt = 42t\sec^2(3t^2)dt.$$

So

$$t\sec^2\left(3t^2\right)dt = \frac{du}{42}.$$

We get

$$\begin{aligned}\int t\sec^2\left(3t^2\right)e^{7\tan\left(3t^2\right)}dt &= \int\frac{1}{42}e^u du\\ &= \frac{1}{42}e^u + C\\ &= \boxed{\frac{1}{42}e^{7\tan\left(3t^2\right)} + C}.\end{aligned}$$

(e) $\displaystyle\int e^x(e^x-2)^{2/3}dx$

Solution. We use the substitution $u = e^x - 2$, so that $du = e^x dx$. This gives

$$\begin{aligned}\int e^x(e^x-2)^{2/3}dx &= \int u^{2/3}du\\ &= \frac{3}{5}u^{5/3} + C\\ &= \boxed{\frac{3}{5}(e^x-2)^{5/3} + C}.\end{aligned}$$

(f) $\displaystyle\int e^{2x}(e^x-2)^{2/3}dx$

Solution. We again use the substitution $u = e^x - 2$, $du = e^x dx$. But this time, we have an extraneous factor e^x since $e^{2x} = e^x e^x$. We can express this extraneous factor in terms of u as $e^x = u+2$. Therefore

$$\begin{aligned}\int e^{2x}(e^x-2)^{2/3}dx &= \int e^x(e^x-2)^{2/3}e^x dx\\ &= \int (u+2)u^{2/3}du\\ &= \int\left(u^{5/3}+2u^{2/3}\right)du\\ &= \frac{3}{8}u^{8/3} + \frac{6}{5}u^{5/3} + C\\ &= \boxed{\frac{3}{8}(e^x-2)^{8/3} + \frac{6}{5}(e^x-2)^{5/3} + C}.\end{aligned}$$

(g) $\displaystyle\int_{e^3}^{e^6} \frac{dt}{t\ln(t)}$

Solution. We use the substitution $u = \ln(t)$, so that $du = \frac{dt}{t}$. The bounds change as follows

$$\begin{aligned} t = e^3 &\Rightarrow u = \ln(e^3) = 3, \\ t = e^6 &\Rightarrow u = \ln(e^6) = 6. \end{aligned}$$

The integral becomes

$$\begin{aligned} \int_{e^3}^{e^6} \frac{dt}{t\ln(t)} &= \int_3^6 \frac{du}{u} \\ &= [\ln|u|]_3^6 \\ &= \ln(6) - \ln(3) \\ &= \ln\left(\frac{6}{3}\right) \\ &= \boxed{\ln(2)}. \end{aligned}$$

(h) $\displaystyle\int \frac{dx}{5x + 4\sqrt{x}}$

Solution. We can first factor out a $\sqrt{x}$ from the denominator, which gives

$$\int \frac{dx}{5x + 4\sqrt{x}} = \int \frac{dx}{\sqrt{x}\,(5\sqrt{x} + 4)}.$$

We can then use the substitution $u = 5\sqrt{x} + 4$, which gives $du = \frac{5dx}{2\sqrt{x}}$. This gives $\frac{dx}{\sqrt{x}} = \frac{2du}{5}$, and the integral becomes

$$\begin{aligned} \int \frac{dx}{\sqrt{x}\,(5\sqrt{x} + 4)} &= \int \frac{2du}{5u} \\ &= \frac{2}{5}\ln|u| + C \\ &= \boxed{\frac{2}{5}\ln|5\sqrt{x} + 4| + C}. \end{aligned}$$

(i) $\displaystyle\int \frac{dx}{\sqrt{2 - x^2}}$

Solution. Recall the reference antiderivative

$$\int \frac{du}{\sqrt{1 - u^2}} = \sin^{-1}(u) + C.$$

We can use this antiderivative after factoring out a 2 from the square root and letting $u = \frac{x}{\sqrt{2}}$. This gives

$$\begin{aligned}
\int \frac{dx}{\sqrt{2-x^2}} &= \int \frac{dx}{\sqrt{2\left(1-\frac{x^2}{2}\right)}} \\
&= \int \frac{dx}{\sqrt{2}\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}} \\
&= \int \frac{du}{\sqrt{1-u^2}} \\
&= \sin^{-1}(u) + C \\
&= \boxed{\sin^{-1}\left(\frac{x}{\sqrt{2}}\right) + C}.
\end{aligned}$$

(j) $\displaystyle\int_0^1 \frac{xdx}{\sqrt{2-x^2}}$

Solution 1. This time, the numerator is (up to a constant factor) the derivative of the inside of the square root. Therefore, we can compute this integral with the substitution $u = 2 - x^2$, $du = -2xdx$. Thus we have $xdx = -\frac{du}{2}$, and the bounds change to

$$\begin{aligned}
x = 0 \;\Rightarrow\; u = 2 - 0^2 = 2, \\
x = 1 \;\Rightarrow\; u = 2 - 1^2 = 1.
\end{aligned}$$

We obtain

$$\begin{aligned}
\int_0^1 \frac{xdx}{\sqrt{2-x^2}} &= \int_2^1 -\frac{du}{2\sqrt{u}} \\
&= \left[-\sqrt{u}\right]_2^1 \\
&= \boxed{1-\sqrt{2}}.
\end{aligned}$$

Solution 2. We can be more ambitious with the substitution and let $u = \sqrt{2-x^2}$. The bounds change to

$$\begin{aligned}
x = 0 \;\Rightarrow\; u = \sqrt{2-0^2} = \sqrt{2}, \\
x = 1 \;\Rightarrow\; u = \sqrt{2-1^2} = 1.
\end{aligned}$$

Differentiating gives $du = -\frac{xdx}{\sqrt{2-x^2}}$, which is the entire integrand up to a negative sign. So the integral becomes

$$\int_0^1 \frac{xdx}{\sqrt{2-x^2}} = \int_{\sqrt{2}}^1 - du = \boxed{\sqrt{2}-1}.$$

(k) $\displaystyle\int_0^{2/3} \frac{dz}{4+9z^2}$

Solution. This integral will make use of the reference antiderivative

$$\int \frac{du}{1+u^2} = \tan^{-1}(u) + C.$$

To get to this form, we factor out a 4 from the denominator to obtain

$$\int_0^{2/3} \frac{dz}{4+9z^2} = \int_0^{2/3} \frac{dz}{4\left(1+\frac{9z^2}{4}\right)} = \int_0^{2/3} \frac{dz}{4\left(1+\left(\frac{3z}{2}\right)^2\right)}$$

We can then use the substitution $u = \frac{3z}{2}$, which gives $du = \frac{3dz}{2}$, so $dz = \frac{2du}{3}$. The bounds change to

$$x = 0 \;\Rightarrow\; u = 0,$$
$$x = \frac{2}{3} \;\Rightarrow\; u = 1.$$

We obtain

$$\begin{aligned}
\int_0^{2/3} \frac{dz}{4\left(1+\left(\frac{3z}{2}\right)^2\right)} &= \int_0^1 \frac{1}{4}\cdot\frac{2}{3}\cdot\frac{du}{1+u^2} \\
&= \left[\frac{1}{6}\tan^{-1}(u)\right]_0^1 \\
&= \frac{1}{6}\left(\tan^{-1}(1) - \tan^{-1}(0)\right) \\
&= \frac{1}{6}\left(\frac{\pi}{4} - 0\right) \\
&= \boxed{\frac{\pi}{24}}.
\end{aligned}$$

(l) $\displaystyle\int \frac{dx}{x^2+6x+34}$

Solution. We will need to complete the square first to see a sum or difference of squares in the denominator:

$$x^2 + 6x + 34 = (x^2 + 6x + 9) - 9 + 34 = (x+3)^2 + 25$$

Therefore the integral becomes

$$\begin{aligned}
\int \frac{dx}{x^2+6x+34} &= \int \frac{dx}{(x+3)^2+25} \\
&= \int \frac{dx}{25\left(\frac{(x+3)^2}{25}+1\right)} \\
&= \frac{1}{25}\int \frac{dx}{\left(\frac{x+3}{5}\right)^2+1} \\
&= \frac{1}{25}\int \frac{5du}{u^2+1} \qquad \left(u = \frac{x+3}{2},\; dx = 5du\right) \\
&= \frac{1}{5}\tan^{-1}(u) + C \\
&= \boxed{\frac{1}{5}\tan^{-1}\left(\frac{x+3}{5}\right) + C}.
\end{aligned}$$

Rutgers University
Math 152

Sections 5.5, 5.6, 8.1: Review of Integration - Worksheet Solutions

1. Evaluate the following antiderivatives.

(a) $\displaystyle\int \frac{dx}{\sqrt{8x - x^2}}$

Solution. We will be able to use the reference antiderivative

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}(u) + C$$

after completing the square in the square root and using a substitution. Completing the square gives

$$8x - x^2 = -(x^2 - 8x) = -(x^2 - 8x + 16) + 16 = 16 - (x-4)^2.$$

Therefore the integral can be written as

$$\int \frac{dx}{\sqrt{8x-x^2}} = \int \frac{dx}{\sqrt{16-(x-4)^2}} = \int \frac{dx}{\sqrt{16\left(1 - \frac{(x+4)^2}{16}\right)}} = \int \frac{dx}{4\sqrt{1-\left(\frac{x-4}{4}\right)^2}}.$$

We substitute $u = \frac{x-4}{4}$, which gives $du = \frac{dx}{4}$. The integral becomes

$$\begin{aligned} \int \frac{dx}{4\sqrt{1-\left(\frac{x-4}{4}\right)^2}} &= \int \frac{du}{\sqrt{1-u^2}} \\ &= \sin^{-1}(u) + C \\ &= \boxed{\sin^{-1}\left(\frac{x-4}{4}\right) + C}. \end{aligned}$$

(b) $\displaystyle\int \frac{\tan^{-1}(t)^3}{1+t^2}dt$

Solution. We use the substitution $u = \tan^{-1}(t)$, so $du = \frac{dt}{1+t^2}$. This gives

$$\begin{aligned} \int \frac{\tan^{-1}(t)^3}{1+t^2}dt &= \int u^3 du \\ &= \frac{1}{4}u^4 + C \\ &= \boxed{\frac{1}{4}\tan^{-1}(t)^4 + C}. \end{aligned}$$

(c) $\displaystyle\int \frac{\tan(3\ln(x))}{x}dx$

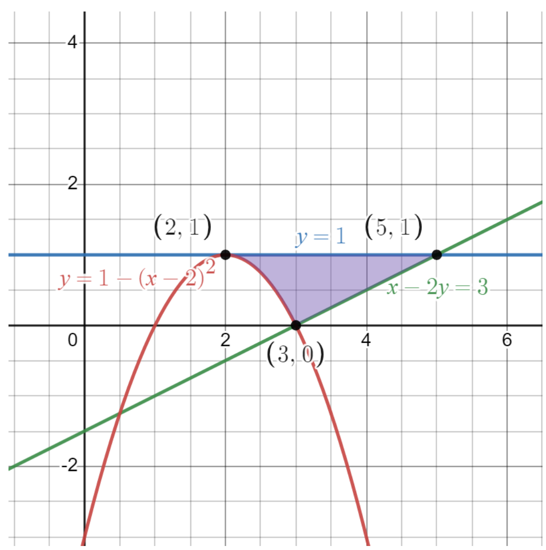
Solution. We use the substitution $u = 3\ln(x)$, so $du = \frac{3dx}{x}$ and the integral becomes

$$\begin{aligned}\int \frac{\tan(3\ln(x))}{x}dx &= \int \frac{\tan(u)}{3}du \\ &= \frac{1}{3}\ln|\sec(u)| + C \\ &= \boxed{\frac{1}{3}\ln|\sec(3\ln(x))| + C}.\end{aligned}$$

2. For each of the regions described below (i) sketch the region, clearly labeling the curves and their intersection points, (ii) calculate the area of the region using an x-integral and (iii) calculate the area of the region using a y-integral.

(a) The region to the right of the parabola $y = 1 - (x-2)^2$, below the line $y = 1$ and to the left of the line $x - 2y = 3$.

Solution. (i)



(ii) The region is not vertically simple, so we will need a sum of x-integrals. For $2 \leqslant x \leqslant 3$, the vertical strip at x is bounded by $y = 1$ on the top and $y = 1-(x-2)^2$ on the bottom. For $3 \leqslant x \leqslant 5$, the vertical strip at x is bounded by $y = 1$ on the top and the line $x - 2y = 3 \;\Rightarrow\; y = \frac{x-3}{2}$ on the bottom. Therefore the area is given by

$$\begin{aligned}A &= \int_2^3 \left(1 - (1-(x-2)^2)\right)dx + \int_3^5 \left(1 - \frac{x-3}{2}\right)dx \\ &= \int_2^3 (x-2)^2 dx + \frac{1}{2}\int_3^5 (5-x)dx \\ &= \left[\frac{(x-2)^3}{3}\right]_2^3 + \frac{1}{2}\left[5x - \frac{x^2}{2}\right]_3^5\end{aligned}$$

$$= \frac{(3-2)^3}{3} - \frac{(2-2)^3}{3} + \frac{1}{2}\left(5\cdot 5 - \frac{5^2}{2} - 5\cdot 3 + \frac{3^2}{2}\right)$$
$$= \boxed{\frac{4}{3} \text{ square units}}.$$

(iii) The region is horizontally simple. The horizontal strip at y is bounded on the right by the line $x - 2y = 3$, which gives $x = 2y + 3$ when expressed as a function of y. The curve bounding on the left is the right branch of the parabola $y = 1 - (x-2)^2$. Expressing this branch as a function of y gives

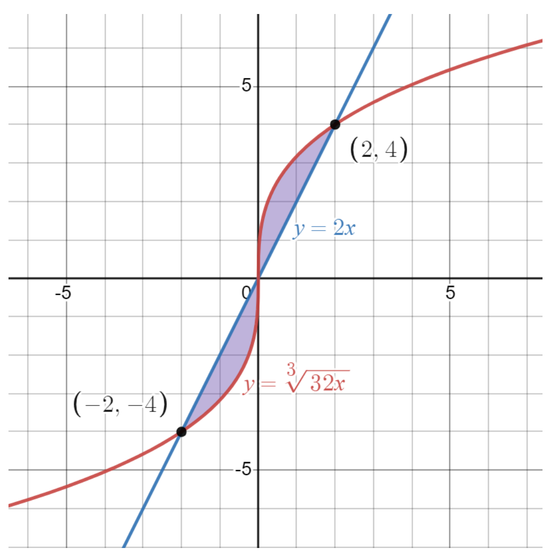
$$y = 1-(x-2)^2 \;\Rightarrow\; (x-2)^2 = 1-y \;\Rightarrow |x-2| = \sqrt{1-y} \;\Rightarrow\; x-2 = \sqrt{1-y} \;\Rightarrow\; x = 2+\sqrt{1-y}.$$

Note that $|x-2| = x-2$ since $x - 2 \geqslant 0$ on the right branch of the parabola. Therefore the area is

$$A = \int_0^1 \left((2y+3) - \left(2+\sqrt{1-y}\right)\right) dy$$
$$= \int_0^1 \left(2y + 1 - \sqrt{1-y}\right) dy$$
$$= \left[y^2 + y + \frac{2}{3}(1-y)^{3/2}\right]_0^1$$
$$= 1 + 1 - \frac{2}{3}$$
$$= \boxed{\frac{4}{3} \text{ square units}}.$$

(b) The region bounded by the curves $y = 2x$ and $y = \sqrt[3]{32x}$.

Solution. (i)



(ii) The region is not vertically simple. For $0 \leqslant x \leqslant 2$, the vertical strip at x is bounded on the top by $y = \sqrt[3]{32x}$ and on the bottom by $y = 2x$. For $-2 \leqslant x \leqslant 0$, the vertical strip at x is bounded on the top by $y = 2x$ and on the bottom by $y = \sqrt[3]{32x}$. Therefore

$$\begin{aligned} A &= \int_{-2}^{0} \left(2x - \sqrt[3]{32x}\right) dx + \int_{0}^{2} \left(\sqrt[3]{32x} - 2x\right) dx \\ &= \left[x^2 - 32^{1/3}\frac{3}{4}x^{4/3}\right]_{-2}^{0} + \left[32^{1/3}\frac{3}{4}x^{4/3} - x^2\right]_{0}^{2} \\ &= -(-2)^2 + 32^{1/3}\frac{3}{4}(-2)^{4/3} + 32^{1/3}\frac{3}{4}2^{4/3} - 2^2 \\ &= \boxed{4 \text{ square units}}. \end{aligned}$$

(iii) We need to express the curves as functions of y to use a y-integral.

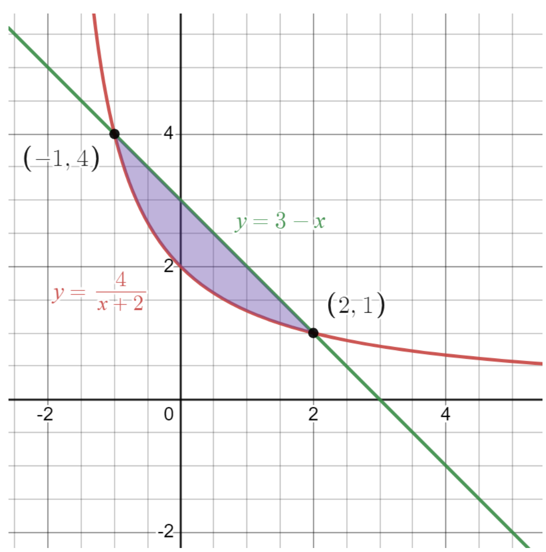
$$\begin{aligned} &y = \sqrt[3]{32x} \;\Rightarrow\; 32x = y^3 \;\Rightarrow\; x = \frac{y^3}{32}, \\ &y = 2x \;\Rightarrow\; x = \frac{y}{2}. \end{aligned}$$

The region is not horizontally simple. For $0 \leqslant y \leqslant 4$, the horizontal strip at y is bounded on the right by $x = \frac{y}{2}$ and on the left by $x = \frac{y^3}{32}$. For $-4 \leqslant y \leqslant 0$, the horizontal strip at y is bounded on the right by $x = \frac{y^3}{32}$ and on the left by $x = \frac{y}{2}$. Therefore

$$\begin{aligned} A &= \int_{-4}^{0} \left(\frac{y^3}{32} - \frac{y}{2}\right) dy + \int_{0}^{4} \left(\frac{y}{2} - \frac{y^3}{32}\right) dy \\ &= \left[\frac{y^4}{128} - \frac{y^2}{4}\right]_{-4}^{0} + \left[\frac{y^2}{4} - \frac{y^4}{128}\right]_{0}^{4} \\ &= -\frac{(-4)^4}{128} + \frac{(-4)^2}{4} + \frac{4^2}{4} - \frac{4^4}{128} \\ &= \boxed{4 \text{ square units}}. \end{aligned}$$

(c) The region bounded by the curves $y = \dfrac{4}{x+2}$ and $y = 3 - x$.

Solution. (i)



(ii) The region is vertically simple. The vertical strip at x is bounded on the top by $y = 3 - x$ and on the bottom by $y = \frac{4}{x+2}$. Therefore the area is

$$\begin{aligned}
A &= \int_{-1}^{2} \left((3 - x) - \frac{4}{x+2} \right) dx \\
&= \left[3x - \frac{1}{2}x^2 - 4\ln|x+2| \right]_{-1}^{2} \\
&= \left(3 \cdot 2 - \frac{1}{2}2^2 - 4\ln(4) \right) - \left(-3 - \frac{1}{2} - 4\ln(1) \right) \\
&= \boxed{\frac{15}{2} - 8\ln(2) \text{ units}^2}.
\end{aligned}$$

(iii) We need to express the curves as functions of y.

$$\begin{aligned}
y &= 3 - x \;\Rightarrow\; x = 3 - y, \\
y &= \frac{4}{x+2} \;\Rightarrow\; x + 2 = \frac{4}{y} \;\Rightarrow\; x = \frac{4}{y} - 2.
\end{aligned}$$

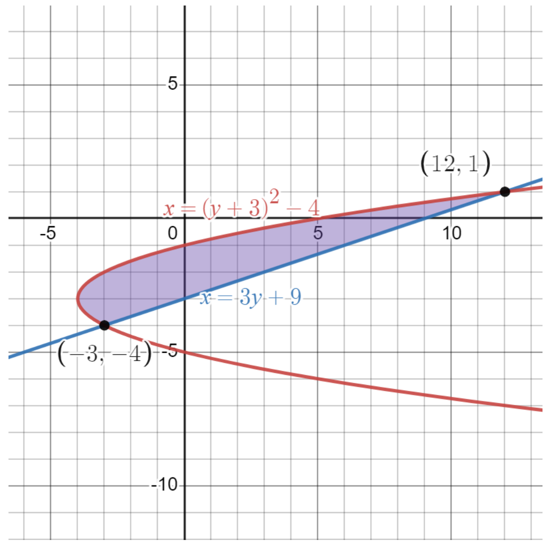
The region is horizontally simple. The horizontal strip at y is bounded on the right by $x = 3 - y$ and on the left by $x = \dfrac{4}{y} - 2$. Therefore

$$\begin{aligned}
A &= \int_{1}^{4} \left((3 - y) - \left(\frac{4}{y} - 2 \right) \right) dy \\
&= \int_{1}^{4} \left(5 - y - \frac{4}{y} \right) dy \\
&= \left[5y - \frac{1}{2}y^2 - 4\ln|y| \right]_{1}^{4} \\
&= \left(5 \cdot 4 - \frac{1}{2}4^2 - 4\ln(4) \right) - \left(5 - \frac{1}{2} - 4\ln(1) \right) \\
&= \boxed{\frac{15}{2} - 8\ln(2) \text{ square units}}.
\end{aligned}$$

3. Calculate the area of the regions described below.

   (a) The region bounded by the parabola $x = (y+3)^2 - 4$ and the line $x = 3y + 9$.

   Solution. A sketch of the region is included below.

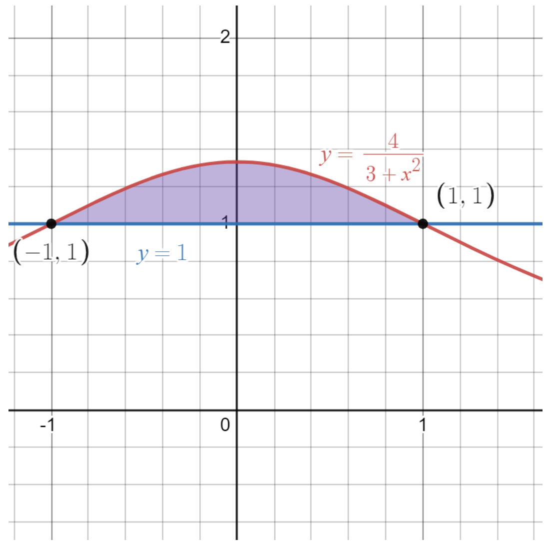


The region is horizontally simple, but not vertically simple. So computing the area using horizontal strips/integration with respect to y will be simpler than using vertical strips/integration with respect to x since we will only need one integral. The horizontal strip at y is bounded on the right by the line $x = 3y + 9$ and on the left by the parabola $x = (y+3)^2 - 4$. Therefore the area is given by

$$\begin{aligned}
A &= \int_{-4}^{1} \big((3y+9) - ((y+3)^2 - 4)\big)\, dy \\
&= \int_{-4}^{1} \big(3y + 13 - (y^2 + 6y + 9)\big)\, dy \\
&= \int_{-4}^{1} \big(4 - 3y - y^2\big)\, dy \\
&= \left[4y - \frac{3}{2}y^2 - \frac{1}{3}y^3\right]_{-4}^{1} \\
&= \left(4 - \frac{3}{2} - \frac{1}{3}\right) - \left(4(-4) - \frac{3}{2}(-4)^2 - \frac{1}{3}(-4)^3\right) \\
&= \boxed{\frac{125}{6} \text{ square units}}.
\end{aligned}$$

(b) The region bounded by $y = \dfrac{4}{3+x^2}$ and $y = 1$.

Solution. A sketch of the region is included below.



Note that the region is both vertically and horizontally simple. So we would need only one integral to compute the area using integration with respect to either x or y. However, integration with respect to x will be simpler here, both to set up the integral and compute the antiderivative. The vertical strip at x is bounded on the top by $y = \frac{4}{3+x^2}$ and on the bottom by $y = 1$. So the area is given by

$$A = \int_{-1}^{1} \left(\frac{4}{3+x^2} - 1 \right) dx = 2 \int_{0}^{1} \left(\frac{4}{3+x^2} - 1 \right) dx,$$

the second equality holding because the integrand is even (or equivalently, because the region is symmetric with respect to the y-axis). To compute the antiderivative of the first term of the integrand, we can factor out a 3 from the denominator and use the reference antiderivative

$$\int \frac{du}{1+u^2} = \tan^{-1}(u) + C.$$

This gives

$$\begin{aligned} \int \frac{4dx}{3+x^2} &= \int \frac{4dx}{3\left(1+\frac{x^2}{3}\right)} \\ &= \frac{4}{3} \int \frac{dx}{1+\left(\frac{x}{\sqrt{3}}\right)^2} \\ &= \frac{4}{3} \int \frac{\sqrt{3}du}{1+u^2} \qquad \left(u = \frac{x}{\sqrt{3}}\right) \\ &= \frac{4\sqrt{3}}{3} \tan^{-1}(u) + C \\ &= \frac{4\sqrt{3}}{3} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C. \end{aligned}$$

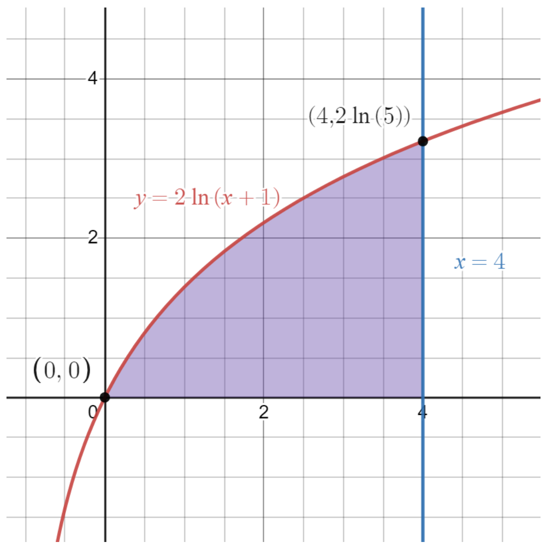
We can now use this to compute the area. We obtain

$$\begin{aligned} A &= 2 \int_{0}^{1} \left(\frac{4}{3+x^2} - 1 \right) dx \\ &= 2 \left[\frac{4\sqrt{3}}{3} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - x \right]_{0}^{1} \end{aligned}$$

$$
\begin{aligned}
&= 2\left(\frac{4\sqrt{3}}{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - 1\right) \\
&= 2\left(\frac{4\sqrt{3}}{3}\cdot\frac{\pi}{6} - 1\right) \\
&= \boxed{2\left(\frac{2\sqrt{3}\pi}{9} - 1\right) \text{ square units}}.
\end{aligned}
$$

(c) The region bounded by $y = 2\ln(x+1)$, the x-axis and the line $x = 4$.

Solution. A sketch of the region is included below.



The region is both vertically and horizontally simple. Calculating the area using an x-integral would require finding an antiderivative of ln, which we do not know how to do (yet! we will learn how to do this in section 8.2). So we will prefer a y-integral here. We can express the curve $y = 2\ln(x+1)$ as a function of y as follows

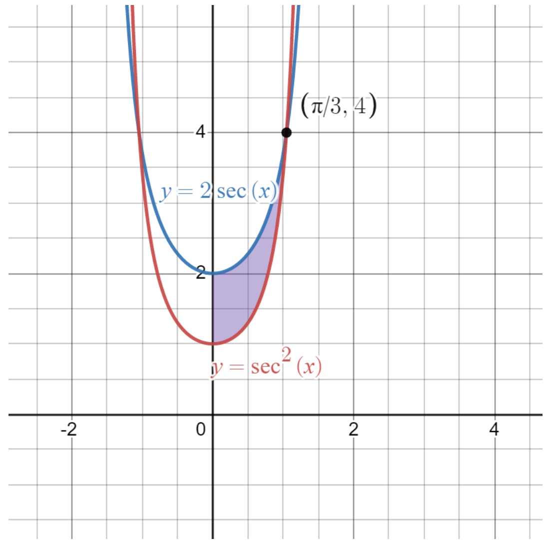
$$
y = 2\ln(x+1) \;\Rightarrow\; \frac{y}{2} = \ln(x+1) \;\Rightarrow\; x + 1 = e^{y/2} \;\Rightarrow x = e^{y/2} - 1.
$$

The horizontal strip at y is bounded by the line $x = 4$ on the right and the curve $x = e^{y/2} - 1$ on the left. Therefore the area is

$$
\begin{aligned}
A &= \int_0^{2\ln(5)} \left(4 - \left(e^{y/2} - 1\right)\right) dy \\
&= \int_0^{2\ln(5)} \left(5 - e^{y/2}\right) dy \\
&= \left[5y - 2e^{y/2}\right]_0^{2\ln(5)} \\
&= \left(10\ln(5) - 2e^{\ln(5)}\right) - (-2) \\
&= \boxed{10\ln(5) - 8 \text{ square units}}.
\end{aligned}
$$

(d) The region to the right of the y-axis, above the graph of $y = \sec(x)^2$ and below the graph of $y = 2\sec(x)$.

Solution. A sketch of the region is included below.



The region is vertically simple. The vertical strip at x is bounded on the top by $y = 2\sec(x)$ and on the bottom by $y = \sec(x)^2$. Therefore the area is given by

$$\begin{aligned} A &= \int_0^{\pi/3} \left(2\sec(x) - \sec(x)^2\right) dx \\ &= \left[2\ln|\sec(x) + \tan(x)| - \tan(x)\right]_0^{\pi/3} \\ &= \left(2\ln\left|\sec\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{3}\right)\right| - \tan\left(\frac{\pi}{3}\right)\right) - (2\ln|\sec(0) + \tan(0)| - \tan(0)) \\ &= \left(2\ln\left|2 + \sqrt{3}\right| - \sqrt{3}\right) - (2\ln|1 + 0| - 0) \\ &= \boxed{2\ln(2 + \sqrt{3}) - \sqrt{3} \text{ square units}}. \end{aligned}$$

4. Suppose that f is an **even** function such that

$$\int_{-9}^{5} f(x)dx = -13 \quad \text{and} \quad \int_0^9 f(x)dx = 4.$$

Evaluate the definite integrals below.

(a) $\displaystyle\int_{-9}^{9} f(x)dx$

Solution. Since f is even, by symmetry we have

$$\int_{-9}^{9} f(x)dx = 2\int_0^9 f(x)dx = \boxed{8}.$$

(b) $\int_0^5 (4x - 3f(x))dx$

Solution. Let us start by calculating $\int_0^5 f(x)dx$. By additivity of the integral, we have

$$\int_{-9}^5 f(x)dx = \int_{-9}^0 f(x)dx + \int_0^5 f(x)dx.$$

Since f is even, we have

$$\int_{-9}^0 f(x)dx = \int_0^9 f(x)dx = 4.$$

So we get

$$-13 = 4 + \int_0^5 f(x)dx \ \Rightarrow \ \int_0^5 f(x)dx = -17.$$

Now using the linearity of the integral, we obtain

$$\begin{aligned}\int_0^5 (4x - 3f(x))dx &= 4\int_0^5 xdx - 3\int_0^5 f(x)dx \\ &= 4\left[\frac{1}{2}x^2\right]_0^5 - 3(-17) \\ &= 4\frac{1}{2}(25) + 51 \\ &= \boxed{101}.\end{aligned}$$

(c) $\int_{-3}^3 xf(x)dx$

Solution. Since f is even, the function $g(x) = xf(x)$ is odd, as shown below:

$$g(-x) = (-x)f(-x) = -xf(x) = -g(x).$$

Since the interval of integration $[-3, 3]$ is centered at 0, we deduce

$$\boxed{\int_{-3}^3 xf(x) = 0}.$$

(d) $\int_0^3 xf(x^2)dx$

Solution. We can evaluate this integral using the substitution $u = x^2$, which gives $du = 2xdx$, or $xdx = \frac{du}{2}$. The bounds become

$$\begin{aligned}x = 0 &\ \Rightarrow \ u = 0^2 = 0, \\ x = 3 &\ \Rightarrow \ u = 3^2 = 9.\end{aligned}$$

Therefore

$$\int_0^3 xf(x^2)dx = \int_0^9 \frac{1}{2}f(u)du$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^9 f(u)du \\
&= \frac{1}{2}4 \\
&= \boxed{2}.
\end{aligned}
$$

5. Find the average value of the following functions on the given interval.

   (a) $f(x) = \dfrac{3}{\sqrt{100-x^2}}$ on $[0, 5]$.

   Solution. The average value is given by

$$
\begin{aligned}
\text{av}(f) &= \frac{1}{5-0}\int_0^5 \frac{3}{\sqrt{100-x^2}}dx \\
&= \frac{3}{5}\int_0^5 \frac{dx}{\sqrt{100\left(1-\frac{x^2}{100}\right)}} \\
&= \frac{3}{5}\int_0^5 \frac{dx}{10\sqrt{1-\left(\frac{x}{10}\right)^2}} \\
&= \frac{3}{5}\int_0^{1/2} \frac{du}{\sqrt{1-u^2}} \qquad \left(u = \frac{x}{10}\right) \\
&= \frac{3}{5}\left[\sin^{-1}(u)\right]_0^{1/2} \\
&= \frac{3}{5}\left(\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)\right) \\
&= \frac{3}{5}\cdot\frac{\pi}{6} \\
&= \boxed{\frac{\pi}{10}}.
\end{aligned}
$$

   (b) $f(x) = x\sqrt[3]{3x-7}$ on $[2, 5]$.

   Solution. The average value is given by

$$
\text{av}(f) = \frac{1}{5-2}\int_2^5 x\sqrt[3]{3x-7}dx = \frac{1}{3}\int_2^5 x\sqrt[3]{3x-7}dx.
$$

   We can calculate the integral using the substitution $u = 3x-7$. This will give $du = 3dx$, or $dx = \frac{du}{3}$. The bounds will become

$$
\begin{aligned}
x = 2 &\Rightarrow u = 3\cdot 2 - 7 = -1, \\
x = 5 &\Rightarrow u = 3\cdot 5 - 7 = 8.
\end{aligned}
$$

   Finally, the extraneous factor x in the integrand can be expressed in terms of u as $x = \frac{u+7}{3}$. We obtain

$$
\text{av}(f) = \frac{1}{3}\int_{-1}^8 \frac{u+7}{3}\sqrt[3]{u}\frac{du}{3}
$$

$$
\begin{aligned}
&= \frac{1}{27}\int_{-1}^{8} \left(u^{4/3} + 7u^{1/3}\right) du \\
&= \frac{1}{27}\left[\frac{3}{7}u^{7/3} + \frac{21}{4}u^{4/3}\right]_{-1}^{8} \\
&= \frac{1}{27}\left(\left(\frac{3}{7}8^{7/3} + \frac{21}{4}8^{4/3}\right) - \left(\frac{3}{7}(-1)^{7/3} + \frac{21}{4}(-1)^{4/3}\right)\right) \\
&= \boxed{\frac{139}{28}}.
\end{aligned}
$$

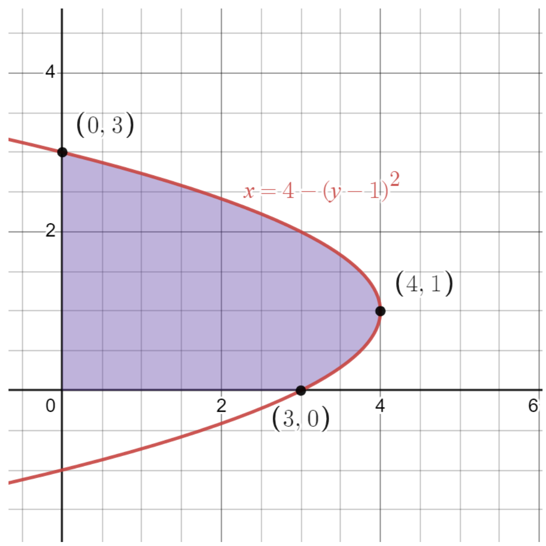
Rutgers University
Math 152

Section 6.1: Volume by Cross-Sections - Worksheet Solutions

1. Consider the region $\mathcal{R}$ in the first quadrant bounded by the curve $x = 4 - (y-1)^2$.

   (a) Sketch the region. Make sure to clearly label the curve and its intercepts.

   Solution.



   (b) A solid has base $\mathcal{R}$ and cross-sections perpendicular to the y-axis. Calculate the volume of the solid if the cross-sections are (i) semi-circles with diameter in the base and (ii) equilateral triangles with a side in the base.

   Solution. The horizontal strip at y in the region is bounded on the right by $x = 4 - (y-1)^2$ and on the left by $x = 0$. Therefore it has length $\ell(y) = 4 - (y-1)^2 - 0 = 4 - (y^2 - 2y + 1) = 3 + 2y - y^2$.

   (i) A semi-circle of diameter ℓ has area $\frac{1}{2}\pi\left(\frac{\ell}{2}\right)^2 = \frac{\pi}{8}\ell^2$. With the length of the strip $\ell(y)$ we found above, we can express the area of the cross-section at y as

$$A(y) = \frac{\pi}{8}\ell(y)^2 = \frac{\pi}{8}\left(3 + 2y - y^2\right)^2 = \frac{\pi}{8}\left(9 + 12y - 2y^2 - 4y^3 + y^4\right).$$

   So the volume of the solid is

$$\begin{aligned} V &= \int_0^3 A(y)dy \\ &= \int_0^3 \frac{\pi}{8}\left(9 + 12y - 2y^2 - 4y^3 + y^4\right) dy \end{aligned}$$

$$
\begin{aligned}
&= \frac{\pi}{8}\left[9y + 6y^2 - \frac{2}{3}y^3 - y^4 + \frac{1}{5}y^5\right]_0^3 \\
&= \frac{\pi}{8}\left(9 \cdot 3 + 6(3)^2 - \frac{2}{3}(3)^3 - 3^4 + \frac{1}{5}(3)^5\right) \\
&= \boxed{\frac{63\pi}{40} \text{ cubic units}}.
\end{aligned}
$$

(ii) An equilateral triangle with side length ℓ has area $\frac{\sqrt{3}}{4}\ell^2$. With the length of the strip $\ell(y)$ we found above, we can express the area of the cross-section at y as

$$
A(y) = \frac{\sqrt{3}}{4}\ell(y)^2 = \frac{\sqrt{3}}{4}\left(3 + 2y - y^2\right)^2 = \frac{\sqrt{3}}{4}\left(9 + 12y - 2y^2 - 4y^3 + y^4\right).
$$

So the volume of the solid is

$$
\begin{aligned}
V &= \int_0^3 A(y)dy \\
&= \int_0^3 \frac{\sqrt{3}}{4}\left(9 + 12y - 2y^2 - 4y^3 + y^4\right) dy \\
&= \frac{\sqrt{3}}{4}\left[9y + 6y^2 - \frac{2}{3}y^3 - y^4 + \frac{1}{5}y^5\right]_0^3 \\
&= \frac{\sqrt{3}}{4}\left(9 \cdot 3 + 6(3)^2 - \frac{2}{3}(3)^3 - 3^4 + \frac{1}{5}(3)^5\right) \\
&= \boxed{\frac{63\sqrt{3}}{20} \text{ cubic units}}.
\end{aligned}
$$

Note: we could have used the integral that we had already compute in part (i) to minimize computations:

$$
\int_0^3 \left(9 + 12y - 2y^2 - 4y^3 + y^4\right) dy = \frac{63}{5}.
$$

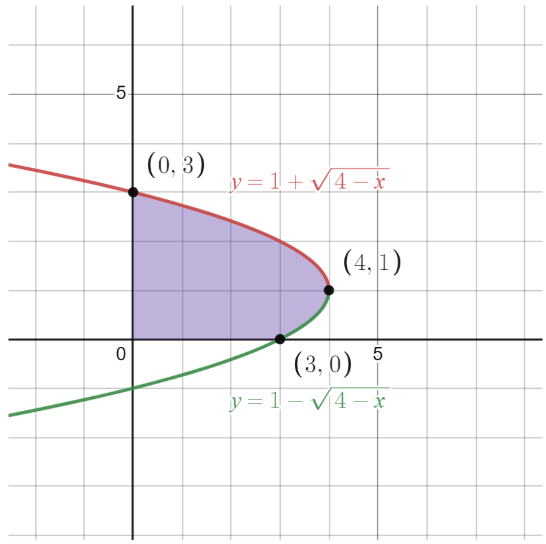
Only the factor in front of the integral changed for part (ii).

(c) A solid has base $\mathcal{R}$ and its cross-sections perpendicular to the x-axis are isosceles right triangles with hypotenuse in the base. Calculate the volume of the solid.

Solution. We need to express the curve as functions of x:

$$
x = 4 - (y-1)^2 \;\Rightarrow\; (y-1)^2 = 4 - x \;\Rightarrow |y - 1| = \sqrt{4-x} \;\Rightarrow\; y = 1 \pm \sqrt{4-x}.
$$

The solution with the positive sign corresponds to the upper branch of the parabola and the negative sign corresponds to the lower branch of the parabola, see figure below.



The vertical strip at x is bounded on the top by $y = 1 + \sqrt{4 - x}$. On the bottom, the vertical strip at x is bounded by $y = 0$ for $0 \leqslant x \leqslant 3$ and $y = 1 - \sqrt{4 - x}$ for $3 \leqslant x \leqslant 4$. Therefore, the length of the strip is

$$\ell(x) = \begin{cases} 1 + \sqrt{4 - x} - 0 = 1 + \sqrt{4 - x} & \text{if } 0 \leqslant x \leqslant 3, \\ (1 + \sqrt{4 - x}) - (1 - \sqrt{4 - x}) = 2\sqrt{4 - x} & \text{if } 3 \leqslant x \leqslant 4. \end{cases}$$

The length of an isosceles right triangle with hypotenuse ℓ is $\frac{1}{4}\ell^2$, so the area of the cross-section at x is

$$A(x) = \frac{1}{4}\ell(x)^2 = \begin{cases} \frac{1}{4}\left(1 + \sqrt{4 - x}\right)^2 = \dfrac{1}{4}\left(5 - x + 2\sqrt{4 - x}\right) & \text{if } 0 \leqslant x \leqslant 3, \\ \frac{1}{4}\left(2\sqrt{4 - x}\right)^2 = 4 - x & \text{if } 3 \leqslant x \leqslant 4. \end{cases}$$

So the volume of the solid is given by

$$\begin{aligned} V &= \int_0^4 A(x)dx \\ &= \int_0^3 \frac{1}{4}\left(5 - x + 2\sqrt{4 - x}\right) dx + \int_3^4 (4 - x)dx \\ &= \frac{1}{4}\left[5x - \frac{1}{2}x^2 - \frac{4}{3}(4 - x)^{3/2}\right]_0^3 + \left[4x - \frac{1}{2}x^2\right]_3^4 \\ &= \frac{1}{4}\left(15 - \frac{9}{2} - \frac{4}{3} + \frac{4}{3}4^{3/2}\right) + \left(16 - \frac{1}{2}16 - 12 + \frac{9}{2}\right) \\ &= \boxed{\frac{131}{24} \text{ cubic units}}. \end{aligned}$$

(d) Calculate the volume of the solid of revolution obtained by revolving $\mathcal{R}$ about (i) the y-axis and (ii) the line $y = -2$.

Solution. (i) Revolving the horizontal strip at y about the y-axis creates a disk of radius $r(y) = 4 - (y - 1)^2 = 3 + 2y - y^2$. So the volume is given by

$$V = \int_0^3 \pi r(y)^2 dy$$

$$= \pi \int_0^3 (3 + 2y - y^2)^2 dy$$
$$= \boxed{\frac{63\pi}{5} \text{ units}^3}.$$

(We have used the integral already computed in part (b)(i).)

(ii) Revolving the vertical strip at x about the line $y = -2$ creates a washer. The outer radius of the washer is

$$r_{\text{out}}(x) = 1 + \sqrt{4-x} - (-2) = 3 + \sqrt{4-x}.$$

The inner radius is

$$r_{\text{in}}(x) = \begin{cases} 0 - (-2) = 2 & \text{if } 0 \leqslant x \leqslant 3, \\ 1 - \sqrt{4-x} - (-2) = 3 - \sqrt{4-x} & \text{if } 3 \leqslant x \leqslant 4. \end{cases}$$

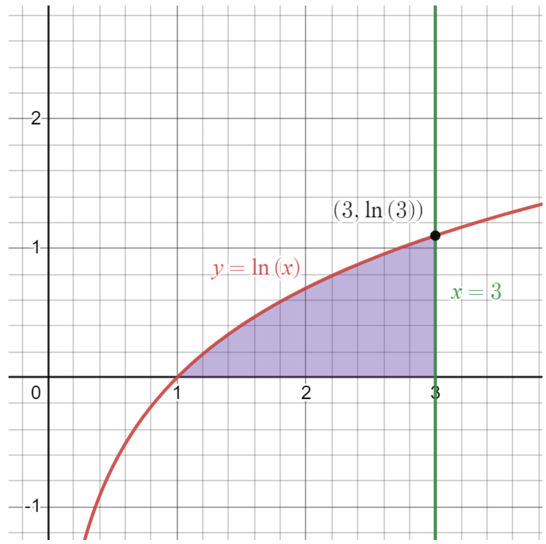
So the volume is given by

$$\begin{aligned} V &= \int_0^4 \pi \left(r_{\text{out}}(x)^2 - r_{\text{in}}(x)^2\right) dx \\ &= \int_0^3 \pi \left((3+\sqrt{4-x})^2 - 2^2\right) dx + \int_3^4 \pi \left((3+\sqrt{4-x})^2 - (3-\sqrt{4-x})^2\right) dx \\ &= \pi \int_0^3 \left(9 + 6\sqrt{4-x} - x\right) dx + \pi \int_3^4 12\sqrt{4-x} dx \\ &= \pi \left(\left[9x - 4(4-x)^{3/2} - \frac{1}{2}x^2\right]_0^3 + \left[-8\sqrt{4-x}\right]_3^4 \right) \\ &= \boxed{\frac{117\pi}{2} \text{ cubic units}}. \end{aligned}$$

2. Use the method of disks/washers to calculate the volume of the solids of revolutions obtained by revolving the regions described below about the given axis.

   (a) The region below the graph of $y = \ln(x)$ on $1 \leqslant x \leqslant 3$ revolved about the line $x = 3$.

   Solution. The region and axis of revolution are sketched below.



To use the method of disks/washers, we need to revolve strips perpendicular to the axis of revolution. Since the axis is vertical here, we revolve horizontal strips and use integration with respect to y. We need to express the curve as a function of y:

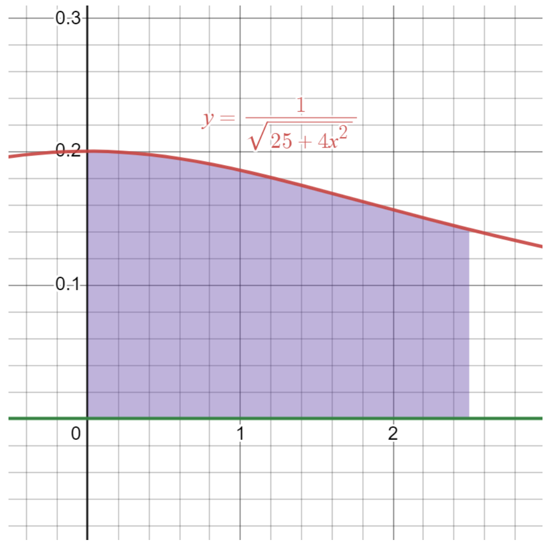
$$y = \ln(x) \;\Rightarrow\; x = e^y.$$

Revolving the horizontal strip at y in the region around $x = 3$ creates a disk with radius $r(y) = 3 - e^y$. So the volume is given by

$$\begin{aligned}
V &= \int_0^{\ln(3)} \pi r(y)^2 dy \\
&= \pi \int_0^{\ln(3)} (3 - e^y)^2 dy \\
&= \pi \int_0^{\ln(3)} (9 - 6e^y + e^{2y}) dy \\
&= \pi \left[9y - 6e^y + \frac{1}{2}e^{2y} \right]_0^{\ln(3)} \\
&= \pi \left(9\ln(3) - 6e^{\ln(3)} + \frac{1}{2}e^{2\ln(3)} + 6e^0 - \frac{1}{2}e^0 \right) \\
&= \pi \left(9\ln(3) - 6 \cdot 3 + \frac{1}{2}3^2 + 6 - \frac{1}{2} \right) \\
&= \boxed{\pi\,(9\ln(3) - 8) \text{ cubic units}}.
\end{aligned}$$

(b) The region below the graph of $y = \dfrac{1}{\sqrt{25 + 4x^2}}$ on $0 \leqslant x \leqslant \dfrac{5}{2}$ revolved about the x-axis.

Solution. The region and axis of revolution are sketched below.

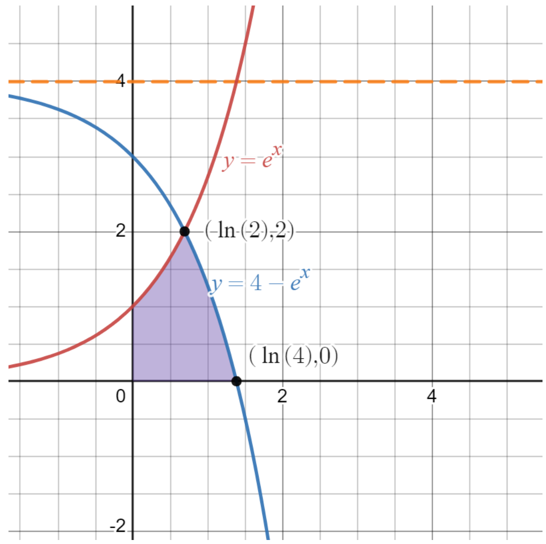


To use the method of disks/washers, we need to revolve strips perpendicular to the axis of revolution. Since the axis is horizontal here, we revolve vertical strips and use integration with respect to x. Revolving the vertical strip at x in the region around the x-axis creates a disk with radius $r(x) = \dfrac{1}{\sqrt{25+4x^2}}$. So the volume is given by

$$\begin{aligned}
V &= \int_0^{5/2} \pi r(x)^2 dx \\
&= \pi \int_0^{5/2} \frac{dx}{25+4x^2} \\
&= \frac{\pi}{25} \int_0^{5/2} \frac{dx}{1+\left(\frac{2x}{5}\right)^2} \\
&= \frac{\pi}{25} \int_0^1 \frac{5du}{2(1+u^2)} \qquad \left(u = \frac{2x}{5}\right) \\
&= \frac{2\pi}{5} \left[\tan^{-1}(u)\right]_0^1 \\
&= \frac{2\pi}{5} \left(\tan^{-1}(1) - \tan^{-1}(0)\right) \\
&= \frac{2\pi}{5} \left(\frac{\pi}{4} - 0\right) \\
&= \boxed{\frac{\pi^2}{10} \text{ cubic units}}.
\end{aligned}$$

(c) The region bounded by $y = e^x$, $y = 4 - e^x$ and the coordinate axes revolved about the line $y = 4$.

Solution. The region and axis of revolution are sketched below.



To use the method of disks/washers, we need to revolve strips perpendicular to the axis of revolution. Since the axis is horizontal here, we revolve vertical strips and use integration with respect to x. Revolving the vertical strip at x in the region creates a washer. The outer radius of the washer is $r_{\text{out}}(x) = 4 - 0 = 4$. the inner radius of the washer is

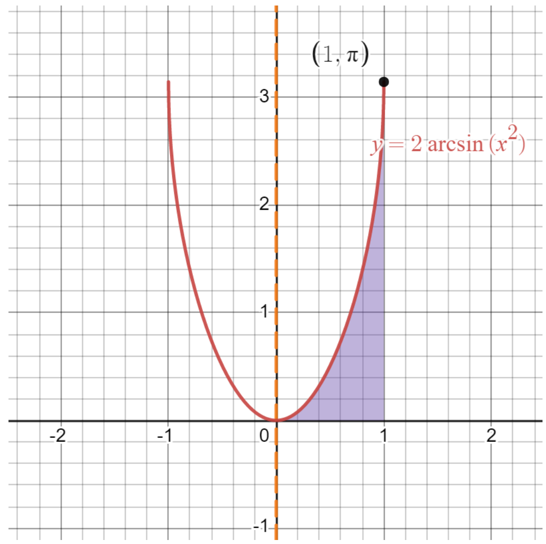
$$r_{\text{in}}(x) = \begin{cases} 4 - e^x & \text{if } 0 \leqslant x \leqslant \ln(2), \\ 4 - (4 - e^x) = e^x & \text{if } \ln(2) \leqslant x \leqslant \ln(4). \end{cases}$$

So the volume is

$$\begin{aligned} V &= \int_0^{\ln(4)} \pi \left(r_{\text{out}}(x)^2 - r_{\text{in}}(x)^2\right) dx \\ &= \int_0^{\ln(2)} \pi \left(4^2 - (4 - e^x)^2\right) dx + \int_{\ln(2)}^{\ln(4)} \pi \left(4^2 - (e^x)^2\right) dx \\ &= \pi \int_0^{\ln(2)} \left(8e^x - e^{2x}\right) dx + \pi \int_{\ln(2)}^{\ln(4)} \left(16 - e^{2x}\right) dx \\ &= \pi \left(\left[8e^x - \frac{1}{2}e^{2x}\right]_0^{\ln(2)} + \left[16x - \frac{1}{2}e^{2x}\right]_{\ln(2)}^{\ln(4)} \right) \\ &= \boxed{\pi \left(16\ln(2) + \frac{1}{2}\right) \text{ cubic units}}. \end{aligned}$$

(d) The region below the graph of $y = 2\sin^{-1}\left(x^2\right)$ on $0 \leqslant x \leqslant 1$ revolved about the y-axis.

Solution. The region and axis of revolution are sketched below.



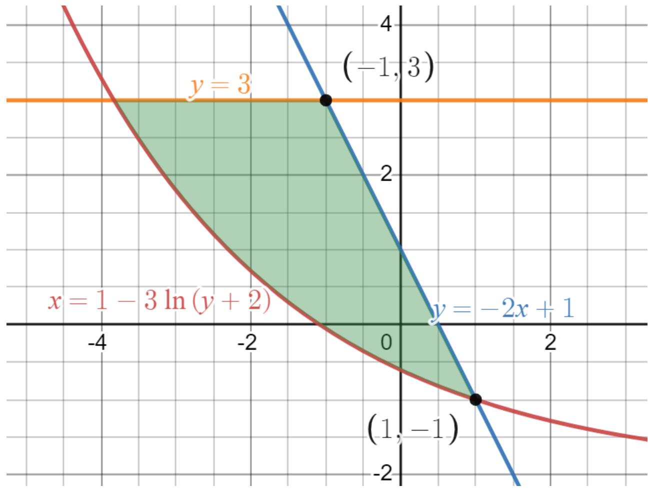
To use the method of disks/washers, we need to revolve strips perpendicular to the axis of revolution. Since the axis is vertical here, we revolve horizontal strips and use integration with respect to y. We need to express the curve as a function of y:

$$y = 2\sin^{-1}(x^2) \ \Rightarrow \ x^2 = \sin\left(\frac{y}{2}\right) \ \Rightarrow \ |x| = \sqrt{\sin\left(\frac{y}{2}\right)} \ \Rightarrow \ x = \sqrt{\sin\left(\frac{y}{2}\right)}$$

where the simplification $|x| = x$ is because $x \geqslant 0$ for the portion of the curve considered. Revolving the horizontal strip at y in the region around the y-axis creates a washer with outer radius $r_{\text{out}}(y) = 1$ and inner radius $r_{\text{in}}(y) = \sqrt{\sin\left(\frac{y}{2}\right)}$. So the volume is

$$\begin{aligned} V &= \int_0^\pi \pi\left(r_{\text{out}}(y)^2 - r_{\text{in}}(y)^2\right) dy \\ &= \int_0^\pi \pi\left(1 - \sin\left(\frac{y}{2}\right)\right) dy \\ &= \pi\left[y + 2\cos\left(\frac{y}{2}\right)\right]_0^\pi \\ &= \pi\left(\pi + 2\cos\left(\frac{\pi}{2} - 2\cos(0)\right)\right) \\ &= \boxed{\pi(\pi - 2) \text{ cubic units}}. \end{aligned}$$

3. Consider the region $\mathcal{R}$ shaded in the figure below.



Use the method of washers to set-up integrals that compute the volume of the solid obtained by revolving $\mathcal{R}$ about the line

(a) $x = 2$

Solution. We need to express the blue line as a function of y:

$$y = -2x + 1 \;\Rightarrow\; x = \frac{1-y}{2}$$

Revolving the horizontal strip at y in the region about the line $x = 2$ creates a washer with inner radius $r_{\text{in}}(y) = 2 - \frac{1-y}{2} = \frac{3+y}{2}$ and outer radius $r_{\text{out}}(y) = 2 - (1 - 3\ln(y+2)) = 1 + 3\ln(y+2)$. So the volume of the solid is given by

$$\boxed{V = \int_{-1}^{3} \pi \left((1 + 3\ln(y+2))^2 - \left(\frac{3+y}{2}\right)^2 \right) dy}.$$

(b) $y = 3$

Solution. We need to express the red curve as a function of x:

$$x = 1 - 3\ln(y+2) \;\Rightarrow\; \ln(y+2) = \frac{1-x}{3} \;\Rightarrow\; y = e^{(1-x)/3} - 2.$$

Revolving the vertical strip at x in the region about $y = 3$ creates a disk of radius

$$r(x) = \begin{cases} 3 - \left(e^{(1-x)/3} - 2\right) = 5 - e^{(1-x)/3} & \text{if } 1 - 3\ln(5) \leqslant x \leqslant -1, \\ 3 - (-2x+1) = 2 + 2x & \text{if } -1 \leqslant x \leqslant 1. \end{cases}$$

So the volume is given by

$$\boxed{V = \int_{1-3\ln(5)}^{-1} \pi \left(5 - e^{(1-x)/3}\right)^2 dx + \int_{-1}^{1} \pi \,(2+2x)^2 \, dx}.$$

(c) $x = -4$

Solution. Revolving the horizontal strip at y in the region about the line $x = -4$ creates a washer with outer radius $r_{\text{out}}(y) = \frac{1-y}{2} - (-4) = \frac{9-y}{2}$ and inner radius $r_{\text{in}}(y) = (1 - 3\ln(y+2)) - (-4) = 5 - 3\ln(y+2)$. So the volume of the solid is given by

$$\boxed{V = \int_{-1}^{3} \pi \left(\left(\frac{9-y}{2} \right)^2 - (5 - 3\ln(y+2))^2 \right) dy}.$$

(d) $y = -2$

Solution. Revolving the vertical strip at x in the region creates a washer with inner radius $r_{\text{in}}(x) = e^{(1-x)/3} - 2 - (-2) = e^{(1-x)/3}$. The outer radius is given by

$$r_{\text{out}}(x) = \begin{cases} 3 - (-2) = 5 & \text{if } 1 - 3\ln(5) \leqslant x \leqslant -1, \\ -2x + 1 - (-2) = 3 - 2x & \text{if } -1 \leqslant x \leqslant 1. \end{cases}$$

So the volume is given by

$$\boxed{V = \int_{1-3\ln(5)}^{-1} \pi \left(5^2 - \left(e^{(1-x)/3} \right)^2 \right) dx + \int_{-1}^{1} \pi \left((3-2x)^2 - \left(e^{(1-x)/3} \right)^2 \right) dx}.$$

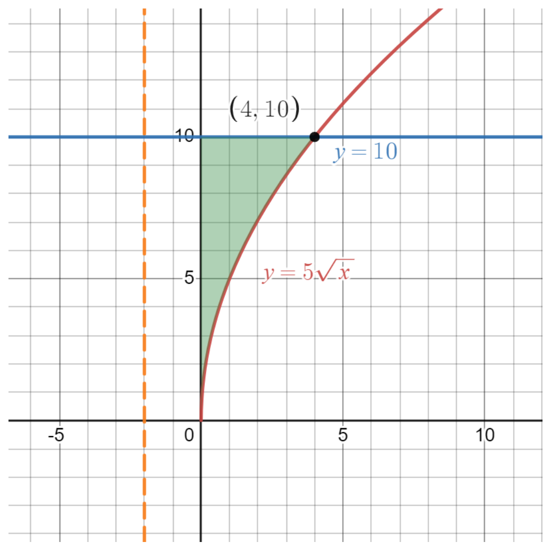
Rutgers University
Math 152

Section 6.2: Volume by Shells - Worksheet

1. Find the volume of the solid of revolution obtained by revolving the given region about the given axis using (i) the method of cylindrical shells and (ii) the method of disks/washers.

   (a) The region bounded by the y-axis, the curve $y = 5\sqrt{x}$ and the line $y = 10$ revolved about the line $x = -2$.

   Solution.



(i) Cylindrical shells are obtained by revolving strips parallel to the axis of revolution. Here the axis is vertical, so we consider the vertical strip at x. The shell radius is the distance between the axis $x = -2$ and the vertical strip at x, so it is $r(x) = x - (-2) = x + 2$. The shell height is the length of strip and is given by $h(x) = y_{\text{top}}(x) - y_{\text{bot}}(x) = 10 - 5\sqrt{x}$. Thus, the volume is

$$
\begin{aligned}
V &= \int_0^4 2\pi r(x)h(x)dx \\
&= \int_0^4 2\pi(x+2)(10-5\sqrt{x})dx \\
&= 10\pi \int_0^4 (x+2)(2-\sqrt{x})dx \\
&= 10\pi \int_0^4 \left(2x + 4 - x^{3/2} - 2\sqrt{x}\right) dx \\
&= 10\pi \left[x^2 + 4x - \frac{2}{5}x^{5/2} - \frac{4}{3}x^{3/2}\right]_0^4 \\
&= 10\pi \left(4^2 + 4\cdot 4 - \frac{2}{5}4^{5/2} - \frac{4}{3}4^{3/2}\right)
\end{aligned}
$$

$$= \boxed{\frac{256\pi}{3} \text{ cubic units}}.$$

(ii) Washers are obtained by revolving strips perpendicular to the axis of revolution. Here the axis is vertical, so we consider the vertical strip at y. We will need to express the curve as a function of y:

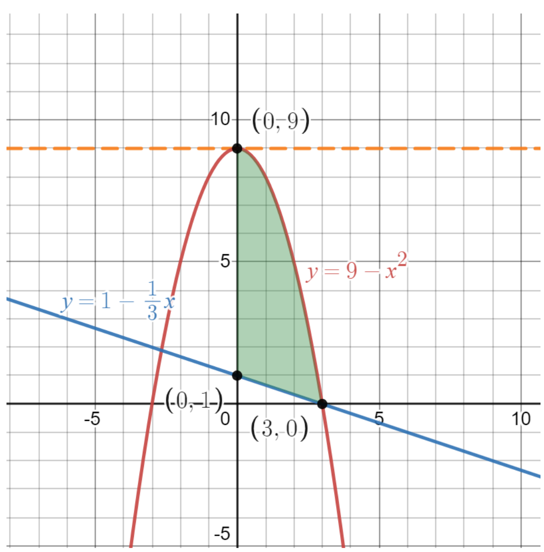
$$y = 5\sqrt{x} \;\Rightarrow\; x = \frac{y^2}{25}$$

The inner radius is $r_{\text{in}}(y) = 0 - (-2) = 2$. The outer radius is $r_{\text{out}} = \frac{y^2}{25} - (-2) = \frac{y^2}{25} + 2$. Therefore, the volume is

$$\begin{aligned}
V &= \int_0^{10} \pi \left(r_{\text{out}}(y)^2 - r_{\text{in}}(y)^2\right) dy \\
&= \int_0^{10} \pi \left(\left(\frac{y^2}{25} + 2\right)^2 - 2^2\right) dy \\
&= \pi \int_0^{10} \left(\frac{y^4}{625} + \frac{4y^2}{25}\right) dy \\
&= \pi \left[\frac{y^5}{3125} + \frac{4y^3}{75}\right]_0^{10} \\
&= \pi \left(\frac{10^5}{3125} + \frac{4 \cdot 10^3}{75}\right) \\
&= \boxed{\frac{256\pi}{3} \text{ cubic units}}.
\end{aligned}$$

(b) The region in the first quadrant bounded by the curves $y = 9 - x^2$ and $y = 1 - \frac{1}{3}x$ revolved about the line $y = 9$.

Solution.



(i) Cylindrical shells are obtained by revolving strips parallel to the axis of revolution. Here the axis is horizontal, so we consider the horizontal strip at y. The shell radius is the distance between the axis $y = 9$ and the horizontal strip at y, which is below $y = 9$, so it is $r(y) = 9 - y$. to find the shell height, we need to express the curves as functions of y:

$$y = 9 - x^2 \;\Rightarrow\; |x| = \sqrt{9-y} \;\Rightarrow\; x = \sqrt{9-y}, y = 1 - \frac{x}{3} \;\Rightarrow\; x = 3 - 3y.$$

The shell height is given by

$$h(y) = \begin{cases} \sqrt{9-y} & \text{if } 1 \leqslant y \leqslant 9, \\ \sqrt{9-y} - (3-3y) = \sqrt{9-y} - 3 + 3y & \text{if } 0 \leqslant y \leqslant 1. \end{cases}$$

Therefore, the volume is

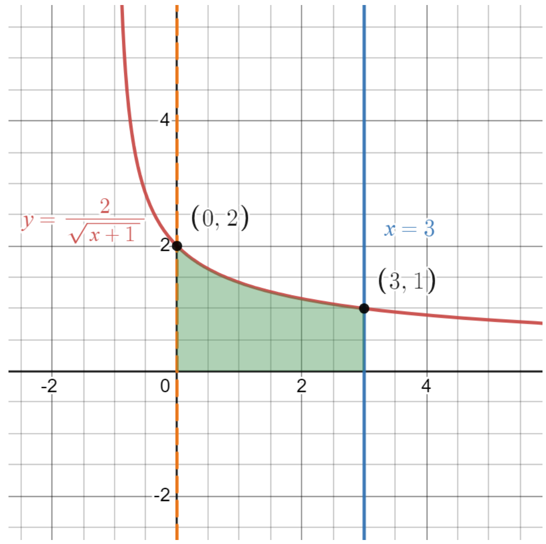
$$\begin{aligned} V &= \int_0^9 2\pi r(y)h(y)dy \\ &= \int_0^1 2\pi(9-y)\left(\sqrt{9-y} - 3 + 3y\right) dy + \int_1^9 2\pi(9-y)\sqrt{9-y}dy \\ &= 2\pi\left(\int_0^1 \left((9-y)^{3/2} + 30y - 3y^2 - 27\right) dy + \int_1^9 (9-y)^{3/2}dy\right) \\ &= 2\pi\left(\int_0^9 (9-y)^{3/2}dy + \int_0^1 \left(30y - 3y^2 - 27\right) dy\right) \\ &= 2\pi\left(\left[-\frac{2}{5}(9-y)^{5/2}\right]_0^9 + \left[15y^2 - y^3 - 27y\right]_0^1\right) \\ &= 2\pi\left(\frac{2}{5}9^{5/2} + 15 - 1 - 27\right) \\ &= \boxed{\frac{842\pi}{5} \text{ cubic units}}. \end{aligned}$$

(ii) Washers are obtained by revolving strips perpendicular to the axis of revolution. Here the axis is horizontal, so we consider the vertical strip at x. The inner radius is $r_{\text{in}}(x) = 9 - (9 - x^2) = x^2$. The outer radius is $r_{\text{out}}(x) = 9 - \left((1 - \frac{x}{3}\right) = 8 + \frac{x}{3}$. Therefore, the volume is

$$\begin{aligned} V &= \int_0^3 \pi\left(r_{\text{out}}(x)^2 - r_{\text{in}}(x)^2\right) dx \\ &= \int_0^3 \pi\left(\left(8 + \frac{x}{3}\right)^2 - (x^2)^2\right) dx \\ &= \pi\int_0^3 \left(64 + \frac{16x}{3} + \frac{x^2}{9} - x^4\right) dx \\ &= \pi\left[64x + \frac{8x^2}{3} + \frac{x^3}{27} - \frac{x^5}{5}\right]_0^3 \\ &= \pi\left(64 \cdot 3 + \frac{8 \cdot 3^2}{3} + \frac{3^3}{27} - \frac{3^5}{5}\right) \\ &= \boxed{\frac{842\pi}{5} \text{ cubic units}}. \end{aligned}$$

(c) The region below the graph of $y = \dfrac{2}{\sqrt{x+1}}$ for $0 \leqslant x \leqslant 3$ revolved about the y-axis.

Solution.



(i) Cylindrical shells are obtained by revolving strips parallel to the axis of revolution. Here the axis is vertical, so we consider the vertical strip at x. The shell radius is the distance between the axis $x = 0$ and the vertical strip at x, which is to the right of $x = 0$, so it is $r(x) = x$. The shell height is the length of the strip and is given by $h(x) = \frac{2}{\sqrt{x+1}}$. Therefore, the volume is

$$\begin{aligned} V &= \int_0^3 2\pi r(x)h(x)dx \\ &= \int_0^3 2\pi x \frac{2}{\sqrt{x+1}} dx \\ &= 4\pi \int_0^3 \frac{x}{\sqrt{x+1}} dx. \end{aligned}$$

We can compute this integral using the substitution $u = x + 1$, which gives $du = dx$. The bounds change as follows

$$\begin{aligned} x = 0 &\Rightarrow u = 0 + 1 = 1, \\ x = 3 &\Rightarrow u = 3 + 1 = 4. \end{aligned}$$

The extraneous factor x in the numerator can be expressed in terms of u as $x = u - 1$. We get

$$\begin{aligned} V &= 4\pi \int_1^4 \frac{u-1}{\sqrt{u}} du \\ &= 4\pi \int_1^4 \left(\sqrt{u} - \frac{1}{\sqrt{u}}\right) du \\ &= 4\pi \left[\frac{2}{3}u^{3/2} - 2\sqrt{u}\right]_1^4 \\ &= 4\pi \left(\frac{2}{3}4^{3/2} - 2\sqrt{4} - \frac{2}{3} + 2\right) \end{aligned}$$

$$= \boxed{\frac{32\pi}{3} \text{ cubic units}}.$$

(ii) Washers are obtained by revolving strips perpendicular to the axis of revolution. Here the axis is vertical, so we consider the horizontal strip at y. The axis of revolution is the left boundary of the region, so the washers are actually disks. To find the radius of the disk, we need to express the graph as a function of y:

$$y = \frac{2}{\sqrt{x+1}} \quad \Rightarrow \quad \sqrt{x+1} = \frac{2}{y} \quad \Rightarrow \quad x = \frac{4}{y^2} - 1.$$

The radius of the disks is

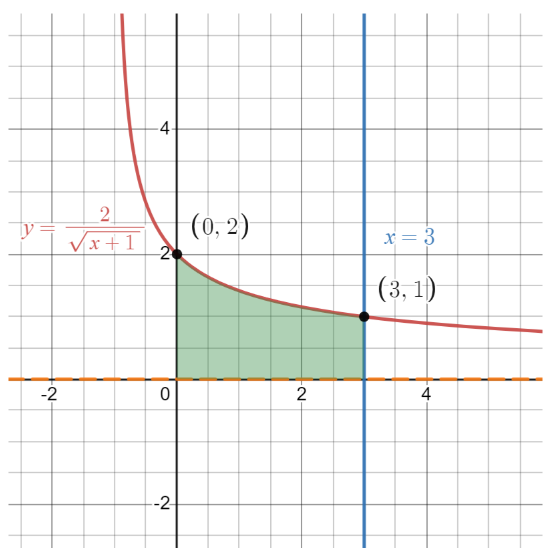
$$r(y) = \begin{cases} 3 & \text{if } 0 \leqslant y \leqslant 1, \\ \frac{4}{y^2} - 1 & \text{if } 1 \leqslant y \leqslant 2. \end{cases}$$

Therefore, the volume is

$$\begin{aligned} V &= \int_0^2 \pi r(y)^2 dy \\ &= \int_0^1 \pi \cdot 3^2 dy + \int_1^2 \pi \left(\frac{4}{y^2} - 1\right)^2 dy \\ &= \pi \left(9 + \int_1^2 \left(\frac{16}{y^4} - \frac{8}{y^2} + 1\right) dy\right) \\ &= \pi \left(9 + \left[-\frac{16}{3y^3} + \frac{8}{y} + y\right]_1^2\right) \\ &= \pi \left(9 - \frac{16}{3 \cdot 2^3} + \frac{8}{2} + 2 + \frac{16}{3} - 8 - 1\right) \\ &= \boxed{\frac{32\pi}{3} \text{ cubic units}}. \end{aligned}$$

(d) The region below the graph of $y = \dfrac{2}{\sqrt{x+1}}$ for $0 \leqslant x \leqslant 3$ revolved about the x-axis.

Solution.



(i) Cylindrical shells are obtained by revolving strips parallel to the axis of revolution. Here the axis is horizontal, so we consider the horizontal strip at y. The shell radius is the distance between the axis $y = 0$ and the horizontal strip at y, which is above $y = 0$, so it is $r(y) = y$. To find the shell height, we need to express the graph as a function of y:

$$y = \frac{2}{\sqrt{x+1}} \Rightarrow \sqrt{x+1} = \frac{2}{y} \Rightarrow x = \frac{4}{y^2} - 1.$$

The shell height is given by

$$h(y) = \begin{cases} \frac{y^2}{4} - 1 & \text{if } 1 \leqslant y \leqslant 2, \\ 3 & \text{if } 0 \leqslant y \leqslant 1. \end{cases}$$

Therefore, the volume is

$$\begin{aligned} V &= \int_0^2 2\pi r(y)h(y)dy \\ &= \int_0^1 2\pi 3y dy + \int_1^2 2\pi y \left(\frac{4}{y^2} - 1\right) dy \\ &= 2\pi \left(\int_0^1 3y dy + \int_1^2 \left(\frac{4}{y} - y\right) dy\right) \\ &= 2\pi \left(\left[\frac{3y^2}{2}\right]_0^1 + \left[4\ln|y| - \frac{y^2}{2}\right]_1^2\right) \\ &= 2\pi \left(\frac{3}{2} + 4\ln(2) - \frac{2^2}{2} + \frac{1}{2}\right) \\ &= \boxed{8\pi \ln(2) \text{ cubic units}}. \end{aligned}$$

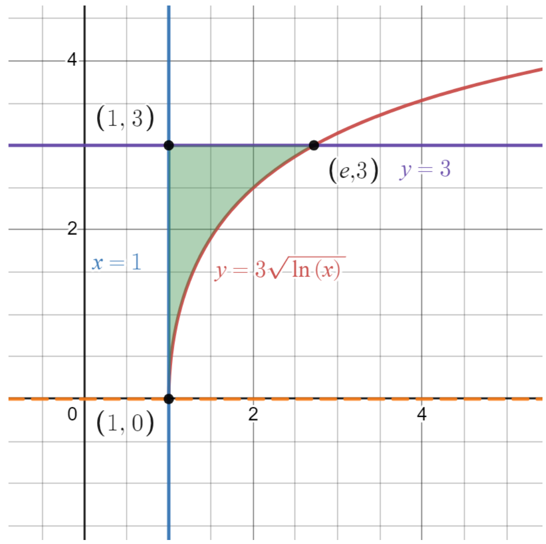
(ii) Washers are obtained by revolving strips perpendicular to the axis of revolution. Here the axis is horizontal, so we consider the vertical strip at x. The axis of revolution is the bottom boundary of the region, so the washers are actually disks of radius $r(x) = \frac{2}{\sqrt{x+1}}$. Therefore, the volume is

$$\begin{aligned} V &= \int_0^3 \pi r(x)^2 dx \\ &= \int_0^3 \pi \left(\frac{2}{\sqrt{x+1}}\right) dx \\ &= 4\pi \int_0^3 \frac{dx}{x+1} \\ &= 4\pi \left[\ln(x+1)\right]_0^3 \\ &= 4\pi \ln(4) \\ &= \boxed{8\pi \ln(2) \text{ cubic units}}. \end{aligned}$$

2. Find the volume of the solid of revolution obtained by revolving the given region about the given axis using the method of cylindrical shells.

   (a) The region bounded by the curve $y = 3\sqrt{\ln(x)}$, the line $y = 3$ and the line $x = 1$ revolved about the x-axis.

   Solution.



(i) Cylindrical shells are obtained by revolving strips parallel to the axis of revolution. Here the axis is horizontal, so we consider the horizontal strip at y. The shell radius is the distance between the axis $y = 0$ and the horizontal strip at y, which is above $y = 0$, so it is $r(y) = y$. To find the shell height, we will need to express the curve as a function of y:

$$y = 3\sqrt{\ln(x)} \Rightarrow \ln(x) = \frac{y^2}{9} \Rightarrow x = e^{y^2/9}.$$

The shell height is the length of the strip, which is $h(y) = e^{y^2/9} - 1$. So the volume is

$$\begin{aligned} V &= \int_0^4 2\pi r(y)h(y)dy \\ &= \int_0^4 2\pi y \left(e^{y^2/9} - 1\right) dy. \end{aligned}$$

We can compute this integral using the substitution $u = \frac{y^2}{9}$, which gives $du = \frac{2ydy}{9}$. The bounds become

$$\begin{aligned} y = 0 &\Rightarrow u = \frac{0^2}{9} = 0, \\ y = 4 &\Rightarrow u = \frac{4^2}{9} = \frac{16}{9}. \end{aligned}$$

We obtain

$$\begin{aligned} V &= 9\pi \int_0^{16/9} (e^u - 1)\, du \\ &= 9\pi \left[e^u - u\right]_0^{16/9} \\ &= 9\pi \left(e^{16/9} - \frac{16}{9} - 1\right) \\ &= \boxed{\pi\left(9e^{16/9} - 25\right) \text{ cubic units}}. \end{aligned}$$

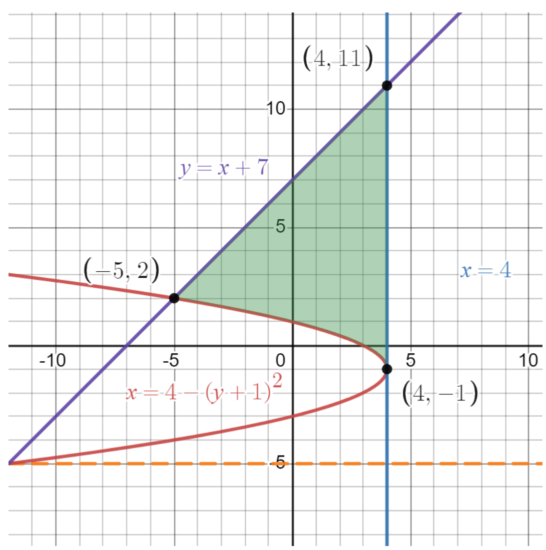
(b) The region below the graph of $y = \dfrac{1}{16+x^4}$ for $0 \leqslant x \leqslant 2$ revolved about the y-axis.

Solution. The cylindrical shell is obtained by revolving the vertical strip at x. It has radius $r(x) = x$ and height $h(x) = \frac{1}{16+x^4}$. Therefore the volume is

$$\begin{aligned} V &= \int_0^2 2\pi r(x)h(x)dx \\ &= \int_0^2 2\pi x \frac{1}{16+x^4} dx \\ &= \int_0^4 \pi \frac{du}{16+u^2} \qquad (u = x^2) \\ &= \pi \left[\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)\right]_0^4 \\ &= \frac{\pi}{4}\tan^{-1}(1) \\ &= \boxed{\frac{\pi^2}{16} \text{ cubic units}}. \end{aligned}$$

(c) The region bounded by the curve $x = 4-(y+1)^2$, the line $x = 4$ and the line $y = x+7$ revolved about the line $y = -5$.

Solution.



(i) Cylindrical shells are obtained by revolving strips parallel to the axis of revolution. Here the axis is horizontal, so we consider the horizontal strip at y. The shell radius is the distance between the axis $y = -5$ and the horizontal strip at y, which is above $y = -5$, so it is $r(y) = y-(-5) = y+5$. The shell height is given by

$$h(y) = \begin{cases} 1-(4-(y+1)^2) = y^2+2y-2 & \text{if } -1 \leqslant y \leqslant 2, \\ 1-(y-7) = 8-y & \text{if } 2 \leqslant y \leqslant 11. \end{cases}$$

So the volume is

$$
\begin{aligned}
V &= \int_{-1}^{11} 2\pi r(y)h(y)dy \\
&= \int_{-1}^{2} 2\pi(y+5)(y^2+2y-2)dy + \int_{2}^{11} 2\pi(y+5)(8-y)dy \\
&= 2\pi\left(\int_{-1}^{2}\left(y^3+7y^2+8y-10\right)dy + \int_{2}^{11}\left(40+3y-y^2\right)dy\right) \\
&= 2\pi\left(\left[\frac{y^4}{4}+\frac{7y^3}{3}+4y^2-10y\right]_{-1}^{2} + \left[40y+\frac{3y^2}{2}-\frac{y^3}{2}\right]_{2}^{11}\right) \\
&= \boxed{\frac{405\pi}{2} \text{ cubic units.}}
\end{aligned}
$$

Rutgers University
Math 152

Section 6.3: Arc Length - Worksheet Solutions

1. Calculate the arc length of the given curves.

   (a) $y = 11 - 2(x-5)^{3/2}$, $5 \leqslant x \leqslant 6$.

   Solution. We have
$$\frac{dy}{dx} = -2 \cdot \frac{3}{2}(x-5)^{1/2} = -3\sqrt{x-5}.$$
So
$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(-3\sqrt{x-5}\right)^2 = 1 + 9(x-5) = 9x - 44.$$
Therefore the arc length is given by
$$\begin{aligned} L &= \int_5^6 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_5^6 \sqrt{9x-44}dx. \end{aligned}$$
We use the substitution $u = 9x - 44$, so that $du = 9dx$. The bounds become
$$\begin{aligned} x = 5 &\Rightarrow u = 1, \\ x = 6 &\Rightarrow u = 10. \end{aligned}$$
So the integral becomes
$$\begin{aligned} L &= \int_1^{10} \frac{1}{9}\sqrt{u}du \\ &= \frac{1}{9}\left[\frac{2}{3}u^{3/2}\right]_1^{10} \\ &= \boxed{\frac{2}{27}\left(10^{3/2} - 1\right) \text{ units}}. \end{aligned}$$

   (b) $x = \frac{1}{4}\sqrt[3]{y} - \frac{9}{5}\sqrt[3]{y^5}$, $1 \leqslant y \leqslant 2$.

   Solution. We have
$$\frac{dx}{dy} = \frac{1}{4} \cdot \frac{1}{3}y^{-2/3} - \frac{9}{5} \cdot \frac{5}{3}y^{2/3} = \frac{1}{12}y^{-2/3} - 3y^{2/3}.$$
So
$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(\frac{1}{12}y^{-2/3} - 3y^{2/3}\right)^2$$

$$
\begin{aligned}
&= 1 + \frac{1}{144}y^{-4/3} + 9y^{4/3} - 2\cdot\frac{1}{12}y^{-2/3}\cdot 3y^{2/3} \\
&= 1 + \frac{1}{144}y^{-4/3} + 9y^{4/3} - \frac{1}{2} \\
&= \frac{1}{144}y^{-4/3} + 9y^{4/3} + \frac{1}{2} \\
&= \left(\frac{1}{12}y^{-2/3}\right)^2 + \left(3y^{2/3}\right)^2 + 2\cdot\frac{1}{12}y^{-2/3}\cdot 3y^{2/3} \\
&= \left(\frac{1}{12}y^{-2/3} + 3y^{2/3}\right)^2.
\end{aligned}
$$

Therefore the arc length is given by

$$
\begin{aligned}
L &= \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy \\
&= \int_1^2 \sqrt{\left(\frac{1}{12}y^{-2/3} + 3y^{2/3}\right)^2}\,dy \\
&= \int_1^2 \left|\frac{1}{12}y^{-2/3} + 3y^{2/3}\right|\,dy \\
&= \int_1^2 \left(\frac{1}{12}y^{-2/3} + 3y^{2/3}\right)dy \\
&= \left[\frac{1}{4}y^{1/3} + \frac{9}{5}y^{5/3}\right]_1^2 \\
&= \left(\frac{1}{4}2^{1/3} + \frac{9}{5}2^{5/3}\right) - \left(\frac{1}{4} + \frac{9}{5}\right) \\
&= \boxed{\frac{5\sqrt[3]{2} + 72\sqrt[3]{4} - 41}{20}\text{ units}}.
\end{aligned}
$$

(c) $x = \sqrt{16y - y^2}$, $4 \leqslant y \leqslant 12$.

Solution. We have

$$
\frac{dx}{dy} = \frac{16 - 2y}{2\sqrt{16y - y^2}} = \frac{8 - y}{\sqrt{16y - y^2}}.
$$

Therefore

$$
\begin{aligned}
1 + \left(\frac{dx}{dy}\right)^2 &= 1 + \left(\frac{8 - y}{\sqrt{16y - y^2}}\right)^2 \\
&= 1 + \frac{(8 - y)^2}{16y - y^2} \\
&= \frac{16y - y^2 + (64 - 16y + y^2)}{16y - y^2} \\
&= \frac{64}{16y - y^2} \\
&= \frac{64}{64 - (y - 8)^2}
\end{aligned}
$$

where we have completed the square in the denominator for the last step. So the arc length integral becomes

$$
\begin{aligned}
L &= \int_4^{12} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \\
&= \int_4^{12} \sqrt{\frac{64}{64 - (y-8)^2}}\, dy \\
&= 8\int_4^{12} \frac{dy}{\sqrt{64 - (y-8)^2}} \\
&= 8\int_4^{12} \frac{dy}{8\sqrt{1 - \left(\frac{y-8}{8}\right)^2}} \\
&= 8\int_{-1/2}^{1/2} \frac{du}{\sqrt{1-u^2}} \qquad \left(u = \frac{y-8}{8}\right) \\
&= 8\left[\sin^{-1}(u)\right]_{-1/2}^{1/2} \\
&= 8\left(\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)\right) \\
&= 8\left(\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right) \\
&= \boxed{\frac{8\pi}{3} \text{ units}}.
\end{aligned}
$$

(d) $y = \dfrac{1}{6}\ln\left(\sin(3x)\cos(3x)\right), \ \dfrac{\pi}{18} \leqslant x \leqslant \dfrac{\pi}{9}$.

Solution. We have

$$
\frac{dy}{dx} = \frac{1}{6}\cdot\frac{3\cos(3x)^2 - 3\sin(3x)^2}{\sin(3x)\cos(3x)} = \frac{3\cos(3x)^2}{6\sin(3x)\cos(3x)} - \frac{3\sin(3x)^2}{6\sin(3x)\cos(3x)} = \frac{1}{2}\cot(3x) - \frac{1}{2}\tan(3x).
$$

So

$$
\begin{aligned}
1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \left(\frac{1}{2}\cot(3x) - \frac{1}{2}\tan(3x)\right)^2 \\
&= 1 + \frac{1}{4}\cot(3x)^2 + \frac{1}{4}\tan(3x)^2 - 2\cdot\frac{1}{2}\cot(3x)\cdot\frac{1}{2}\tan(3x) \\
&= 1 + \frac{1}{4}\cot(3x)^2 + \frac{1}{4}\tan(3x)^2 - \frac{1}{2} \\
&= \frac{1}{4}\cot(3x)^2 + \frac{1}{4}\tan(3x)^2 + \frac{1}{2} \\
&= \frac{1}{4}\cot(3x)^2 + \frac{1}{4}\tan(3x)^2 + 2\cdot\frac{1}{2}\cot(3x)\cdot\frac{1}{2}\tan(3x) \\
&= \left(\frac{1}{2}\cot(3x) + \frac{1}{2}\tan(3x)\right)^2.
\end{aligned}
$$

Therefore the arc length is given by

$$
L = \int_{\pi/18}^{\pi/9} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx
$$

$$
\begin{aligned}
&= \int_{\pi/18}^{\pi/9} \sqrt{\left(\frac{1}{2}\cot(3x) + \frac{1}{2}\tan(3x)\right)^2}\, dx \\
&= \int_{\pi/18}^{\pi/9} \left|\frac{1}{2}\cot(3x) + \frac{1}{2}\tan(3x)\right| dx \\
&= \frac{1}{2}\int_{\pi/18}^{\pi/9} (\cot(3x) + \tan(3x))\, dx \\
&= \frac{1}{6}\left[\ln|\sin(3x)| + \ln|\sec(3x)|\right]_{\pi/18}^{\pi/9} \\
&= \frac{1}{6}\left(\ln\left|\sin\left(\frac{\pi}{3}\right)\right| + \ln\left|\sec\left(\frac{\pi}{3}\right)\right| - \ln\left|\sin\left(\frac{\pi}{6}\right)\right| - \ln\left|\sec\left(\frac{\pi}{6}\right)\right|\right) \\
&= \frac{1}{6}\left(\ln\left(\frac{\sqrt{3}}{2}\right) + \ln(2) - \ln\left(\frac{1}{2}\right) - \ln\left(\frac{2}{\sqrt{3}}\right)\right) \\
&= \boxed{\frac{1}{6}\ln(3) \text{ units}}.
\end{aligned}
$$

(e) $y = \dfrac{e^{5x} + e^{-5x}}{10}$, $0 \leqslant x \leqslant \dfrac{1}{5}$.

Solution. We have

$$\frac{dy}{dx} = \frac{5e^{5x} - 5e^{5x}}{10} = \frac{e^{5x} - e^{5x}}{2}.$$

So

$$
\begin{aligned}
1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \left(\frac{e^{5x} - e^{-5x}}{2}\right)^2 \\
&= 1 + \frac{1}{4}e^{10x} + \frac{1}{4}e^{-10x} - 2 \cdot \frac{1}{2}e^{5x} \cdot \frac{1}{2}e^{-5x} \\
&= 1 + \frac{1}{4}e^{10x} + \frac{1}{4}e^{-10x} - \frac{1}{2} \\
&= \frac{1}{4}e^{10x} + \frac{1}{4}e^{-10x} + \frac{1}{2} \\
&= \left(\frac{1}{2}e^{5x}\right)^2 + \left(\frac{1}{2}e^{-5x}\right)^2 + 2 \cdot \frac{1}{2}e^{5x} \cdot \frac{1}{2}e^{-5x} \\
&= \left(\frac{e^{5x} + e^{-5x}}{2}\right)^2.
\end{aligned}
$$

Therefore the arc length is given by

$$
\begin{aligned}
L &= \int_0^{1/5} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int_0^{1/5} \sqrt{\left(\frac{e^{5x} + e^{-5x}}{2}\right)^2}\, dx \\
&= \int_0^{1/5} \left|\frac{e^{5x} + e^{-5x}}{2}\right| dx \\
&= \int_0^{1/5} \left(\frac{e^{5x} + e^{-5x}}{2}\right) dx
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{10}\left[e^{5x} - e^{-5x}\right]_0^{1/5} \\
&= \frac{1}{10}\left(e - e^{-1} - 1 + 1\right) \\
&= \boxed{\frac{e - e^{-1}}{10} \text{ units}}.
\end{aligned}
$$

(f) $x = \frac{4}{5}y^{5/4}$, $0 \leqslant y \leqslant 9$.

Solution. We have $\frac{dx}{dy} = y^{1/4}$, so

$$
\begin{aligned}
L &= \int_0^9 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \\
&= \int_0^9 \sqrt{1 + y^{1/2}} dy.
\end{aligned}
$$

In this last integral, we substitute $u = 1 + y^{1/2}$, so $du = \frac{dy}{2y^{1/2}}$. Observing that $y^{1/2} = u - 1$, we have

$$dy = 2y^{1/2} du = 2(u-1) du.$$

The bounds become

$$
\begin{aligned}
y = 0 \;&\Rightarrow\; u = 1 + 0^{1/2} = 1, \\
y = 9 \;&\Rightarrow\; u = 1 + 9^{1/2} = 4.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
L &= \int_1^4 \sqrt{u} 2(u-1) du \\
&= 2\int_1^4 \left(u^{3/2} - u^{1/2}\right) du \\
&= 2\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^4 \\
&= 4\left(\frac{4^{5/2}}{5} - \frac{4^{3/2}}{3} - \frac{1}{5} + \frac{1}{3}\right) \\
&= \boxed{\frac{232}{15}}.
\end{aligned}
$$

2. Find a curve of the form $y = f(x)$ passing through $(4, 13)$, having negative derivative, and whose length integral on $1 \leqslant x \leqslant 7$ is given by

$$L = \int_1^7 \sqrt{1 + \frac{25}{x^3}} dx.$$

Solution. The arc length being given by

$$L = \int_1^7 \sqrt{1 + f'(x)^2} dx,$$

we can deduce that $f'(x)^2 = \frac{25}{x^3}$. Taking square roots on both sides gives

$$\sqrt{f'(x)^2} = \sqrt{\frac{25}{x^3}}$$
$$\Rightarrow \quad |f'(x)| = \frac{5}{x^{3/2}}.$$

Since f has negative derivative, $|f'(x)| = -f'(x)$ and we deduce

$$f'(x) = -\frac{5}{x^{3/2}} = -5x^{-3/2}.$$

Taking an antiderivative gives

$$f(x) = \int -5x^{-3/2}dx = -5(-2)x^{-1/2} + C = \frac{10}{\sqrt{x}} + C.$$

To find the constant C, we can use the fact that the curve passes through $(4, 13)$, which gives

$$\frac{10}{\sqrt{4}} + C = 13 \;\Rightarrow\; C = 8.$$

Therefore, the solution to the problem is the curve

$$\boxed{y = \frac{10}{\sqrt{x}} + 8}.$$

Rutgers University
Math 152

Section 6.4: Areas of Surfaces of Revolution - Worksheet Solutions

1. Find the surface area obtained by revolving the given curve about the given axis.

   (a) The curve $y = \sqrt{3x-5}$, $2 \leqslant x \leqslant 3$, revolved about the x-axis.

   Solution. Method 1: we use an x-integral. The area for a surface of revolution about the x-axis is given by

$$\begin{aligned} A &= \int_{\text{curve}} 2\pi y ds \\ &= \int_2^3 2\pi y(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx \\ &= \int_2^3 2\pi\sqrt{3x-5}\sqrt{1+\left(\frac{3}{2\sqrt{3x-5}}\right)^2}dx \\ &= 2\pi\int_2^3\sqrt{(3x-5)\left(1+\frac{9}{4(3x-5)}\right)}dx \\ &= 2\pi\int_2^3\sqrt{3x-5+\frac{9}{4}}dx \\ &= 2\pi\int_2^3\sqrt{3x-\frac{11}{4}}dx. \end{aligned}$$

   We can finish evaluating this integral using the substitution $u = 3x - \frac{11}{4}$, which gives $du = 3dx$. The bounds change as follows

$$\begin{aligned} x = 2 &\Rightarrow u = 6 - \frac{11}{4} = \frac{13}{4}, \\ x = 3 &\Rightarrow u = 9 - \frac{11}{4} = \frac{25}{4}. \end{aligned}$$

   So the integral becomes

$$\begin{aligned} A &= 2\pi\int_{13/4}^{25/4}\frac{1}{3}\sqrt{u}du \\ &= \frac{2\pi}{3}\left[\frac{2}{3}u^{3/2}\right]_{13/4}^{25/4} \\ &= \frac{2\pi}{3}\cdot\frac{2}{3}\left(\left(\frac{25}{4}\right)^{3/2}-\left(\frac{13}{4}\right)^{3/2}\right) \\ &= \boxed{\frac{\pi\left(125-13^{3/2}\right)}{18}\text{ square units}}. \end{aligned}$$

Method 2: we use a y-integral, observing that we can express the curve as a function of y as $x = \frac{y^2-5}{3}$, $1 \leqslant y \leqslant 2$. The area for a surface of revolution about the x-axis is given by

$$\begin{aligned} A &= \int_{\text{curve}} 2\pi y ds \\ &= \int_1^2 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \\ &= \int_1^2 2\pi y \sqrt{1 + \left(\frac{2y}{3}\right)^2} dy \\ &= 2\pi \int_1^2 y \sqrt{1 + \frac{4y^2}{9}} dy. \end{aligned}$$

We can finish evaluating this integral using the substitution $u = 1 + \frac{4y^2}{9}$, which gives $du = \frac{8ydy}{9}$. The bounds change as follows

$$\begin{aligned} y = 1 &\Rightarrow u = 1 + \frac{4}{9} = \frac{13}{9}, \\ y = 2 &\Rightarrow u = 1 + \frac{16}{9} = \frac{25}{9}. \end{aligned}$$

So the integral becomes

$$\begin{aligned} A &= 2\pi \int_{13/9}^{25/9} \frac{9}{8} \sqrt{u} du \\ &= \frac{9\pi}{4} \left[\frac{2}{3} u^{3/2}\right]_{13/9}^{25/9} \\ &= \frac{9\pi}{4} \cdot \frac{2}{3} \left(\left(\frac{25}{9}\right)^{3/2} - \left(\frac{13}{9}\right)^{3/2}\right) \\ &= \boxed{\frac{\pi\left(125 - 13^{3/2}\right)}{18} \text{ square units}}. \end{aligned}$$

(b) The curve $x = \sqrt{16y - y^2}$, $0 \leqslant y \leqslant 8$, revolved about the y-axis.

Solution. We use integration with respect to y. The area for a surface of revolution about the y-axis is given by

$$\begin{aligned} A &= \int_{\text{curve}} 2\pi x ds \\ &= \int_0^8 2\pi x(y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \\ &= \int_0^8 2\pi \sqrt{16y - y^2} \sqrt{1 + \left(\frac{16 - 2y}{2\sqrt{16y - y^2}}\right)^2} dy \end{aligned}$$

$$
\begin{aligned}
&= 2\pi \int_0^8 \sqrt{(16y - y^2)\left(1 + \frac{(8-y)^2}{16y - y^2}\right)} dy \\
&= 2\pi \int_0^8 \sqrt{16y - y^2 + (8-y)^2} dy \\
&= 2\pi \int_0^8 \sqrt{16y - y^2 + 64 - 16y + y^2} dy \\
&= 2\pi \int_0^8 8 dy \\
&= \boxed{128\pi \text{ square units}}.
\end{aligned}
$$

(c) The curve $x = 2\sqrt[3]{y}$, $0 \leqslant y \leqslant 1$, revolved about the x-axis.

Solution. Method 1: we use an x-integral. The curve can be expressed as a function of x as $y = \frac{x^3}{8}$, $0 \leqslant x \leqslant 2$. The area for a surface of revolution about the x-axis is given by

$$
\begin{aligned}
A &= \int_{\text{curve}} 2\pi y ds \\
&= \int_0^2 2\pi y(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
&= \int_0^2 2\pi \frac{x^3}{8} \sqrt{1 + \left(\frac{3x^2}{8}\right)^2} dx \\
&= \frac{\pi}{4} \int_0^2 x^3 \sqrt{1 + \frac{9x^4}{64}} dx.
\end{aligned}
$$

We can finish evaluating this integral using the substitution $u = 1 + \frac{9x^4}{64}$, which gives $du = \frac{9x^3 dx}{16}$. The bounds change as follows

$$
\begin{aligned}
x = 0 &\Rightarrow u = 1 + \frac{9 \cdot 0}{64} = 1, \\
x = 2 &\Rightarrow u = 1 + \frac{9 \cdot 2^4}{64} = \frac{13}{4}.
\end{aligned}
$$

So the integral becomes

$$
\begin{aligned}
A &= \frac{\pi}{4} \int_1^{13/4} \frac{16}{9} \sqrt{u} du \\
&= \frac{4\pi}{9} \left[\frac{2}{3} u^{3/2}\right]_1^{13/4} \\
&= \frac{4\pi}{9} \cdot \frac{2}{3} \left(\left(\frac{13}{4}\right)^{3/2} - 1\right) \\
&= \boxed{\frac{\pi\left(13^{3/2} - 8\right)}{27} \text{ square units}}.
\end{aligned}
$$

Method 2: we use a y-integral. The area for a surface of revolution about the x-axis is given by

$$\begin{aligned}
A &= \int_{\text{curve}} 2\pi y ds \\
&= \int_0^1 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \\
&= \int_0^1 2\pi y \sqrt{1 + \left(\frac{2}{3y^{2/3}}\right)^2} dy \\
&= 2\pi \int_0^1 y \sqrt{1 + \frac{4}{9y^{4/3}}} dy \\
&= 2\pi \int_0^1 y \frac{\sqrt{y^{4/3} + \frac{4}{9}}}{y^{2/3}} dy \\
&= 2\pi \int_0^1 y^{1/3} \sqrt{y^{4/3} + \frac{4}{9}} dy.
\end{aligned}$$

We can finish evaluating this integral using the substitution $u = y^{4/3} + \frac{4}{9}$, which gives $du = \frac{4y^{1/3}dy}{3}$. The bounds change as follows

$$\begin{aligned}
y = 0 &\Rightarrow u = 0^{4/3} + \frac{4}{9} = \frac{4}{9}, \\
y = 1 &\Rightarrow u = 1^{4/3} + \frac{4}{9} = \frac{13}{9}.
\end{aligned}$$

So the integral becomes

$$\begin{aligned}
A &= 2\pi \int_{4/9}^{13/9} \frac{3}{4} \sqrt{u} du \\
&= \frac{3\pi}{2} \left[\frac{2}{3} u^{3/2}\right]_{4/9}^{13/9} \\
&= \frac{3\pi}{2} \cdot \frac{2}{3} \left(\left(\frac{13}{9}\right)^{3/2} - \left(\frac{4}{9}\right)^{3/2}\right) \\
&= \boxed{\frac{\pi\left(13^{3/2} - 8\right)}{27} \text{ square units}}.
\end{aligned}$$

(d) The curve $x = \frac{3}{5} y^{5/3}$, $0 \leqslant y \leqslant 1$, revolved about the y-axis.

Solution. We use integration with respect to y. The area for a surface of revolution about the y-axis is given by

$$\begin{aligned}
A &= \int_{\text{curve}} 2\pi x ds \\
&= \int_0^1 2\pi x(y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 2\pi \frac{3}{5} y^{5/3} \sqrt{1 + \left(y^{2/3}\right)^2} dy \\
&= \frac{6\pi}{5} \int_0^1 y^{5/3} \sqrt{1 + y^{4/3}} dy.
\end{aligned}
$$

We finish evaluating this integral using the substitution $u = 1 + y^{4/3}$, which gives $du = \frac{4y^{1/3}dy}{3}$. The bounds change as follows

$$
\begin{aligned}
x = 0 &\Rightarrow u = 1 + 0^{4/3} = 1, \\
x = 1 &\Rightarrow u = 1 + 1^{4/3} = 2.
\end{aligned}
$$

There will be an extraneous factor $y^{4/3}$ in the integrand, which we can express in terms of u as $y^{4/3} = u - 1$. So the area becomes

$$
\begin{aligned}
A &= \frac{6\pi}{15} \int_0^1 y^{4/3} \sqrt{1 + y^{4/3}} y^{1/3} dy \\
&= \frac{6\pi}{15} \int_1^2 (u - 1)\sqrt{u} \frac{3}{4} dy \\
&= \frac{3\pi}{10} \int_1^2 \left(u^{3/2} - \sqrt{u}\right) du \\
&= \frac{3\pi}{10} \left[\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2}\right]_1^2 \\
&= \boxed{\frac{6\pi}{75} \left(\sqrt{2} + 1\right) \text{ units}}.
\end{aligned}
$$

(e) The curve $y = x^{3/2}$, $1 \leqslant x \leqslant 4$, revolved about the y-axis.

Solution. We use an x-integral. The area for a surface of revolution about the y-axis is given by

$$
\begin{aligned}
A &= \int_{\text{curve}} 2\pi x ds \\
&= \int_1^4 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
&= \int_1^4 2\pi x \sqrt{1 + \left(\frac{3x^{1/2}}{2}\right)^2} dx \\
&= 2\pi \int_1^4 x \sqrt{1 + \frac{9x}{4}} dx.
\end{aligned}
$$

We can finish evaluating this integral using the substitution $u = 1 + \frac{9x}{4}$, which gives $du = \frac{9dx}{4}$. The bounds change as follows

$$
\begin{aligned}
x = 1 &\Rightarrow u = 1 + \frac{9}{4} = \frac{13}{4}, \\
x = 4 &\Rightarrow u = 1 + \frac{9 \cdot 4}{4} = 10.
\end{aligned}
$$

We have an extraneous factor x in the integrand which we express in terms of u as

$$x = \frac{4}{9}(u-1).$$

So the integral becomes

$$\begin{aligned} A &= 2\pi \int_{13/4}^{10} \frac{4}{9} \cdot \frac{4}{9}(u-1)\sqrt{u}du \\ &= \frac{32\pi}{81} \int_{13/4}^{10} \left(u^{3/2} - \sqrt{u}\right) du \\ &= \frac{32\pi}{81} \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_{13/4}^{10} \\ &= \boxed{\frac{32\pi}{81} \left(\frac{2}{5}\left(10^{5/2} - \left(\frac{25}{4}\right)^{5/2}\right) - \frac{2}{3}\left(10^{3/2} - \left(\frac{25}{4}\right)^{3/2}\right)\right) \text{ square units}}. \end{aligned}$$

Rutgers University
Math 152

Section 8.2: Integration by Parts - Worksheet Solutions

1. Evaluate the following antiderivatives.

   (a) $\int x^3 \cos(5x)dx$

   Solution. We will evaluate this integral with three consecutive IBPs, taking u to be the power of x each time. For the first IBP, we take

$$u = x^3 \;\Rightarrow\; du = 3x^2 dx,$$
$$dv = \cos(5x)dx \;\Rightarrow\; v = \frac{1}{5}\sin(5x).$$

   This gives

$$\int x^3 \cos(5x)dx = \frac{x^3\sin(5x)}{5} - \int 3x^2 \frac{\sin(5x)}{5}dx$$
$$= \frac{x^3\sin(5x)}{5} - \frac{3}{5}\int x^2 \sin(5x)dx.$$

   For the second IBP, we take

$$u = x^2 \;\Rightarrow\; du = 2xdx,$$
$$dv = \sin(5x)dx \;\Rightarrow\; v = -\frac{1}{5}\cos(5x).$$

   This gives

$$\int x^3 \cos(5x)dx = \frac{x^3\sin(5x)}{5} - \frac{3}{5}\left(-\frac{x^2\cos(5x)}{5} - \int -2x\frac{\cos(5x)}{5}dx\right)$$
$$= \frac{x^3\sin(5x)}{5} + \frac{3x^2\cos(5x)}{25} - \frac{6}{25}\int x\cos(5x)dx.$$

   Finally, for the third and final IBP, we take

$$u = x \;\Rightarrow\; du = dx,$$
$$dv = \cos(5x)dx \;\Rightarrow\; v = \frac{1}{5}\sin(5x),$$

   and we obtain

$$\int x^3 \cos(5x)dx = \frac{x^3\sin(5x)}{5} + \frac{3x^2\cos(5x)}{25} - \frac{6}{25}\left(\frac{x\sin(5x)}{5} - \int \frac{\sin(5x)}{5}dx\right)$$
$$= \frac{x^3\sin(5x)}{5} + \frac{3x^2\cos(5x)}{25} - \frac{6x\sin(5x)}{125} + \frac{6}{125}\int \sin(5x)dx$$
$$= \boxed{\frac{x^3\sin(5x)}{5} + \frac{3x^2\cos(5x)}{25} - \frac{6x\sin(5x)}{125} - \frac{6\cos(5x)}{625} + C}.$$

(b) $\int x^2 \sin^{-1}(x)dx$

Solution. We start with an IBP taking

$$u = \sin^{-1}(x) \;\Rightarrow\; du = \frac{dx}{\sqrt{1-x^2}},$$
$$dv = x^2 dx \;\Rightarrow\; v = \frac{x^3}{3},$$

which gives

$$\int x^2 \sin^{-1}(x)dx = \frac{x^3 \sin^{-1}(x)}{3} - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}dx.$$

For this last integral, we perform the substitution $u = 1 - x^2$, $du = -2xdx$. The numerator is $x^2 xdx$, and we will replace $x^2 = 1 - u$ and $xdx = -\frac{du}{2}$. We obtain

$$\begin{aligned}
\int \frac{x^2 xdx}{\sqrt{1-x^2}} &= \int -\frac{1-u}{2\sqrt{u}}du \\
&= \frac{1}{2}\int \left(u^{1/2} - u^{-1/2}\right) du \\
&= \frac{1}{2}\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right) \\
&= \frac{(1-x^2)^{3/2}}{3} - (1-x^2)^{1/2}.
\end{aligned}$$

Plugging back in the above equation gives

$$\begin{aligned}
\int x^2 \sin^{-1}(x)dx &= \frac{x^3 \sin^{-1}(x)}{3} - \frac{1}{3}\left(\frac{(1-x^2)^{3/2}}{3} - (1-x^2)^{1/2}\right) \\
&= \boxed{\frac{x^3 \sin^{-1}(x)}{3} - \frac{(1-x^2)^{3/2}}{9} + \frac{(1-x^2)^{1/2}}{3} + C}.
\end{aligned}$$

(c) $\int \frac{\ln(x)}{x^5}dx$

Solution. We use an IBP with

$$u = \ln(x) \;\Rightarrow\; du = \frac{dx}{x},$$
$$dv = \frac{dx}{x^5}, \;\Rightarrow\; v = -\frac{1}{4x^4}.$$

This gives

$$\begin{aligned}
\int \frac{\ln(x)}{x^5}dx &= -\frac{\ln(x)}{4x^4} + \frac{1}{4}\int \frac{1}{x^5}dx \\
&= \boxed{-\frac{\ln(x)}{4x^4} - \frac{1}{16x^4} + C}.
\end{aligned}$$

(d) $\int x^3 e^{-x^2} dx$

Solution. We can start with the substitution $t = -x^2$, $dt = -2xdx$, which gives

$$\int x^3 e^{-x^2} dx = \int x^2 e^{-x^2} x dx = \int (-t) e^t \frac{dt}{-2} = \frac{1}{2} \int t e^t dt$$

. We compute this new integral with an IBP taking

$$u = t \;\Rightarrow\; du = dt,$$
$$dv = e^t dt \;\Rightarrow\; v = e^t,$$

which gives

$$\int t e^t dt = t e^t - \int e^t dt = t e^t - e^t = e^t (t-1).$$

We plug back and replace t by $-x^2$:

$$\begin{aligned}\int x^3 e^{-x^2} dx &= \frac{e^t}{2} (t-1) \\ &= \boxed{\frac{e^{-x^2}}{2} \left(-x^2 - 1\right) + C}.\end{aligned}$$

(e) $\int e^{-2x} \sin(3x) dx$

Solution. We perform an IBP twice and then solve for the unknown integral. For the first IBP, the parts are

$$u = \sin(3x) \;\Rightarrow\; du = 3\cos(3x) dx,$$
$$dv = e^{-2x} dx \;\Rightarrow\; v = -\frac{e^{-2x}}{2}.$$

This gives

$$\begin{aligned}\int e^{-2x} \sin(3x) dx &= -\frac{e^{-2x}\sin(3x)}{2} - \int -\frac{e^{-2x}}{2} 3\cos(3x) dx \\ &= -\frac{e^{-2x}\sin(3x)}{2} + \frac{3}{2} \int e^{-2x} \cos(3x) dx.\end{aligned}$$

For the second IBP, the parts are

$$u = \cos(3x) \;\Rightarrow\; du = -3\sin(3x) dx,$$
$$dv = e^{-2x} dx \;\Rightarrow\; v = -\frac{e^{-2x}}{2}.$$

This gives

$$\begin{aligned}\int e^{-2x} \sin(3x) dx &= -\frac{e^{-2x}\sin(3x)}{2} + \frac{3}{2}\left(-\frac{e^{-2x}\cos(3x)}{2} - \int -\frac{e^{-2x}}{2} (-3\sin(3x)) dx \right) \\ &= -\frac{e^{-2x}\sin(3x)}{2} - \frac{3e^{-2x}\cos(3x)}{4} - \frac{9}{4} \int e^{-2x} \sin(3x) dx.\end{aligned}$$

We can now solve the relation for the unknown integral. This gives

$$\begin{aligned}
\left(1+\frac{9}{4}\right)\int e^{-2x}\sin(3x)dx &= -\frac{e^{-2x}\sin(3x)}{2}-\frac{3e^{-2x}\cos(3x)}{4}\\
\frac{13}{4}\int e^{-2x}\sin(3x)dx &= -\frac{e^{-2x}\sin(3x)}{2}-\frac{3e^{-2x}\cos(3x)}{4}\\
\int e^{-2x}\sin(3x)dx &= \boxed{-\frac{2e^{-2x}\sin(3x)}{13}-\frac{3e^{-2x}\cos(3x)}{13}+C}
\end{aligned}$$

(f) $\int x\sec(5x)^2dx$

*Solution.*We use an IBP with the parts

$$\begin{aligned}
&u = x \;\Rightarrow\; du = dx,\\
&dv = \sec(5x)^2dx \;\Rightarrow\; v = \frac{\tan(5x)}{5},
\end{aligned}$$

which gives

$$\begin{aligned}
\int x\sec(5x)^2dx &= \frac{x\tan(5x)}{5}-\frac{1}{5}\int\tan(5x)dx\\
&= \boxed{\frac{x\tan(5x)}{5}-\frac{\ln|\sec(5x)|}{25}+C}.
\end{aligned}$$

2. Calculate the volume of the solid obtained by revolving the given region about the given axis using (i) the method of disks/washers and (ii) the method of cylindrical shells.

(a) The region between the graph of $y=\sqrt{\tan^{-1}(x)}$ and the x-axis for $0\leqslant x\leqslant 1$ revolved about the x-axis.

Solution. (i) Revolving the vertical strip at x in the region about the x-axis forms a disk of radius $r(x)=\sqrt{\tan^{-1}(x)}$. So the volume is

$$\begin{aligned}
V &= \int_0^1 \pi r(x)^2dx\\
&= \int_0^1 \pi\sqrt{\tan^{-1}(x)}^2\,dx\\
&= \pi\int_0^1 \tan^{-1}(x)dx.
\end{aligned}$$

We calculate this integral with an IBP using the parts

$$\begin{aligned}
&u = \tan^{-1}(x) \;\Rightarrow\; du = \frac{dx}{x^2+1},\\
&dv = dx \;\Rightarrow\; v = x.
\end{aligned}$$

We obtain

$$V = \pi\left(\left[x\tan^{-1}(x)\right]_0^1-\int_0^1\frac{x}{x^2+1}dx\right)$$

$$= \pi \left(\frac{\pi}{4} - \int_0^1 \frac{x}{x^2+1} dx \right).$$

This remaining integral can be evaluated with the substitution $u = x^2 + 1$, so that $du = 2xdx$. The bounds are $u = 1$ when $x = 0$ and $u = 2$ when $x = 1$. We get

$$\begin{aligned} V &= \pi \left(\frac{\pi}{4} - \int_1^2 \frac{du}{2u} \right) \\ &= \pi \left(\frac{\pi}{4} - \frac{1}{2} [\ln |u|]_1^2 \right) \\ &= \boxed{\pi \left(\frac{\pi}{4} - \frac{\ln(2)}{2} \right) \text{ cubic units}}. \end{aligned}$$

(ii) We will need to express the curve as a function of y:

$$y = \sqrt{\tan^{-1}(x)} \;\Rightarrow\; \tan^{-1}(x) = y^2 \;\Rightarrow\; x = \tan\left(y^2\right).$$

Then the region is bounded between the curves $x = \tan\left(y^2\right)$ and $x = 1$ for $0 \leqslant y \leqslant \sqrt{\frac{\pi}{4}}$. Revolving the horizontal strip at y in the region about the x-axis forms a cylindrical shell of radius $r(y) = y$ and height $h(y) = 1 - \tan(y^2)$. So the volume is

$$\begin{aligned} V &= \int_0^{\sqrt{pi/4}} 2\pi r(y) h(y) dy \\ &= \int_0^{\sqrt{pi/4}} 2\pi y \left(1 - \tan\left(y^2\right)\right) dy. \end{aligned}$$

We can calculate this integral with the substituion $u = y^2$, which gives $du = 2ydy$. The new bounds are $u = 0$ to $u = \frac{\pi}{4}$. We get

$$\begin{aligned} V &= \int_0^{\pi/4} \pi \left(1 - \tan(u)\right) du \\ &= \pi \left[u - \ln|\sec(u)|\right]_0^{\pi/4} \\ &= \pi \left(\frac{\pi}{4} - \ln\left(\sec\left(\frac{\pi}{4}\right)\right) + \ln\left(\sec(0)\right) \right) \\ &= \pi \left(\frac{\pi}{4} - \ln\left(\sqrt{2}\right) + \ln(1) \right) \\ &= \boxed{\pi \left(\frac{\pi}{4} - \frac{\ln(2)}{2} \right) \text{ cubic units}}. \end{aligned}$$

(b) The region bounded by the y-axis, the graph of $y = \sin(x)$ and the line $y = 1$ revolved about the y-axis.

Solution. (i) The region can be described as the region between the y-axis and $x = \sin^{-1}(y)$ for $0 \leqslant y \leqslant 1$. Revolving the horizontal strip at y in the region will form a disk of radius $r(y) = \sin^{-1}(y)$. So the volume is

$$V = \int_0^1 \pi r(y)^2 dy$$

$$= \int_0^1 \pi \sin^{-1}(y)^2 dy.$$

We can calculate this integral with two successive IBPs. The first one uses the parts

$$u = \sin^{-1}(y)^2 \;\Rightarrow\; du = \frac{2\sin^{-1} dy}{\sqrt{1-y^2}},$$
$$dv = dy \;\Rightarrow\; v = y.$$

This gives

$$\begin{aligned} V &= \pi\left(\left[y\sin^{-1}(y)^2\right]_0^1 - \int_0^1 \frac{2y\sin^{-1}(y)}{\sqrt{1-y^2}}dy\right) \\ &= \pi\left(\frac{\pi^2}{4} - 2\int_0^1 \frac{y\sin^{-1}(y)}{\sqrt{1-y^2}}dy\right). \end{aligned}$$

In this last integral, we use an IBP with parts

$$u = \sin^{-1}(y) \;\Rightarrow\; du = \frac{dy}{\sqrt{1-y^2}},$$
$$dv = \frac{ydy}{\sqrt{1-y^2}} \;\Rightarrow\; v = -\sqrt{1-y^2}.$$

We get

$$\begin{aligned} V &= \pi\left(\frac{\pi^2}{4} - 2\left(\left[-\sin^{-1}(y)\sqrt{1-y^2}\right]_0^1 - \int_0^1 \frac{-\sqrt{1-y^2}}{\sqrt{1-y^2}}dy\right)\right) \\ &= \pi\left(\frac{\pi^2}{4} - 2\left(0 + \int_0^1 dy\right)\right) \\ &= \boxed{\pi\left(\frac{\pi^2}{4} - 2\right) \text{ cubic units}}. \end{aligned}$$

(ii) Revolving the vertical strip at x in the region about the y-axis creates a cylindrical shell of radius $r(x) = x$ and height $h(x) = 1 - \sin(x)$. Therefore the volume is

$$\begin{aligned} V &= \int_0^{\pi/2} 2\pi r(x)h(x)dx \\ &= 2\pi\int_0^{\pi/2} x\left(1 - \sin(x)\right)dx. \end{aligned}$$

We can compute this integral with an IBP, taking the parts

$$u = x \;\Rightarrow\; du = dx,$$
$$dv = 1 - \sin(x) \;\Rightarrow\; v = x + \cos(x).$$

We get

$$V = 2\pi\left(\left[x(x+\cos(x))\right]_0^{\pi/2} - \int_0^{\pi/2} (x + \cos(x))dx\right)$$

$$
\begin{aligned}
&= 2\pi \left(\frac{\pi}{2}\left(\frac{\pi}{2}+0\right) - 0 - \left[\frac{x^2}{2}+\sin(x)\right]_0^{\pi/2} \right) \\
&= 2\pi \left(\frac{\pi^2}{4} - \frac{\pi^2}{8} - 1 \right) \\
&= \boxed{\pi\left(\frac{\pi^2}{4}-2\right) \text{ cubic units}}.
\end{aligned}
$$

(c) The region between the graph of $y = \ln(x)$ and the x-axis for $1 \leqslant x \leqslant e$ revolved about the line $x = -2$.

(i) The region can be described as the region between $x = e^y$ and $x = e$ for $0 \leqslant y \leqslant 1$. Revolving the horizontal strip at y in the region about the line $x = -2$ forms a washer with inner radius $r_{\text{in}}(y) = e^y - (-2) = e^y + 2$ and outer radius $r_{\text{out}}(y) = e - (-2) = e + 2$. So the volume is

$$
\begin{aligned}
V &= \int_0^1 \pi \left(r_{\text{out}}(y)^2 - r_{\text{in}}(y)^2\right) dy \\
&= \int_0^1 \pi \left((e+2)^2 - (e^y+2)^2\right) dy \\
&= \pi \int_0^1 \left((e+2)^2 - 4 - e^{2y} - 4e^y\right) dy \\
&= \pi \left[((e+2)^2-4)y - \frac{e^{2y}}{2} - 4e^y\right]_0^1 \\
&= \pi \left((e+2)^2 - 4 - \frac{e^2}{2} - 4e + \frac{1}{2} + 4 \right) \\
&= \boxed{\frac{\pi(e^2+9)}{9} \text{ cubic units}}.
\end{aligned}
$$

(ii) Revolving the vertical strip at x about the line $x = -2$ forms a cylindrical shell with radius $r(x) = x - (-2) = x + 2$ and height $h(x) = \ln(x)$. So the volume is

$$
\begin{aligned}
V &= \int_1^e 2\pi r(x)h(x)dx \\
&= 2\pi \int_1^e (x+2)\ln(x)dx.
\end{aligned}
$$

We compute this integral with an IBP, taking the parts

$$
\begin{aligned}
u &= \ln(x) \ \Rightarrow\ du = \frac{dx}{x}, \\
dv &= (x+2)dx \ \Rightarrow\ v = \frac{x^2}{2} + 2x.
\end{aligned}
$$

This gives

$$
\begin{aligned}
V &= 2\pi \left(\left[\left(\frac{x^2}{2}+2x\right)\ln(x)\right]_1^e - \int_1^e \left(\frac{x^2}{2}+2x\right)\frac{1}{x}dx \right) \\
&= 2\pi \left(\frac{e^2}{2} + 2e - \int_1^e \left(\frac{x}{2}+2\right) dx \right)
\end{aligned}
$$

$$
\begin{aligned}
&= 2\pi\left(\frac{e^2}{2} + 2e - \left[\frac{x^2}{4} + 2x\right]_1^e\right)\\
&= 2\pi\left(\frac{e^2}{2} + 2e - \frac{e^2}{4} - 2e + \frac{1}{4} + 2\right)\\
&= \boxed{\frac{\pi(e^2+9)}{9}\text{ cubic units}}.
\end{aligned}
$$

3. Find reduction formulas for the following integrals.

   (a) $\displaystyle\int \cos(3x)^n dx$

   Solution. We split off a factor $\cos(3x)$ and use IBP with parts

$$
\begin{aligned}
u &= \cos(3x)^{n-1} \;\Rightarrow\; du = -3(n-1)\cos(3x)^{n-2}\sin(3x)dx,\\
dv &= \cos(3x)dx \;\Rightarrow\; v = \frac{\sin(3x)}{3}.
\end{aligned}
$$

   This gives

$$
\begin{aligned}
\int \cos(3x)^n dx &= \int \cos(3x)^{n-1}\cos(3x)dx\\
\int \cos(3x)^n dx &= \frac{\cos(3x)^{n-1}\sin(3x)}{3} - \int -3(n-1)\cos(3x)^{n-2}\sin(3x)\frac{\sin(3x)}{3}dx\\
\int \cos(3x)^n dx &= \frac{\cos(3x)^{n-1}\sin(3x)}{3} + (n-1)\int \cos(3x)^{n-2}\sin(3x)^2 dx.
\end{aligned}
$$

   In this last integral, we use the Phytagorean identity $\sin(3x)^2 = 1-\cos(3x)^2$ to obtain

$$
\begin{aligned}
\int \cos(3x)^n dx &= \frac{\cos(3x)^{n-1}\sin(3x)}{3} + (n-1)\int \cos(3x)^{n-2}\left(1-\cos(3x)^2\right)dx\\
\int \cos(3x)^n dx &= \frac{\cos(3x)^{n-1}\sin(3x)}{3} + (n-1)\int \cos(3x)^{n-2}dx - (n-1)\int \cos(3x)^n dx
\end{aligned}
$$

   We can now solve for the original integral by moving the term $-(n-1)\displaystyle\int \cos(3x)^n dx$ to the left-hand side.

$$
\begin{aligned}
\int \cos(3x)^n dx + (n-1)\int \cos(3x)^n dx &= \frac{\cos(3x)^{n-1}\sin(3x)}{3} + (n-1)\int \cos(3x)^{n-2}dx\\
n\int \cos(3x)^n dx &= \frac{\cos(3x)^{n-1}\sin(3x)}{3} + (n-1)\int \cos(3x)^{n-2}dx
\end{aligned}
$$

   We now divide by n to obtain the reduction formula

$$
\boxed{\int \cos(3x)^n dx = \frac{\cos(3x)^{n-1}\sin(3x)}{3n} + \frac{(n-1)}{n}\int \cos(3x)^{n-2}dx}.
$$

(b) $\displaystyle\int \ln(x)^n dx$

Solution. We use IBP with parts

$$\begin{aligned} u = \ln(x)^n &\Rightarrow du = \frac{n\ln(x)^{n-1}dx}{x}, \\ dv = dx &\Rightarrow v = x. \end{aligned}$$

This gives

$$\int \ln(x)^n dx = x\ln(x)^n - \int x\frac{n\ln(x)^{n-1}}{x}dx$$

$$\boxed{\int \ln(x)^n dx = x\ln(x)^n - n\int \ln(x)^{n-1}dx}$$

(c) $\displaystyle\int \sec(5x)^n dx$

Solution. We separate a factor $\sec(5x)^2$ and use IBP with parts

$$\begin{aligned} u = \sec(5x)^{n-2} &\Rightarrow du = 5(n-2)\sec(5x)^{n-2}\tan(5x)dx, \\ dv = \sec(5x)^2dx &\Rightarrow v = \frac{\tan(5x)}{5}. \end{aligned}$$

We get

$$\begin{aligned} \int \sec(5x)^n dx &= \int \sec(5x)^{n-2}\sec(5x)^2dx \\ Int\sec(5x)^n dx &= \frac{\sec(5x)^{n-2}\tan(5x)}{5} - \int 5(n-2)\sec(5x)^{n-2}\tan(5x)\frac{\tan(5x)}{5}dx \\ \int \sec(5x)^n dx &= \frac{\sec(5x)^{n-2}\tan(5x)}{5} - (n-2)\int \sec(5x)^{n-2}\tan(5x)^2dx \end{aligned}$$

In this last integral, we use the Pythagorean identity $\tan(5x)^2 = \sec(5x)^2 - 1$, which gives

$$\begin{aligned} \int \sec(5x)^n dx &= \frac{\sec(5x)^{n-2}\tan(5x)}{5} - (n-2)\int \sec(5x)^{n-2}\left(\sec(5x)^2 - 1\right)dx \\ \int \sec(5x)^n dx &= \frac{\sec(5x)^{n-2}\tan(5x)}{5} - (n-2)\int \sec(5x)^n dx + (n-2)\int \sec(5x)^{n-2}dx \end{aligned}$$

We can now solve for the original integral by moving the term $-(n-2)\displaystyle\int \sec(5x)^n dx$ to the left-hand side.

$$\begin{aligned} \int \sec(5x)^n dx + (n-2)\int \sec(5x)^n dx &= \frac{\sec(5x)^{n-2}\tan(5x)}{5} + (n-2)\int \sec(5x)^{n-2}dx \\ (n-1)\int \sec(5x)^n dx &= \frac{\sec(5x)^{n-2}\tan(5x)}{5} + (n-2)\int \sec(5x)^{n-2}dx \end{aligned}$$

Dividing by $n-1$ gives the reduction formula

$$\boxed{\int \sec(5x)^n \, dx = \frac{\sec(5x)^{n-2}\tan(5x)}{5(n-1)} + \frac{n-2}{n-1}\int \sec(5x)^{n-2}\, dx}.$$

Rutgers University
Math 152

Section 8.3: Trigonometric Integrals - Worksheet Solutions

1. Calculate the following integrals.

   (a) $\displaystyle\int \sin(5x)^2 dx$

   Solution. We use the double angle formula

$$\sin(\theta)^2 = \frac{1-\cos(2\theta)}{2},$$

   which gives

$$\begin{aligned}\int \sin(5x)^2 dx &= \int \frac{1-\cos(10x)}{2} dx \\ &= \boxed{\frac{x}{2} - \frac{\sin(10x)}{20} + C}.\end{aligned}$$

   (b) $\displaystyle\int \sec(2x)^4 \tan(2x)^6 dx$

   Solution. The exponent of secant is even, so we can split off a factor $\sec(2x)^2$, rewrite the rest of the integrand in terms of $\tan(2x)$ using the Pythagorean identity $\sec(2x)^2 = \tan(2x)^2 + 1$, and use the substitution $u = \tan(2x)$, $du = 2\sec(2x)^2 dx$. This gives

$$\begin{aligned}\int \sec(2x)^4 \tan(2x)^6 dx &= \int \sec(2x)^2 \tan(2x)^6 \sec(2x)^2 dx \\ &= \int (\tan(2x)^2 + 1) \tan(2x)^6 \sec(2x)^2 dx \\ &= \int (u^2+1)u^6 \frac{du}{2} \\ &= \frac{1}{2}\int (u^8 + u^6) du \\ &= \frac{1}{2}\left(\frac{u^9}{9} + \frac{u^7}{7}\right) + C \\ &= \boxed{\frac{1}{2}\left(\frac{\tan(2x)^9}{9} + \frac{\tan(2x)^7}{7}\right) + C}.\end{aligned}$$

   (c) $\displaystyle\int_0^{\pi/21} \tan(7\theta)^3 d\theta$

Solution. We can split off a factor $\tan(7\theta)^2$, replace it by $\sec(7\theta)^2 - 1$ and distribute. This gives

$$\begin{aligned}\int_0^{\pi/21} \tan(7\theta)^3 d\theta &= \int_0^{\pi/21} \tan(7\theta)\tan(7\theta)^2 d\theta \\ &= \int_0^{\pi/21} \tan(7\theta)\left(\sec(7\theta)^2 - 1\right) d\theta \\ &= \int_0^{\pi/21} \tan(7\theta)\sec(7\theta)^2 d\theta - \int_0^{\pi/21} \tan(7\theta) d\theta.\end{aligned}$$

The first integral can be evaluated using the substitution $u = \tan(7\theta)$, which gives $du = 7\sec(7\theta)^2 d\theta$. The bounds become

$$\begin{aligned}\theta = 0 &\Rightarrow u = \tan(0) = 0, \\ \theta = \frac{\pi}{21} &\Rightarrow u = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}.\end{aligned}$$

So we get

$$\begin{aligned}\int_0^{\pi/21} \tan(7\theta)^3 d\theta &= \int_0^{\sqrt{3}} \frac{u}{7} du - \int_0^{\pi/21} \tan(7\theta) d\theta \\ &= \left[\frac{u^2}{21}\right]_0^{\sqrt{3}} - \left[\frac{\ln|\sec(7\theta)|}{7}\right]_0^{\pi/21} \\ &= \frac{\sqrt{3}^2}{21} - \frac{1}{7}\left(\ln\left(\sec\left(\frac{\pi}{3}\right)\right) - \ln(\sec(0))\right) \\ &= \boxed{\frac{1 - \ln(2)}{7}}.\end{aligned}$$

(d) $\displaystyle\int \sec(3x)^2 \ln(\sec(3x)) dx$

Solution. We can start with an IBP with parts

$$\begin{aligned}u = \ln(\sec(3x)) &\Rightarrow du = \frac{3\sec(3x)\tan(3x)}{\sec(3x)} dx = 3\tan(3x) dx, \\ dv = \sec(3x)^2 dx &\Rightarrow v = \frac{1}{3}\tan(3x).\end{aligned}$$

We get

$$\begin{aligned}\int \sec(3x)^2 \ln(\sec(3x)) dx &= \frac{\tan(3x)\ln(\sec(3x))}{3} - \int 3\tan(3x)\frac{\tan(3x)}{3} dx \\ &= \frac{\tan(3x)\ln(\sec(3x))}{3} - \int \tan(3x)^2 dx.\end{aligned}$$

This last integral can be computed using the Pythagorean identity $\tan(3x)^2 = \sec(3x)^2 - 1$, which gives

$$\begin{aligned}\int \sec(3x)^2 \ln(\sec(3x)) dx &= \frac{\tan(3x)\ln(\sec(3x))}{3} - \int \left(\sec(3x)^2 - 1\right) dx \\ &= \boxed{\frac{\tan(3x)\ln(\sec(3x))}{3} - \frac{\tan(3x)}{3} - x + C}.\end{aligned}$$

(e) $\displaystyle\int_{\pi}^{3\pi/2} \cos(z)^5 \sin(z)^8 dz$

Solution. Since the power of cos is odd, we can compute this integral by splitting off a factor $\cos(z)$, rewriting the remaining factors in terms of $\sin(z)$ using the Pythagorean identity $\cos(z)^2 = 1-\sin(z)^2$ and substituting $u = \sin(z)$, $du = \cos(z)dz$. The bounds will change to

$$\begin{aligned} z = \pi &\;\Rightarrow\; u = \sin(\pi) = 0, \\ z = \frac{3\pi}{2} &\;\Rightarrow\; u = \sin\left(\frac{3\pi}{2}\right) = -1. \end{aligned}$$

This yields

$$\begin{aligned} \int_{\pi}^{3\pi/2} \cos(z)^5 \sin(z)^8 dz &= \int_{\pi}^{3\pi/2} \cos(z)^4 \sin(z)^8 \cos(z) dz \\ &= \int_{\pi}^{3\pi/2} \left(1 - \sin(z)^2\right)^2 \sin(z)^8 \cos(z) dz \\ &= \int_{0}^{-1} \left(1 - u^2\right)^2 u^8 du \\ &= \int_{0}^{-1} \left(u^8 - u^{10}\right) du \\ &= \left[\frac{u^9}{9} - \frac{u^{11}}{11}\right]_0^{-1} \\ &= \frac{(-1)^9}{9} - \frac{(-1)^{11}}{11} \\ &= \boxed{-\frac{2}{99}}. \end{aligned}$$

(f) $\displaystyle\int_{\pi/3}^{\pi/2} \sqrt{\frac{1+\cos(t)}{1-\cos(t)}} dt$

Solution. We can rewrite the inside of the square root as a perfect square using the double angle formulas. From

$$\cos(\theta)^2 = \frac{1+\cos(2\theta)}{2}, \quad \sin(\theta)^2 = \frac{1-\cos(2\theta)}{2},$$

we obtain

$$1+\cos(t) = 2\cos\left(\frac{t}{2}\right)^2, \quad 1-\cos(t) = 2\sin\left(\frac{t}{2}\right)^2.$$

Using this for the integral gives

$$\begin{aligned} \int_{\pi/3}^{\pi/2} \sqrt{\frac{1+\cos(t)}{1-\cos(t)}} dt &= \int_{\pi/3}^{\pi/2} \sqrt{\frac{2\cos\left(\frac{t}{2}\right)^2}{2\sin\left(\frac{t}{2}\right)^2}} dt \\ &= \int_{\pi/3}^{\pi/2} \sqrt{\cot\left(\frac{t}{2}\right)^2} dt \\ &= \int_{\pi/3}^{\pi/2} \left|\cot\left(\frac{t}{2}\right)\right| dt \end{aligned}$$

$$
\begin{aligned}
&= \int_{\pi/3}^{\pi/2} \cot\left(\frac{t}{2}\right) dt \\
&= \left[2\ln\left|\sin\left(\frac{t}{2}\right)\right|\right]_{\pi/3}^{\pi/2} \\
&= 2\left(\ln\left(\sin\left(\frac{\pi}{4}\right)\right) - \ln\left(\sin\left(\frac{\pi}{6}\right)\right)\right) \\
&= 2\left(\ln\left(\frac{\sqrt{2}}{2}\right) - \ln\left(\frac{1}{2}\right)\right) \\
&= \boxed{\ln(2)}.
\end{aligned}
$$

2. Express $\int \sin(3x)^n dx$ in terms of $\int \sin(3x)^{n-2} dx$.

Solution. We split off a factor $\sin(3x)$ and use IBP with parts

$$
\begin{aligned}
u &= \sin(3x)^{n-1} \ \Rightarrow \ du = 3(n-1)\sin(3x)^{n-2}\cos(3x)dx, \\
dv &= \sin(3x)dx \ \Rightarrow \ v = -\frac{\cos(3x)}{3}.
\end{aligned}
$$

This gives

$$
\begin{aligned}
\int \sin(3x)^n dx &= \int \sin(3x)^{n-1}\sin(3x)dx \\
\int \sin(3x)^n dx &= -\frac{\sin(3x)^{n-1}\cos(3x)}{3} - \int 3(n-1)\sin(3x)^{n-2}\cos(3x)\frac{-\cos(3x)}{3}dx \\
\int \sin(3x)^n dx &= -\frac{\sin(3x)^{n-1}\cos(3x)}{3} + (n-1)\int \sin(3x)^{n-2}\cos(3x)^2 dx.
\end{aligned}
$$

In this last integral, we use the Phytagorean identity $\cos(3x)^2 = 1 - \sin(3x)^2$ to obtain

$$
\begin{aligned}
\int \sin(3x)^n dx &= -\frac{\sin(3x)^{n-1}\cos(3x)}{3} + (n-1)\int \sin(3x)^{n-2}\left(1 - \sin(3x)^2\right) dx \\
\int \sin(3x)^n dx &= -\frac{\sin(3x)^{n-1}\cos(3x)}{3} + (n-1)\int \sin(3x)^{n-2}dx - (n-1)\int \sin(3x)^n dx
\end{aligned}
$$

We can now solve for the original integral by moving the term $-(n-1)\int \sin(3x)^n dx$ to the left-hand side.

$$
\begin{aligned}
\int \sin(3x)^n dx + (n-1)\int \sin(3x)^n dx &= -\frac{\sin(3x)^{n-1}\cos(3x)}{3} + (n-1)\int \sin(3x)^{n-2}dx \\
n\int \sin(3x)^n dx &= -\frac{\sin(3x)^{n-1}\cos(3x)}{3} + (n-1)\int \sin(3x)^{n-2}dx
\end{aligned}
$$

We now divide by n to obtain the reduction formula

$$
\boxed{\int \sin(3x)^n dx = -\frac{\sin(3x)^{n-1}\cos(3x)}{3n} + \frac{(n-1)}{n}\int \sin(3x)^{n-2}dx}.
$$

3. Consider the region bounded by the x-axis, the graph of $y = \sec(x)^2 \tan(x)$ and the lines $x = 0$, $x = \frac{\pi}{4}$. Calculate the volume of the solid obtained by revolving $\mathcal{R}$ about (a) the x-axis, (b) the y-axis.

Solution. (a) We use the disk method. Revolving the vertical strip at x about the x-axis forms a disk with radius $r(x) = \sec(x)^2 \tan(x)$. Therefore the volume is

$$\begin{aligned} V &= \int_0^{\pi/4} \pi r(x)^2 dx \\ &= \pi \int_0^{\pi/4} \sec(x)^4 \tan(x)^2 dx. \end{aligned}$$

The exponent of secant is even, so we can split off a factor $\sec(x)^2$, rewrite the rest of the integrand in terms of $\tan(x)$ using the Pythagorean identity $\sec(x)^2 = \tan(x)^2 + 1$, and use the substitution $u = \tan(x)$, $du = \sec(x)^2 dx$. The bounds become

$$\begin{aligned} x = 0 &\;\Rightarrow\; u = \tan(0) = 0, \\ x = \frac{\pi}{4} &\;\Rightarrow\; u = \tan\left(\frac{\pi}{4}\right) = 1. \end{aligned}$$

This gives

$$\begin{aligned} V &= \pi \int_0^{\pi/4} \sec(x)^2 \tan(x)^2 \sec(x)^2 dx \\ &= \pi \int_0^{\pi/4} \sec(x)^2 \tan(x)^2 \sec(x)^2 dx \\ &= \pi \int_0^{\pi/4} \left(\tan(x)^2 + 1\right) \tan(x)^2 \sec(x)^2 dx \\ &= \pi \int_0^1 \left(u^2 + 1\right) u^2 du \\ &= \pi \int_0^1 \left(u^4 + u^2\right) du \\ &= \pi \left[\frac{u^5}{5} + \frac{u^3}{3}\right]_0^1 \\ &= \pi \left(\frac{1}{5} + \frac{1}{3}\right) \\ &= \boxed{\frac{8\pi}{15} \text{ cubic units}}. \end{aligned}$$

(b) We use the method of cylindrical shells. Revolving the vertical strip at x about the y-axis forms a shell with radius $r(x) = x$ and height $h(x) = \sec(x)^2 \tan(x)$. So the volume is

$$\begin{aligned} V &= \int_0^{\pi/4} 2\pi r(x) h(x) \\ &= 2\pi \int_0^{\pi/4} x \sec(x)^2 \tan(x) dx. \end{aligned}$$

We can compute this integral with an IBP taking the parts

$$u = x \;\Rightarrow\; du = dx,$$

$$dv = \sec(x)^2\tan(x)dx \;\Rightarrow\; v = \frac{\tan(x)^2}{2}.$$

We get

$$\begin{aligned}
V &= 2\pi\left(\left[\frac{x\tan(x)^2}{2}\right]_0^{\pi/4} - \int_0^{\pi/4}\frac{\tan(x)^2}{2}dx\right)\\
&= 2\pi\left(\frac{\pi}{8} - \frac{1}{2}\int_0^{\pi/4}\left(\sec(x)^2 - 1\right)dx\right)\\
&= 2\pi\left(\frac{\pi}{8} - \frac{1}{2}\left[\tan(x) - x\right]_0^{\pi/4}\right)\\
&= 2\pi\left(\frac{\pi}{8} - \frac{1}{2}\left(1 - \frac{\pi}{4}\right)\right)\\
&= \boxed{\frac{\pi(\pi-2)}{2}\text{ cubic units}}.
\end{aligned}$$

4. Evaluate $\int \sec(\theta)^3 d\theta$ and $\int \sec(\theta)\tan(\theta)^2 d\theta$.

Solution. We can evaluate $\int \sec(\theta)^3 d\theta$ with an IBP and solving for the unknown integral when it reappears on the right-hand side. For the IBP we use the parts

$$\begin{aligned}
u = \sec(\theta) &\;\Rightarrow\; du = \sec(\theta)\tan(\theta)d\theta,\\
dv = \sec(\theta)^2 d\theta &\;\Rightarrow\; v = \tan(\theta).
\end{aligned}$$

We get

$$\begin{aligned}
\int \sec(\theta)^3 d\theta &= \int \sec(\theta)^2\sec(\theta)d\theta\\
\int \sec(\theta)^3 d\theta &= \tan(\theta)\sec(\theta) - \int \tan(\theta)\sec(\theta)\tan(\theta)d\theta\\
\int \sec(\theta)^3 d\theta &= \tan(\theta)\sec(\theta) - \int \tan(\theta)^2\sec(\theta)d\theta
\end{aligned}$$

We will use the Pythagorean identity $\tan(\theta)^2 = \sec(\theta)^2 - 1$ to see the original integral reappear on the right-hand side.

$$\begin{aligned}
\int \sec(\theta)^3 d\theta &= \tan(\theta)\sec(\theta) - \int \left(\sec(\theta)^2 - 1\right)\sec(\theta)d\theta\\
\int \sec(\theta)^3 d\theta &= \tan(\theta)\sec(\theta) - \int \sec(\theta)^3 d\theta + \int \sec(\theta)d\theta\\
\int \sec(\theta)^3 d\theta &= \tan(\theta)\sec(\theta) - \int \sec(\theta)^3 d\theta + \ln|\sec(\theta) + \tan(\theta)|
\end{aligned}$$

We can now move the term $-\int \sec(\theta)^3 d\theta$ to the left hand side and finish solving

$$2\int \sec(\theta)^3 d\theta = \tan(\theta)\sec(\theta) + \ln|\sec(\theta) + \tan(\theta)|$$

$$\Rightarrow \int \sec(\theta)^3 d\theta = \boxed{\frac{1}{2}\left(\tan(\theta)\sec(\theta) + \ln|\sec(\theta)+\tan(\theta)|\right) + C}.$$

For the other integral, we can use the Pythagorean identity and the integral of $\sec^3$ that we just computed as follows:

$$\begin{aligned}
\int \tan(\theta)^2 \sec(\theta) d\theta &= \int \left(\sec(\theta)^2 - 1\right)\sec(\theta) d\theta \\
&= \int \sec(\theta)^3 d\theta - \int \sec(\theta) d\theta \\
&= \frac{1}{2}\left(\tan(\theta)\sec(\theta) + \ln|\sec(\theta)+\tan(\theta)|\right) - \ln|\sec(\theta)+\tan(\theta)| \\
&= \boxed{\frac{1}{2}\left(\tan(\theta)\sec(\theta) - \ln|\sec(\theta)+\tan(\theta)|\right) + C}.
\end{aligned}$$

5. Calculate the arc length of the curve $y = x + \cos(x)\sin(x) - \frac{1}{8}\tan(x)$, $0 \leqslant x \leqslant \frac{\pi}{4}$.

Solution. We have

$$\begin{aligned}
\frac{dy}{dx} &= 1 - \sin(x)^2 + \cos(x)^2 - \frac{1}{8}\sec(x)^2 \\
&= \cos(x)^2 + \cos(x)^2 - \frac{1}{8}\sec(x)^2 \\
&= 2\cos(x)^2 - \frac{1}{8}\sec(x)^2.
\end{aligned}$$

So

$$\begin{aligned}
1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \left(2\cos(x)^2 - \frac{1}{8}\sec(x)^2\right)^2 \\
&= 1 + 4\cos(x)^4 + \frac{1}{64}\sec(x)^4 - 2\cdot 2\cos(x)^2 \cdot \frac{1}{8}\sec(x)^2 \\
&= 1 + 4\cos(x)^4 + \frac{1}{64}\sec(x)^4 - \frac{1}{2} \\
&= 4\cos(x)^4 + \frac{1}{64}\sec(x)^4 + \frac{1}{2} \\
&= 4\cos(x)^4 + \frac{1}{64}\sec(x)^4 + 2\cdot 2\cos(x)^2 \cdot \frac{1}{8}\sec(x)^2 \\
&= \left(2\cos(x)^2 + \frac{1}{8}\sec(x)^2\right)^2.
\end{aligned}$$

Therefore the arc length is given by

$$\begin{aligned}
L &= \int_0^{\pi/4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
&= \int_0^{\pi/4} \sqrt{\left(2\cos(x)^2 + \frac{1}{8}\sec(x)^2\right)^2} dx \\
&= \int_0^{\pi/4} \left|2\cos(x)^2 + \frac{1}{8}\sec(x)^2\right| dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\pi/4} \left(2\cos(x)^2 + \frac{1}{8}\sec(x)^2\right) dx \\
&= \int_0^{\pi/4} \left(2\frac{1+\cos(2x)}{2} + \frac{1}{8}\sec(x)^2\right) dx \\
&= \int_0^{\pi/4} \left(1+\cos(2x) + \frac{1}{8}\sec(x)^2\right) dx \\
&= \left[x + \frac{1}{2}\sin(2x) + \frac{1}{8}\tan(x)\right]_0^{\pi/4} \\
&= \left(\frac{\pi}{4} + \frac{1}{2}\sin\left(\frac{\pi}{2}\right) + \frac{1}{8}\tan\left(\frac{\pi}{4}\right)\right) - 0 \\
&= \boxed{\frac{\pi}{4} + \frac{5}{8} \text{ units}}.
\end{aligned}
$$

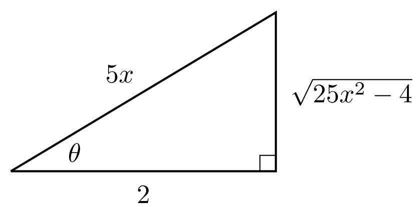
Rutgers University
Math 152

Section 8.4: Trigonometric Substitution - Worksheet Solutions

1. Calculate the following integrals.

(a) $\displaystyle\int \frac{\sqrt{25x^2-4}}{x}dx$ for $x > \frac{2}{5}$.

Solution. We want $25x^2 - 4 = 4\sec(\theta)^2 - 4$, so we substitute $x = \frac{2}{5}\sec(\theta)$ and $dx = \dfrac{2}{5}\sec(\theta)\tan(\theta)d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\sec(\theta) = \frac{5x}{2}$ as shown below.



We get $\sqrt{25x^2-4} = \sqrt{4\sec(\theta)^2 - 4} = 2\tan(\theta)$ and the integral becomes

$$\begin{aligned}\int \frac{\sqrt{25x^2-4}}{x}dx &= \int \frac{2\tan(\theta)}{\frac{2}{5}\sec(\theta)} \cdot \frac{2}{5}\tan(\theta)\sec(\theta)d\theta \\ &= 2\int \tan(\theta)^2 d\theta \\ &= 2\int \left(\sec(\theta)^2 - 1\right) d\theta \\ &= 2\left(\tan(\theta) - \theta\right) + C.\end{aligned}$$

We need to express this result in terms of x. Using the right triangle above, we see that $\tan(\theta) = \frac{\sqrt{25x^2-4}}{2}$ and $\theta = \sec^{-1}\left(\frac{5x}{2}\right)$. Thus

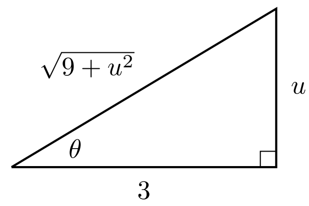
$$\boxed{\int \frac{\sqrt{25x^2-4}}{x}dx = \sqrt{25x^2-4} - 2\sec^{-1}\left(\frac{5x}{2}\right) + C}.$$

(b) $\displaystyle\int \frac{dt}{t\sqrt{9+\ln(t)^2}}$.

Solution. We stat by using the substitution $u = \ln(t)$, which gives $du = \frac{dt}{t}$ and

$$\int \frac{dt}{t\sqrt{9+\ln(t)^2}} = \int \frac{du}{\sqrt{9+u^2}}.$$

We compute this last integral using a trigonometric substitution. We want $9+u^2 = 9+9\tan(\theta)^2$, so we substitute $u = 3\tan(\theta)$ and $du = 3\sec(\theta)^2 d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\tan(\theta) = \frac{u}{3}$ as shown below.



We get $\sqrt{9+u^2} = \sqrt{9+9\tan(\theta)^2} = 3\sec(\theta)$ and the integral becomes

$$\begin{aligned}\int \frac{dt}{t\sqrt{9+\ln(t)^2}} &= \int \frac{du}{\sqrt{9+u^2}} \\ &= \int \frac{3\sec(\theta)^2 d\theta}{3\sec(\theta)} \\ &= \int \sec(\theta) d\theta \\ &= \ln|\tan(\theta)+\sec(\theta)| + C.\end{aligned}$$

We express this result in terms of u using the right triangle above, from which we see that

$$\tan(\theta) = \frac{u}{3}, \quad \sec(\theta) = \frac{\sqrt{9+u^2}}{3}.$$

We get

$$\begin{aligned}\int \frac{dt}{t\sqrt{9+\ln(t)^2}} &= \ln\left|\frac{u}{3} + \frac{\sqrt{9+u^2}}{3}\right| + C \\ &= \ln\left|u+\sqrt{9+u^2}\right| + C.\end{aligned}$$

We now finish by replacing u by $\ln(t)$ and we obtain

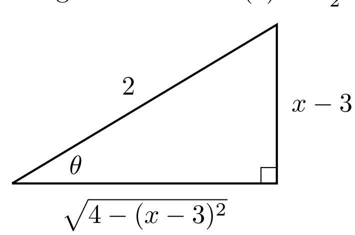
$$\boxed{\int \frac{dt}{t\sqrt{9+\ln(t)^2}} = \ln\left|\ln(t)+\sqrt{9+\ln(t)^2}\right| + C}.$$

(c) $\displaystyle\int \frac{dx}{(6x-x^2-5)^{5/2}}$.

Solution. We start by completing the square in the denominator:

$$6x-x^2-5 = -(x^2-6x)-5 = -(x^2-6x+9)+9-5 = 4-(x-3)^2.$$

We can now use a trigonometric substitution. We want $4-(x-3)^2 = 4-4\sin(\theta)^2$, so we substitute $x-3 = 2\sin(\theta)$ or $x = 3+2\sin(\theta)$. This gives $dx = 2\cos(\theta)d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\sin(\theta) = \frac{x-3}{2}$ as shown below.



We get $\left(4-(x+3)^2\right)^{5/2} = \left(4-4\sin(\theta)^2\right)^{5/2} = (4\cos(\theta)^2)^{5/2} = 32\cos(\theta)^5$. The integral becomes

$$\begin{aligned}\int \frac{dx}{(6x-x^2-5)^{5/2}} &= \int \frac{dx}{\left(4-(x+3)^2\right)^{5/2}} \\ &= \int \frac{2\cos(\theta)d\theta}{32\cos(\theta)^5} \\ &= \frac{1}{16}\int \frac{d\theta}{\cos(\theta)^4} \\ &= \frac{1}{16}\int \sec(\theta)^4 d\theta.\end{aligned}$$

Since the exponent of sec is even, we can split off a factor $\sec(\theta)^2$, rewrite the remaining factors using the Pythagorean identity $\sec(\theta)^2 = \tan(\theta)^2+1$ and then use the substitution $u = \tan(\theta)$, $du = \sec(\theta)^2 d\theta$. This gives

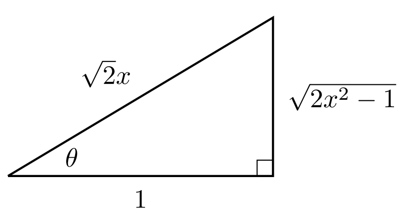
$$\begin{aligned}\int \frac{dx}{(6x-x^2-5)^{5/2}} &= \frac{1}{16}\int \sec(\theta)^2\sec(\theta)^2 d\theta \\ &= \frac{1}{16}\int \left(\tan(\theta)^2+1\right)\sec(\theta)^2 d\theta \\ &= \frac{1}{16}\int (u^2+1)du \\ &= \frac{1}{16}\left(\frac{u^3}{3}+u\right)+C \\ &= \frac{1}{16}\left(\frac{\tan(\theta)^3}{3}+\tan(\theta)\right)+C \\ &= \frac{\tan(\theta)}{16}\left(\frac{\tan(\theta)^2}{3}+1\right)+C.\end{aligned}$$

To express this antiderivative in terms of x, we use the right triangle above, from which we see that $\tan(\theta) = \frac{x-3}{\sqrt{4-(x-3)^2}}$. So we get

$$\boxed{\int \frac{dx}{(6x-x^2-5)^{3/2}} = \frac{x-3}{16\sqrt{4-(x-3)^2}}\left(\frac{(x-3)^2}{3(4-(x-3)^2)}+1\right)+C}.$$

(d) $\displaystyle\int_1^{\sqrt{2}} \frac{dx}{x(2x^2-1)^{3/2}}$.

Solution. We want $2x^2-1 = \sec(\theta)^2-1$, so we substitute $x = \frac{\sec(\theta)}{\sqrt{2}}$ and $dx = \frac{\sec(\theta)\tan(\theta)}{\sqrt{2}}d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\sec(\theta) = \sqrt{2}x$ as shown below.



We get $\left(2x^2-1\right)^{3/2} = \left(\sec(\theta)^2-1\right)^{3/2} = (\tan(\theta)^2)^{3/2} = \tan(\theta)^3$. The bounds change as follows:

$$x = 1 \;\Rightarrow\; \sec(\theta) = \sqrt{2}\cdot 1 = \sqrt{2} \;\Rightarrow\; \theta = \sec^{-1}(\sqrt{2}) = \frac{\pi}{4},$$
$$x = \sqrt{2} \;\Rightarrow\; \sec(\theta) = \sqrt{2}\cdot\sqrt{2} = 2 \;\Rightarrow\; \theta = \sec^{-1}(2) = \frac{\pi}{3}.$$

The integral becomes

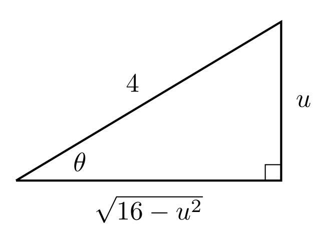
$$\begin{aligned}
\int_1^{\sqrt{2}} \frac{dx}{x(2x^2-1)^{3/2}} &= \int_{\pi/4}^{\pi/3} \frac{\frac{\sec(\theta)\tan(\theta)}{\sqrt{2}}d\theta}{\frac{\sec(\theta)}{\sqrt{2}}\tan(\theta)^3} \\
&= \int_{\pi/4}^{\pi/3} \frac{d\theta}{\tan(\theta)^2} \\
&= \int_{\pi/4}^{\pi/3} \cot(\theta)^2 d\theta \\
&= \int_{\pi/4}^{\pi/3} \left(\csc(\theta)^2 - 1\right) d\theta \\
&= [-\cot(\theta) - \theta]_{\pi/4}^{\pi/3} \\
&= -\cot\left(\frac{\pi}{3}\right) - \frac{\pi}{3} + \cot\left(\frac{\pi}{4}\right) + \frac{\pi}{4} \\
&= \boxed{1 - \frac{1}{\sqrt{3}} - \frac{\pi}{12}}.
\end{aligned}$$

(e) $\displaystyle\int \frac{e^{6x}}{\sqrt{16-e^{4x}}}dx.$

Solution. We start with the substitution $u = e^{2x}$, so that $du = 2e^{2x}dx$. The extraneous factor e^{4x} in the numerator can be expressed as $e^{4x} = (e^{2x})^2 = u^2$. So the integral becomes

$$\int \frac{e^{6x}}{\sqrt{16-e^{4x}}}dx = \int \frac{e^{4x}}{\sqrt{16-e^{4x}}}e^{2x}dx = \int \frac{u^2}{2\sqrt{16-u^2}}du.$$

We can now use a trigonometric substitution. We want $16 - u^2 = 16 - 16\sin(\theta)^2$, so we substitute $u = 4\sin(\theta)$ and $du = 4\cos(\theta)d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\sin(\theta) = \frac{u}{4}$ as shown below.



We get $\sqrt{16-u^2} = \sqrt{16-16\sin(\theta)^2} = \sqrt{16\cos(\theta)^2} = 4\cos(\theta)$. The integral becomes

$$\begin{aligned}
\int \frac{e^{6x}}{\sqrt{16-e^{4x}}}dx &= \int \frac{u^2}{2\sqrt{16-u^2}}du \\
&= \int \frac{(4\sin(\theta))^2}{2(4\cos(\theta))}4\cos(\theta)d\theta
\end{aligned}$$

$$= 8\int \sin(\theta)^2 d\theta.$$

We can compute this integral using the double angle formulas $\sin(\theta)^2 = \dfrac{1-\cos(2\theta)}{2}$. We get

$$\begin{aligned}
\int \frac{e^{6x}}{\sqrt{16-e^{4x}}}dx &= 8\int \frac{1-\cos(2\theta)}{2} 2d\theta \\
&= 4\left(\theta - \frac{\sin(2\theta)}{2}\right) + C \\
&= 4\left(\theta - \cos(\theta)\sin(\theta)\right) + C
\end{aligned}$$

where we have used the trigonometric identity $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$ in the last step. We can use the right triangle above to express this result in terms of u, observing that

$$\theta = \sin^{-1}\left(\frac{u}{4}\right), \quad \cos(\theta) = \frac{\sqrt{16-u^2}}{4}, \quad \sin(\theta) = \frac{u}{4}.$$

We can then replace $u = e^{2x}$ and we get

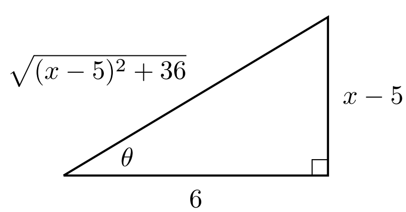
$$\begin{aligned}
\int \frac{e^{6x}}{\sqrt{16-e^{4x}}}dx &= 4\left(\sin^{-1}\left(\frac{u}{4}\right) - \frac{\sqrt{16-u^2}}{4}\frac{u}{4}\right) + C \\
&= \boxed{\frac{1}{4}\sin^{-1}\left(\frac{e^{2x}}{4}\right) - \frac{e^{2x}\sqrt{16-e^{4x}}}{4} + C}.
\end{aligned}$$

(f) $\displaystyle\int_5^{11} \frac{dx}{(x^2-10x+61)^{5/2}}$.

Solution. We start by completing the square in the denominator:

$$x^2 - 10x + 61 = (x^2 - 10x + 25) - 25 + 61 = (x-5)^2 + 36.$$

We can now use a trigonometric substitution. We want $(x-5)^2 + 36 = 36\tan(\theta)^2 + 36$, so we substitute $x - 5 = 6\tan(\theta)$, or $x = 5 + 6\tan(\theta)$. This gives $dx = 6\sec(\theta)^2 d\theta$ and the following right triangle with base angle θ such that $\tan(\theta) = \frac{x-5}{6}$.



Then $((x-5)^2+36)^{5/2} = (36\tan(\theta)^2+36)^{5/2} = (36\sec(\theta)^2)^{5/2} = 6^5\sec(\theta)^5$. The bounds change as follows:

$$\begin{aligned}
x = 5 &\;\Rightarrow\; \tan(\theta) = \frac{5-5}{6} = 0 \;\Rightarrow\; \theta = \tan^{-1}(0) = 0, \\
x = 11 &\;\Rightarrow\; \tan(\theta) = \frac{11-5}{6} = 1 \;\Rightarrow\; \theta = \tan^{-1}(1) = \frac{\pi}{4}.
\end{aligned}$$

The integral becomes

$$\begin{aligned}\int_5^{11} \frac{dx}{(x^2-10x+61)^{5/2}} &= \int_5^{11} \frac{dx}{((x-5)^2+36)^{5/2}} \\ &= \int_0^{\pi/4} \frac{6\sec(\theta)^2 d\theta}{6^5 \sec(\theta)^5} \\ &= \frac{1}{1296}\int_0^{\pi/4} \frac{d\theta}{\sec(\theta)^3} \\ &= \frac{1}{1296}\int_0^{\pi/4} \cos(\theta)^3 d\theta.\end{aligned}$$

Since the exponent of cos is odd, we can compute this integral by splitting off a factor $\cos(\theta)$, rewriting the remaining factors with the trigonometric identity $\cos(\theta)^2 = 1 - \sin(\theta)^2$ and using the substitution $u = \sin(\theta)$, $du = \cos(\theta)d\theta$. The bounds will change as follows

$$\begin{aligned}\theta = 0 &\;\Rightarrow\; u = \sin(0) = 0, \\ \theta = \frac{\pi}{4} &\;\Rightarrow\; u = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.\end{aligned}$$

The integral becomes

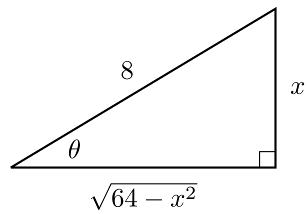
$$\begin{aligned}\int_5^{11} \frac{dx}{(x^2-10x+61)^{5/2}} &= \frac{1}{1296}\int_0^{\pi/4} \cos(\theta)^2\cos(\theta)d\theta \\ &= \frac{1}{1296}\int_0^{\pi/4} \left(1-\sin(\theta)^2\right)\cos(\theta)d\theta \\ &= \frac{1}{1296}\int_0^{\sqrt{2}/2} \left(1-u^2\right)du \\ &= \frac{1}{1296}\left[u - \frac{u^3}{3}\right]_0^{\sqrt{2}/2} \\ &= \frac{1}{1296}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}^3}{2^3\cdot 3}\right) \\ &= \boxed{\frac{5\sqrt{2}}{15552}}.\end{aligned}$$

2. Calculate the average value of the function $f(x) = \dfrac{1}{x\sqrt{64-x^2}}$ on the interval $[4, 4\sqrt{2}]$.

Solution. The average value on the interval $[4, 4\sqrt{2}]$ is given by

$$\text{av}(f) = \frac{1}{4\sqrt{2}-4}\int_4^{4\sqrt{2}} \frac{dx}{x\sqrt{64-x^2}}.$$

We compute this integral using the substitution $x = 8\sin(\theta)$ and $dx = 8\cos(\theta)d\theta$. The right triangle for this trigonometric substitution has base angle θ such that $\sin(\theta) = \frac{x}{8}$ as shown below.



Then $\sqrt{64-x^2} = \sqrt{64-64\sin(\theta)^2} = \sqrt{64\cos(\theta)^2} = 8\cos(\theta)$. The bounds change as follows:

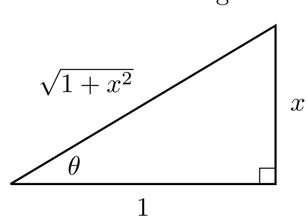
$$x = 4 \;\Rightarrow\; \sin(\theta) = \frac{4}{8} = \frac{1}{2} \;\Rightarrow\; \theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6},$$
$$x = 4\sqrt{2} \;\Rightarrow\; \sin(\theta) = \frac{4\sqrt{2}}{8} = \frac{\sqrt{2}}{2} \;\Rightarrow\; \theta = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.$$

Therefore

$$\begin{aligned}
\text{av}(f) &= \frac{1}{4(\sqrt{2}-1)}\int_{\pi/6}^{\pi/4} \frac{8\cos(\theta)d\theta}{8\sin(\theta)8\cos(\theta)} \\
&= \frac{1}{32(\sqrt{2}-1)}\int_{\pi/6}^{\pi/4} \frac{d\theta}{\sin(\theta)} \\
&= \frac{1}{32(\sqrt{2}-1)}\int_{\pi/6}^{\pi/4} \csc(\theta)d\theta \\
&= \frac{1}{32(\sqrt{2}-1)}\left[\ln|\csc(\theta)-\cot(\theta)|\right]_{\pi/6}^{\pi/4} \\
&= \frac{1}{32(\sqrt{2}-1)}\left(\ln\left|\csc\left(\frac{\pi}{4}\right)-\cot\left(\frac{\pi}{4}\right)\right| - \ln\left|\csc\left(\frac{\pi}{6}\right)-\cot\left(\frac{\pi}{6}\right)\right|\right) \\
&= \boxed{\frac{1}{32(\sqrt{2}-1)}\left(\ln\left(\sqrt{2}-1\right) - \ln(2-\sqrt{(3)})\right)}.
\end{aligned}$$

3. (a) Evaluate $\int \sqrt{1+x^2}dx$.

Solution. We want $1+x^2 = 1+\tan(\theta)^2$, so we substitute $x = \tan(\theta)$, $dx = \sec(\theta)^2 d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\tan(\theta) = x$, as shown below.



Then $\sqrt{1+x^2} = \sqrt{1+\tan(\theta)^2} = \sqrt{\sec(\theta)^2} = \sec(\theta)$. The integral becomes

$$\int \sqrt{1+x^2}dx = \int \sec(\theta)\sec(\theta)^2 d\theta$$

$$= \int \sec(\theta)^3 d\theta.$$

We can evaluate $\int \sec(\theta)^3 d\theta$ with an IBP and solving for the unknown integral when it reappears on the right-hand side. For the IBP we use the parts

$$\begin{aligned} u &= \sec(\theta) \;\Rightarrow\; du = \sec(\theta)\tan(\theta)d\theta, \\ dv &= \sec(\theta)^2 d\theta \;\Rightarrow\; v = \tan(\theta). \end{aligned}$$

We get

$$\begin{aligned} \int \sec(\theta)^3 d\theta &= \int \sec(\theta)^2 \sec(\theta) d\theta \\ \int \sec(\theta)^3 d\theta &= \tan(\theta)\sec(\theta) - \int \tan(\theta)\sec(\theta)\tan(\theta)d\theta \\ \int \sec(\theta)^3 d\theta &= \tan(\theta)\sec(\theta) - \int \tan(\theta)^2 \sec(\theta)d\theta \end{aligned}$$

We will use the Pythagorean identity $\tan(\theta)^2 = \sec(\theta)^2 - 1$ to see the original integral reappear on the right-hand side.

$$\begin{aligned} \int \sec(\theta)^3 d\theta &= \tan(\theta)\sec(\theta) - \int \left(\sec(\theta)^2 - 1\right)\sec(\theta)d\theta \\ \int \sec(\theta)^3 d\theta &= \tan(\theta)\sec(\theta) - \int \sec(\theta)^3 d\theta + \int \sec(\theta)d\theta \\ \int \sec(\theta)^3 d\theta &= \tan(\theta)\sec(\theta) - \int \sec(\theta)^3 d\theta + \ln|\sec(\theta) + \tan(\theta)| \end{aligned}$$

We can now move the term $-\int \sec(\theta)^3 d\theta$ to the left hand side and finish solving

$$\begin{aligned} 2\int \sec(\theta)^3 d\theta &= \tan(\theta)\sec(\theta) + \ln|\sec(\theta) + \tan(\theta)| \\ \Rightarrow \int \sec(\theta)^3 d\theta &= \frac{1}{2}\left(\tan(\theta)\sec(\theta) + \ln|\sec(\theta) + \tan(\theta)|\right) + C. \end{aligned}$$

Using the right triangle above, we can express this result in terms of x, observing that $\tan(\theta) = x$ and $\sec(\theta) = \sqrt{x^2+1}$. We get

$$\boxed{\int \sqrt{1+x^2}dx = \frac{1}{2}\left(x\sqrt{x^2+1} + \ln\left|x + \sqrt{x^2+1}\right|\right) + C}.$$

(b) Use your result from part (a) for the following applications.

(i) Calculate the length of the curve $y = x^2$, $0 \leqslant x \leqslant 1$.

Solution. The arc length is given by

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$
\begin{aligned}
&= \int_0^1 \sqrt{1+(2x)^2}\,dx \\
&= \frac{1}{2}\int_0^2 \sqrt{1+u^2}\,du \quad (u = 2x) \\
&= \frac{1}{2}\left[\frac{1}{2}\left(u\sqrt{u^2+1} + \ln\left|u+\sqrt{u^2+1}\right|\right)\right]_0^2 \\
&= \boxed{\frac{1}{4}\left(2\sqrt{5} + \ln\left(2+\sqrt{5}\right)\right) \text{ units}}.
\end{aligned}
$$

(ii) Calculate the area of the surface obtained by revolving the curve $y = e^x$, $0 \leqslant x \leqslant \ln(2)$, about the x-axis.

Solution. The area of a surface of revolution about the x-axis is given by

$$
\begin{aligned}
L &= \int_0^{\ln(2)} 2\pi y(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\
&= \int_0^{\ln(2)} 2\pi e^x \sqrt{1+e^{2x}}\,dx \\
&= 2\pi \int_1^2 \sqrt{1+u^2}\,du \quad (u = e^x) \\
&= 2\pi \left[\frac{1}{2}\left(u\sqrt{u^2+1} + \ln\left|u+\sqrt{u^2+1}\right|\right)\right]_1^2 \\
&= \boxed{\pi\left(2\sqrt{5} + \ln(2+\sqrt{5}) - \sqrt{2} - \ln(1+\sqrt{2})\right) \text{ square units}}.
\end{aligned}
$$

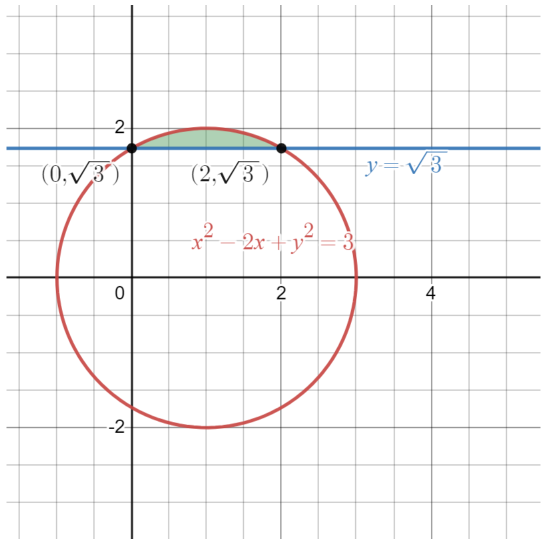
(iii) Calculate the area of the surface obtained by revolving the curve $y = \sin^{-1}(x)$, $0 \leqslant x \leqslant 1$ about the y-axis.

Solution. Note that the curve can be expressed as a function of y as $x = \sin(y)$, $0 \leqslant y \leqslant \frac{\pi}{2}$. The area of a surface of revolution about the y-axis is given by

$$
\begin{aligned}
L &= \int_0^{\pi/2} 2\pi x(y)\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \\
&= \int_0^{\pi/2} 2\pi \sin(y)\sqrt{1+\cos(y)^2}\,dy \\
&= 2\pi \int_1^0 -\sqrt{1+u^2}\,du \quad (u = \cos(y)) \\
&= 2\pi \int_0^1 \sqrt{1+u^2}\,du \\
&= 2\pi \left[\frac{1}{2}\left(u\sqrt{u^2+1} + \ln\left|u+\sqrt{u^2+1}\right|\right)\right]_0^1 \\
&= \boxed{\pi\left(\sqrt{2} + \ln(1+\sqrt{2})\right) \text{ square units}}.
\end{aligned}
$$

4. Calculate the area of the region inside the circle of equation $x^2 - 2x + y^2 = 3$ and above the line $y = \sqrt{3}$.

Solution. The region is sketched below.



Note that the upper half semi-circle can be expressed as a function of x as $y = \sqrt{3 + 2x - x^2}$. We will compute the area using vertical strips. The vertical strip at x in the region has length $\ell(x) = \sqrt{3 + 2x - x^2} - \sqrt{3}$. Therefore, the area is given by

$$A = \int_0^2 \ell(x)dx = \int_0^2 \left(\sqrt{3 + 2x - x^2} - \sqrt{3}\right) dx = \int_0^2 \sqrt{3 + 2x - x^2}dx - 2\sqrt{3}.$$

To compute the remaining integral, we start by completing the square in the square root:

$$3 + 2x - x^2 = 3 - (x^2 - 2x) = 3 - (x^2 - 2x + 1) + 1 = 4 - (x - 1)^2.$$

We can then use a trigonometric substitution. We want $4 - (x - 1)^2 = 4 - 4\sin(\theta)^2$, so we substitute $x - 1 = 2\sin(\theta)$ or $x = 1 + 2\sin(\theta)$. This gives $dx = 2\cos(\theta)d\theta$ and $\sqrt{4 - (x - 1)^2} = \sqrt{4 - 4\sin(\theta)^2} = \sqrt{4\cos(\theta)^2} = 2\cos(\theta)$. The bounds of the integral become

$$x = 0 \;\Rightarrow\; \sin(\theta) = \frac{0 - 1}{2} = -\frac{1}{2} \;\Rightarrow\; \theta = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6},$$
$$x = 2 \;\Rightarrow\; \sin(\theta) = \frac{2 - 1}{2} = \frac{1}{2} \;\Rightarrow\; \theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}.$$

Therefore

$$\begin{aligned}\int_0^2 \sqrt{3 + 2x - x^2}dx &= \int_0^2 \sqrt{4 - (x - 1)^2}dx \\ &= \int_{-\pi/6}^{\pi/6} 2\cos(\theta)2\cos(\theta)d\theta \\ &= 4\int_{-\pi/6}^{\pi/6} \cos(\theta)^2 d\theta \\ &= 8\int_0^{\pi/6} \cos(\theta)^2 d\theta.\end{aligned}$$

where we have used the fact that the integrand is even in the last step. We can now compute this integral with the double angle formula as follows:

$$\begin{aligned}\int_0^2 \sqrt{3+2x-x^2}dx &= 8\int_0^{\pi/6} \frac{1+\cos(2\theta)}{2}2d\theta \\ &= 4\left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{\pi/6} \\ &= 4\left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right) \\ &= \frac{2\pi}{3} + \sqrt{3}.\end{aligned}$$

In conclusion, the area is

$$\begin{aligned}A &= \int_0^2 \sqrt{3+2x-x^2}dx - 2\sqrt{3} \\ &= \frac{2\pi}{3} + \sqrt{3} - 2\sqrt{3} \\ &= \boxed{\frac{2\pi}{3} - \sqrt{3} \text{ square units}}.\end{aligned}$$

5. Consider the region $\mathcal{R}$ bounded between the graph of $y = \dfrac{1}{16-x^2}$ and the x-axis for $0 \leqslant x \leqslant 2$. Find the volume of the solid obtained by revolving $\mathcal{R}$ about the line $x = -3$.

Solution. We use the shell method. Revolving the vertical strip at x about the line $x = -3$ forms a cylindrical shell of radius $r(x) = x + 3$ and height $h(x) = \frac{1}{16-x^2}$. Therefore

$$\begin{aligned}V &= \int_0^2 2\pi r(x)h(x)dx \\ &= 2\pi\int_0^2 \frac{x+3}{16-x^2}dx.\end{aligned}$$

We can evaluate this integral with a trigonometric substitution. We substitute $x = 4\sin(\theta)$, so $dx = 4\cos(\theta)d\theta$ and $16 - x^2 = 16 - 16\sin(\theta)^2 = 16\cos(\theta)^2$. The bounds become

$$\begin{aligned}x = 0 \;&\Rightarrow\; \sin(\theta) = \frac{0}{4} = 0 \;\Rightarrow\; \theta = \sin^{-1}(0) = 0, \\ x = 2 \;&\Rightarrow\; \sin(\theta) = \frac{2}{4} = \frac{1}{2} \;\Rightarrow\; \theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}.\end{aligned}$$

So

$$\begin{aligned}V &= 2\pi\int_0^{\pi/6} \frac{4\sin(\theta)+3}{16\cos(\theta)^2}4\cos(\theta)d\theta \\ &= \frac{\pi}{2}\int_0^{\pi/6} \frac{4\sin(\theta)+3}{\cos(\theta)}d\theta \\ &= \frac{\pi}{2}\int_0^{\pi/6} (4\tan(\theta) + 3\sec(\theta))\, d\theta\end{aligned}$$

$$
\begin{aligned}
&= \frac{\pi}{2}\left[4\ln|\sec(\theta)| + 3\ln|\sec(\theta)+\tan(\theta)|\right]_0^{\pi/6} \\
&= \frac{\pi}{2}\left(4\ln\left(\frac{2}{\sqrt{3}} + 3\ln\left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right)\right)\right) \\
&= \boxed{\frac{\pi}{2}\left(4\ln(2) - \frac{1}{2}\ln(3)\right)\ \text{cubic units}}.
\end{aligned}
$$

Rutgers University
Math 152

Section 8.8: Improper Integrals - Worksheet Solutions

1. Calculate the following integrals or determine if they diverge.

   (a) $\displaystyle\int_0^\infty e^{-5x}dx$

   Solution.

$$\begin{aligned}\int_0^\infty e^{-5x}dx &= \lim_{b\to\infty}\int_0^b e^{-5x}dx \\ &= \lim_{b\to\infty}\left[-\frac{1}{5}e^{-5x}\right]_0^b \\ &= \lim_{b\to\infty}\left(-\frac{1}{5}e^{-5b}+\frac{1}{5}e^0\right) \\ &= \left(-\frac{1}{5}\cdot 0+\frac{1}{5}\right) \\ &= \boxed{\frac{1}{5}}\end{aligned}$$

   (b) $\displaystyle\int_0^{\pi/4}\csc(x)dx$

   Solution. This is a type II improper integral due to the vertical asymptote of $y = \csc(x)$ at $x = 0$.

$$\begin{aligned}\int_0^{\pi/4}\csc(x)dx &= \lim_{a\to 0^+}\int_0^{\pi/4}\csc(x)dx \\ &= \lim_{a\to 0^+}\left[-\ln|\csc(x)+\cot(x)|\right]_a^{\pi/4} \\ &= \lim_{a\to 0^+}\left(-\ln(\sqrt{2}+1)+\ln|\csc(a)+\cot(a)|\right) \\ &= \infty\end{aligned}$$

   since $\cot(a), \csc(a) \to \infty$ when $a \to 0^+$, so $\ln|\csc(a)+\cot(a)| \to \infty$ when $a \to 0^+$. Therefore $\boxed{\displaystyle\int_0^{\pi/4}\csc(x)dx \text{ diverges}}$.

   (c) $\displaystyle\int_{-\infty}^0 xe^{3x}dx$

   Solution. We can start by finding an antiderivative using integration by parts. We use the parts

$$\begin{aligned}u = x &\Rightarrow du = dx \\ dv = e^{3x}dx &\Rightarrow v = \frac{e^{3x}}{3}.\end{aligned}$$

We obtain

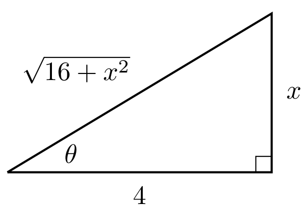
$$\begin{aligned}\int xe^{3x}dx &= \frac{xe^{3x}}{3} - \int \frac{e^{3x}}{3}dx \\ &= \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + C \\ &= \frac{(3x-1)e^{3x}}{9} + C.\end{aligned}$$

We can now compute the improper integral.

$$\begin{aligned}\int_{-\infty}^{0} xe^{3x}dx &= \lim_{a\to-\infty} \int_{a}^{0} xe^{3x}dx \\ &= \lim_{a\to-\infty} \left[\frac{(3x-1)e^{3x}}{9}\right]_a^0 \\ &= \lim_{a\to-\infty} \left(-\frac{1}{9} - \frac{(3a-1)e^{3a}}{9}\right) \\ &= -\frac{1}{9} - \lim_{a\to-\infty} \frac{(3a-1)}{9e^{-3a}} \\ &\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} -\frac{1}{9} - \lim_{a\to-\infty} \frac{3}{-27e^{-3a}} \\ &= -\frac{1}{9} - 0 \\ &= \boxed{-\frac{1}{9}}.\end{aligned}$$

(d) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(16+x^2)^{3/2}}$

Solution. We can start by finding an antiderivative of the integrand. For this, we can use the trigonometric substitution $x = 4\tan(\theta)$, $dx = 4\sec(\theta)^2 d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\tan(\theta) = \frac{x}{4}$ as shown below.



We get $(16+u^2)^{3/2} = (16+16\tan(\theta)^2)^{3/2} = (16\sec(\theta)^2)^{3/2} = 64\sec(\theta)^3$, and the integral becomes

$$\begin{aligned}\int \frac{dx}{(16+x^2)^{3/2}} &= \int \frac{4\sec(\theta)^2 d\theta}{64\sec(\theta)^3} \\ &= \frac{1}{16}\int \frac{d\theta}{\sec(\theta)} \\ &= \frac{1}{16}\int \cos(\theta)d\theta \\ &= \frac{1}{16}\sin(\theta) + C.\end{aligned}$$

In this antiderivative, we can express $\sin(\theta)$ in terms of x using the right triangle above, in which we see that $\sin(\theta) = \frac{x}{\sqrt{16+x^2}}$. Thus

$$\int \frac{dx}{(16+x^2)^{3/2}} = \frac{x}{16\sqrt{16+x^2}} + C.$$

We can now compute the improper integral. Observe that the integrand is even, so the integral on $(-\infty, \infty)$ is equal to two times the integral on $[0, \infty)$.

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx}{(16+x^2)^{3/2}} &= 2\int_0^{\infty} \frac{dx}{(16+x^2)^{3/2}} \\
&= 2 \lim_{b\to\infty} \int_0^b \frac{dx}{(16+x^2)^{3/2}} \\
&= 2 \lim_{b\to\infty} \left[\frac{x}{16\sqrt{16+x^2}}\right]_0^b \\
&= 2 \lim_{b\to\infty} \frac{b}{16\sqrt{16+b^2}} \cdot \frac{\frac{1}{b}}{\frac{1}{b}} \\
&= 2 \lim_{b\to\infty} \frac{1}{16\sqrt{\frac{16+}{b^2}1}} \\
&= 2\frac{1}{16} \\
&= \boxed{\frac{1}{8}}.
\end{aligned}$$

(e) $\displaystyle\int_0^1 \ln(x)dx$

Solution. This is a type II improper integral due to the vertical asymptote of $y = \ln(x)$ at $x = 0$. First we compute an antiderivative using integration by parts with parts

$$\begin{aligned}
u = \ln(x) \;&\Rightarrow\; du = \frac{dx}{x} \\
dv = dx \;&\Rightarrow\; v = x.
\end{aligned}$$

We obtain

$$\begin{aligned}
\int \ln(x)dx &= x\ln(x) - \int x\frac{1}{x}dx \\
&= x\ln(x) - \int dx \\
&= x\ln(x) - x + C.
\end{aligned}$$

Next we compute the improper integral.

$$\begin{aligned}
\int_0^1 \ln(x)dx &= \lim_{a\to 0^+} \int_a^1 \ln(x)dx \\
&= \lim_{a\to 0^+} [x\ln(x) - x]_a^1 \\
&= \lim_{a\to 0^+} (-1 - a\ln(a) + a) \\
&= -1 - \lim_{a\to 0^+} a\ln(a) + 0 \\
&= -1 - \lim_{a\to 0^+} \frac{\ln(a)}{\frac{1}{a}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} -1 - \lim_{a\to 0^+} \frac{\frac{1}{a}}{-\frac{1}{a^2}} \\
&= -1 - \lim_{a\to 0^+} (-a) \\
&= -1 - 0 \\
&= \boxed{-1}.
\end{aligned}$$

(f) $\displaystyle\int_{-2}^{1} \frac{dx}{\sqrt[3]{3x-2}}$

Solution. This is a type II improper integral due to the vertical asymptote of $y = \frac{1}{\sqrt[3]{3x-2}}$ at $x = \frac{2}{3}$. Because the vertical asymptote is in the interior of the interval of integration, we need to break-up the integral into a sum of two integrals and compute each of them as a limit. We get

$$\begin{aligned}
\int_{-2}^{1} \frac{dx}{\sqrt[3]{3x-2}} &= \int_{-2}^{2/3} \frac{dx}{\sqrt[3]{3x-2}} + \int_{2/3}^{1} \frac{dx}{\sqrt[3]{3x-2}} \\
&= \lim_{b\to \frac{2}{3}^-} \int_{-2}^{b} \frac{dx}{\sqrt[3]{3x-2}} + \lim_{a\to \frac{2}{3}^+} \int_{a}^{1} \frac{dx}{\sqrt[3]{3x-2}} \\
&= \lim_{b\to \frac{2}{3}^-} \left[\frac{(3x-2)^{2/3}}{2}\right]_{-2}^{b} + \lim_{a\to \frac{2}{3}^+} \left[\frac{(3x-2)^{2/3}}{2}\right]_{a}^{1} \\
&= \lim_{b\to \frac{2}{3}^-} \left(\frac{(3b-2)^{2/3}}{2} - 2\right) + \lim_{a\to \frac{2}{3}^+} \left(1 - \frac{(3a-2)^{2/3}}{2}\right) \\
&= (0-2) + (1-0) \\
&= \boxed{-1}.
\end{aligned}$$

(g) $\displaystyle\int_{0}^{3/2} \frac{dx}{\sqrt{9-4x^2}}$

Solution. This is a type II improper integral due to the vertical asymptote of $y = \frac{1}{\sqrt{9-4x^2}}$ at $x = \frac{3}{2}$.

$$\begin{aligned}\int_0^{3/2} \frac{dx}{\sqrt{9-4x^2}} &= \lim_{b\to\frac{3}{2}^-} \int_0^b \frac{dx}{\sqrt{9-4x^2}} \\ &= \lim_{b\to\frac{3}{2}^-} \left[\frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right)\right]_0^b \\ &= \lim_{b\to\frac{3}{2}^-} \left(\frac{1}{2}\sin^{-1}\left(\frac{2b}{3}\right) - 0\right) \\ &= \frac{1}{2}\sin^{-1}(1) \\ &= \boxed{\frac{\pi}{4}}.\end{aligned}$$

(h) $\displaystyle\int_e^\infty \frac{dx}{x\ln(x)}$

Solution. We use the substitution $u = \ln(x)$, $du = \frac{dx}{x}$ to compute the antiderivative.

$$\begin{aligned}\int \frac{dx}{x\ln(x)} &= \int \frac{du}{u} \\ &= \ln|u| + C \\ &= \ln|\ln(x)| + C.\end{aligned}$$

We can now use this antiderivative to compute the improper integral.

$$\begin{aligned}\int_e^\infty \frac{dx}{x\ln(x)} &= \lim_{b\to\infty} \int_e^b \frac{dx}{x\ln(x)} \\ &= \lim_{b\to\infty} [\ln|\ln(x)|]_e^b \\ &= \lim_{b\to\infty} (\ln|\ln(b)| - \ln|\ln(e)|) \\ &= \infty\end{aligned}$$

since $\ln(b) \to \infty$ when $b \to \infty$. Therefore $\boxed{\displaystyle\int_e^\infty \frac{dx}{x\ln(x)} \text{ diverges}}$.

(i) $\displaystyle\int_0^\infty e^{-x}\sin(x)dx$

Solution. We start by computing an antiderivative, using two successive integration by parts. For the first IBP, the parts are

$$\begin{aligned}u &= \sin(x) \;\Rightarrow\; du = \cos(x)dx, \\ dv &= e^{-x}dx \;\Rightarrow\; v = -e^{-x}.\end{aligned}$$

This gives

$$\begin{aligned}\int e^{-x}\sin(x)dx &= -e^{-x}\sin(x) - \int -e^{-x}\cos(x)dx \\ &= -e^{-x}\sin(x) + \int e^{-x}\cos(x)dx.\end{aligned}$$

The second IBP uses the parts

$$u = \cos(x) \;\Rightarrow\; du = -\sin(x)dx,$$
$$dv = e^{-x}dx \;\Rightarrow\; v = -e^{-x}.$$

We get

$$\int e^{-x}\sin(x)dx = -e^{-x}\sin(x) - e^{-x}\cos(x) - \int (-e^{-x})(-\sin(x))dx \quad = -e^{-x}\sin(x) - e^{-x}\cos(x) - \int e^{-x}\sin(x)dx.$$

We can solve this relation for the unknown integral by moving the term $-\int e^{-x}\sin(x)dx$ to the left-hand side and we get

$$2\int e^{-x}\sin(x)dx = -e^{-x}\sin(x) - e^{-x}\cos(x)$$
$$\Rightarrow \int e^{-x}\sin(x)dx = -\frac{e^{-x}(\sin(x)+\cos(x))}{2} + C$$

We can now compute the improper integral.

$$\begin{aligned}\int_0^\infty e^{-x}\sin(x)dx &= \lim_{b\to\infty}\int_0^b e^{-x}\sin(x)dx \\ &= \lim_{b\to\infty}\left[-\frac{e^{-x}(\sin(x)+\cos(x))}{2}\right]_0^b \\ &= \lim_{b\to\infty}\left(-\frac{e^{-b}(\sin(b)+\cos(b))}{2} + \frac{1}{2}\right) \\ &= \frac{1}{2} - \lim_{b\to\infty}\frac{\sin(b)+\cos(b)}{e^b}.\end{aligned}$$

This last limit can be computed using the Sandwich Theorem. We have the inequalities

$$-2 \leqslant \sin(b)+\cos(b) \leqslant 2$$
$$\Rightarrow \quad -\frac{2}{e^b} \leqslant \frac{\sin(b)+\cos(b)}{e^b} \leqslant \frac{2}{e^b}$$

Since $\lim_{b\to\infty}\frac{2}{e^b} = 0 = \lim_{b\to\infty} -\frac{2}{e^b}$, it follows that $\lim_{b\to\infty}\frac{\sin(b)+\cos(b)}{e^b} = 0$. Hence

$$\boxed{\int_0^\infty e^{-x}\sin(x)dx = \frac{1}{2}}.$$

2. Use a convergence test to determine if the following improper integrals converge or diverge.

(a) $\displaystyle\int_3^\infty \frac{dx}{xe^x}$

Solution. We use the DCT. Observe that for x in $[3,\infty)$, we have

$$\begin{aligned} &0 \leqslant e^x \leqslant xe^x \\ \Rightarrow\quad &0 \leqslant \frac{1}{xe^x} \leqslant \frac{1}{e^x}. \end{aligned}$$

Furthermore, $\int_3^\infty \frac{dx}{e^x}$ converges since

$$\begin{aligned} \int_3^\infty \frac{dx}{e^x} &= \lim_{b\to\infty} \int_3^b e^{-x}dx \\ &= \lim_{b\to\infty} \left(-e^{-b} + e^3\right) \\ &= e^3. \end{aligned}$$

Thus, $\boxed{\int_3^\infty \frac{dx}{xe^x} \text{ converges}}$ as well.

Remark: the inequality

$$0 \leqslant \frac{1}{xe^x} \leqslant \frac{1}{x}$$

is also true, but it does not help establish the convergence of $\int_3^\infty \frac{dx}{xe^x}$ since the integral of the "bigger function" $\int_3^\infty \frac{dx}{x}$ diverges (type I p-integral with $p = 1$).

(b) $\int_1^\infty \frac{dx}{x^2+3x+1}$

Solution. We use the DCT. Observe that for x in $[1,\infty)$ we have the inequalities

$$\begin{aligned} &0 \leqslant x^2 \leqslant x^2+3x+1 \\ \Rightarrow\quad &0 \leqslant \frac{1}{x^2+3x+1} \leqslant \frac{1}{x^2}. \end{aligned}$$

Furthermore, the integral $\int_1^\infty \frac{dx}{x^2}$ converges since it is a type I p-integral with $p = 2 > 1$. Therefore, $\boxed{\int_1^\infty \frac{dx}{x^2+3x+1} \text{ converges}}$ as well.

Remark. We can also use the LCT, observing that

$$\lim_{x\to\infty} \frac{\frac{1}{x^2+3x+1}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{1+3/x+1/x^2} = 1 > 0$$

and the integral $\int_1^\infty \frac{dx}{x^2}$ converges since it is a type I p-integral with $p = 2 > 1$.

(c) $\int_4^\infty \frac{\cos(x)+5}{x^{3/5}}dx$

Solution. We use the DCT, observing that for x in $[4, \infty)$ we have

$$\begin{aligned} & -1 \leqslant \cos(x) \\ \Rightarrow \quad & 0 \leqslant 4 \leqslant \cos(x) + 5 \\ \Rightarrow \quad & 0 \leqslant \frac{4}{x^{3/5}} \leqslant \frac{\cos(x)+5}{x^{3/5}} \end{aligned}$$

Furthermore, the integral $\int_4^\infty \frac{4}{x^{3/5}} dx$ diverges since it is a type I p-integral with $p = \frac{3}{5} \leqslant 1$. It follows that $\boxed{\int_4^\infty \frac{\cos(x)+5}{x^{3/5}} dx \text{ diverges}}$ as well.

Remark: we would not be able to use the LCT to compare with the divergent p-integral integral $\int_4^\infty \frac{dx}{x^{3/5}}$ since

$$\lim_{x\to\infty} \frac{\frac{\cos(x)+5}{x^{3/5}}}{\frac{1}{x^{3/5}}} = \lim_{x\to\infty} (\cos(x) + 5) \quad \text{does not exist.}$$

(d) $\int_0^1 \frac{dx}{\sqrt{x} + x^2}$

Solution. We use the DCT, observing that for x in $(0, 1]$ we have

$$\begin{aligned} & 0 \leqslant \sqrt{x} \leqslant \sqrt{x} + x^2 \\ \Rightarrow \quad & 0 \leqslant \frac{1}{x^2 + \sqrt{x}} \leqslant \frac{1}{\sqrt{x}} \end{aligned}$$

Furthermore, the integral $\int_0^1 \frac{dx}{\sqrt{x}}$ converges since it is a type II p-integral with $p = \frac{1}{2} < 1$. It follows that $\boxed{\int_0^1 \frac{dx}{x^2 + \sqrt{x}} \text{ converges}}$ as well.

Remark 1. The inequality

$$0 \leqslant \frac{1}{x^2 + \sqrt{x}} \leqslant \frac{1}{x^2}$$

is also true, but it does not help establish the convergence of $\int_0^1 \frac{dx}{x^2 + \sqrt{x}}$ since the integral of the "bigger function" $\int_0^1 \frac{dx}{x^2}$ diverges (type II p-integral with $p = 2 \geqslant 1$).

Remark 2. We could have also used the LCT to compare with the convergent type II p-integral

$\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$, remarking that

$$\begin{aligned}\lim_{x\to 0^+} \frac{\frac{1}{x^2+\sqrt{x}}}{\frac{1}{\sqrt{x}}} &= \lim_{x\to 0^+} \frac{\sqrt{x}}{x^2+\sqrt{x}} \\ &= \lim_{x\to 0^+} \frac{\sqrt{x}}{x^2+\sqrt{x}} \cdot \frac{\frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x}}} \\ &= \lim_{x\to 0^+} \frac{1}{x^{3/2}+1} \\ &= 1 > 0.\end{aligned}$$

(e) $\displaystyle\int_5^\infty \frac{xdx}{x^4-1}$

*Solution.*We use the LCT, comparing with $\dfrac{1}{x^3}$. We have

$$\begin{aligned}\lim_{x\to\infty} \frac{\frac{x}{x^4-1}}{\frac{1}{x^3}} &= \lim_{x\to\infty} \frac{x^4}{x^4-1} \\ &= \lim_{x\to\infty} \frac{x^4}{x^4-1} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}} \\ &= \lim_{x\to\infty} \frac{1}{1-\frac{1}{x^4}} \\ &= 1 > 0.\end{aligned}$$

Furthermore, the integral $\displaystyle\int_5^\infty \frac{dx}{x^3}$ converges since it is a type I p-integral with $p = 3 > 1$. Hence, $\boxed{\displaystyle\int_5^\infty \frac{xdx}{x^4-1} \text{ converges}}$ as well.

Remark. The DCT cannot be used to compare with the convergent type *I* p-integral $\displaystyle\int_5^\infty \frac{dx}{x^3}$ since we have the inequalities

$$\begin{aligned} & 0 \leqslant x^4 - 1 \leqslant x^4 \\ \Rightarrow\ & 0 \leqslant \frac{1}{x^4} \leqslant \frac{1}{x^4-1} \\ \Rightarrow\ & 0 \leqslant \frac{1}{x^3} \leqslant \frac{x}{x^4-1}\end{aligned}$$

and knowing that the integral of the "smaller function" converges does not say anything about the integral of the "bigger function".

(f) $\displaystyle\int_1^\infty \frac{x^3+5x^2+1}{\sqrt{x^7+4x+2}}dx$

Solution. We use the LCT. To find a good function to compare to, we keep the terms of the numerator and denominator that are dominant when $x \to \infty$:

$$\frac{x^3+5x^2+1}{\sqrt{x^7+4x+2}} \sim \frac{x^3}{\sqrt{x^7}} = \frac{x^3}{x^{7/2}} = \frac{1}{x^{1/2}}.$$

Now that we have found our reference function, we properly establish the limit comparison.

$$\begin{aligned}
\lim_{x\to\infty} \frac{\frac{x^3+5x^2+1}{\sqrt{x^7+4x+2}}}{\frac{1}{x^{1/2}}} &= \lim_{x\to\infty} \frac{x^{7/2}+5x^{5/2}+x^{1/2}}{\sqrt{x^7+4x+2}} \\
&= \lim_{x\to\infty} \frac{x^{7/2}+5x^{5/2}+x^{1/2}}{\sqrt{x^7+4x+2}} \cdot \frac{\frac{1}{x^{7/2}}}{\frac{1}{x^{7/2}}} \\
&= \lim_{x\to\infty} \frac{1+\frac{5}{x^2}+\frac{1}{x^3}}{\sqrt{1+\frac{4}{x^6}+\frac{2}{x^7}}} \\
&= 1 > 0.
\end{aligned}$$

We also know that the integral $\int_1^\infty \frac{dx}{x^{1/2}}$ diverges since it is a type I p-integral with $p = \frac{1}{2} \leqslant 1$. Therefore, $\boxed{\int_1^\infty \frac{x^3+5x^2+1}{\sqrt{x^7+4x+2}}dx \text{ diverges}}$ as well.

3. Consider the unbounded region $\mathcal{R}$ between the graph of $y = \frac{\ln(x)}{x}$ and the x-axis for $x \geqslant 1$.

(a) Find the area of the region $\mathcal{R}$ or determine if $\mathcal{R}$ has infinite area.

Solution. The area of $\mathcal{R}$ is given by

$$A = \int_1^\infty \frac{\ln(x)}{x}dx.$$

The antiderivative of the integrand can be found with the substitution $u = \ln(x)$, $du = \frac{dx}{x}$, which gives

$$\int \frac{\ln(x)}{x}dx = \int u du = \frac{u^2}{2} + C = \frac{\ln(x)^2}{2} + C.$$

We can use this to compute the area, as follows

$$\begin{aligned}
A &= \int_1^\infty \frac{\ln(x)}{x}dx \\
&= \lim_{b\to\infty} \int_1^b \frac{\ln(x)}{x}dx \\
&= \lim_{b\to\infty} \left[\frac{\ln(x)^2}{2}\right]_1^b \\
&= \lim_{b\to\infty} \frac{\ln(b)^2}{2} \\
&= \infty.
\end{aligned}$$

So $\boxed{\mathcal{R} \text{ has infinite area}}$.

(b) We now revolve the region $\mathcal{R}$ about the x-axis to form a solid of revolution. Calculate the volume of the solid or determine if the solid has infinite volume.

Solution. Revolving the vertical strip at x in the region about the x-axis forms a disk of radius $r(x) = \frac{\ln(x)}{x}$. So the volume is given by

$$V = \int_1^\infty \pi r(x)^2 dx = \pi \int_1^\infty \frac{\ln(x)^2}{x^2} dx.$$

To compute the antiderivative of the integrand, we use two successive IPBs. The first one will use the parts

$$\begin{aligned} u = \ln(x)^2 \ &\Rightarrow \ du = \frac{2\ln(x)dx}{x}, \\ dv = \frac{dx}{x^2} \ &\Rightarrow \ v = -\frac{1}{x}. \end{aligned}$$

This gives

$$\begin{aligned} \int \frac{\ln(x)^2}{x^2} dx &= -\frac{\ln(x)^2}{x} - \int \frac{2\ln(x)}{x}\left(-\frac{1}{x}\right) dx \\ &= -\frac{\ln(x)^2}{x} + 2\int \frac{\ln(x)}{x^2} dx. \end{aligned}$$

For the second IBP, we take

$$\begin{aligned} u = \ln(x) \ &\Rightarrow \ du = \frac{dx}{x}, \\ dv = \frac{dx}{x^2} \ &\Rightarrow \ v = -\frac{1}{x}. \end{aligned}$$

We obtain

$$\begin{aligned} \int \frac{\ln(x)^2}{x^2} dx &= -\frac{\ln(x)^2}{x} + 2\left(-\frac{\ln(x)}{x} - \int \frac{1}{x}\left(-\frac{1}{x}\right) dx\right) \\ &= -\frac{\ln(x)^2}{x} - 2\frac{\ln(x)}{x} + 2\int \frac{1}{x^2} dx \\ &= -\frac{\ln(x)^2}{x} - 2\frac{\ln(x)}{x} - \frac{2}{x} + C \\ &= -\frac{\ln(x)^2 + 2\ln(x) + 2}{x} + C. \end{aligned}$$

We can now use this to compute the volume.

$$\begin{aligned} V &= \pi \int_1^\infty \frac{\ln(x)^2}{x^2} dx \\ &= \pi \lim_{b\to\infty} \int_1^b \frac{\ln(x)^2}{x^2} dx \\ &= \pi \lim_{b\to\infty} \left[-\frac{\ln(x)^2 + 2\ln(x) + 2}{x}\right]_1^b \\ &= \pi \lim_{b\to\infty} \left(2 - \frac{\ln(b)^2 + 2\ln(b) + 2}{b}\right) \\ &= \pi \left(2 - \lim_{b\to\infty} \frac{\ln(b)^2 + 2\ln(b) + 2}{b}\right). \end{aligned}$$

To compute the remaining limit, we use L'Hôpital's Rule twice for the indeterminate form $\frac{\infty}{\infty}$.

$$\begin{aligned}
V &\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \pi\left(2 - \lim_{b\to\infty} \frac{2\frac{\ln(b)}{b} + \frac{2}{b}}{1}\right) \\
&= \pi\left(2 - \lim_{b\to\infty} \frac{2\ln(b) + 2}{b}\right) \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \pi\left(2 - \lim_{b\to\infty} \frac{\frac{2}{b}}{1}\right) \\
&= \pi(2 - 0) \\
&= \boxed{2\pi \text{ cubic units}}.
\end{aligned}$$

(c) We now revolve the region $\mathcal{R}$ about the y-axis to form a solid of revolution. Calculate the volume of the solid or determine if the solid has infinite volume.

Solution. Revolving the vertical strip at x about the y-axis forms a shell with radius $r(x) = x$ and height $h(x) = \frac{\ln(x)}{x}$. So the volume is

$$V = \int_1^\infty 2\pi r(x)h(x)dx = \int_1^\infty 2\pi x \frac{\ln(x)}{x} dx = 2\pi \int_1^\infty \ln(x)dx.$$

We have previously computed the antiderivative of the integrand using integration by parts and found that

$$\int \ln(x)dx = x(\ln(x) - 1) + C.$$

So the volume is

$$\begin{aligned}
V &= 2\pi \lim_{b\to\infty} \int_1^b \ln(x)dx \\
&= 2\pi \lim_{b\to\infty} \left[x(\ln(x) - 1)\right]_1^b \\
&= 2\pi \lim_{b\to\infty} b(\ln(b) - 1) \\
&= \infty,
\end{aligned}$$

so $\boxed{\text{the solid has infinite volume}}$.

Rutgers University
Math 152

Section 10.1: Sequences - Worksheet Solutions

1. Determine if the sequences below converge or diverge. In case of convergence, find the limit.

   (a) $a_n = \dfrac{\sqrt{1+16n^4}}{n^2+1}$

   Solution. We can compute the limit by dividing numerator and denominator by n^2 as follows:

$$\begin{aligned}\lim_{n\to\infty}\frac{\sqrt{1+16n^4}}{n^2+1} &= \lim_{n\to\infty}\frac{\sqrt{1+16n^4}}{n^2+1}\cdot\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\\ &= \lim_{n\to\infty}\frac{\sqrt{\frac{1}{n^4}+16}}{1+\frac{1}{n^2}}\\ &= \frac{\sqrt{0+16}}{1+0}\\ &= 4.\end{aligned}$$

   Since the limit exists and is finite, we conclude that $\boxed{\text{the sequence } \{a_n\} \text{ converges to the limit 4}}$.

   (b) $a_n = \dfrac{5n+4}{2\cos(n)^2+3n}$

   Solution. We can start the limit computation by dividing numerator and denominator by n as follows:

$$\begin{aligned}\lim_{n\to\infty}\frac{5n+4}{2\cos(n)^2+3n} &= \lim_{n\to\infty}\frac{5n+4}{2\cos(n)^2+3n}\cdot\frac{\frac{1}{n}}{\frac{1}{n}}\\ &= \lim_{n\to\infty}\frac{5+\frac{4}{n}}{\frac{2\cos(n)^2}{n}+3}.\end{aligned}$$

   Now we have

$$\begin{aligned}0 \leqslant 2\cos(n)^2 \leqslant 2\\ \Rightarrow\ 0 \leqslant \frac{2\cos(n)^2}{n} \leqslant \frac{2}{n}\end{aligned}$$

   and $\lim\limits_{n\to\infty}\frac{2}{n}=0$. Therefore, $\lim\limits_{n\to\infty}\frac{2\cos(n)^2}{n}=0$ by the Sandwich Theorem. It follows that

$$\begin{aligned}\lim_{n\to\infty}\frac{5n+4}{2\cos(n)^2+3n} &= \lim_{n\to\infty}\frac{5+\frac{4}{n}}{\frac{2\cos(n)^2}{n}+3}\\ &= \frac{5+0}{0+3}\\ &= \frac{5}{3}.\end{aligned}$$

Since the limit exists and is finite, we conclude that $\boxed{\text{the sequence } \{a_n\} \text{ converges to the limit } \frac{5}{3}}$.

(c) $a_n = \tan^{-1}(1-\sqrt{n})$

Solution. We have $\lim\limits_{n\to\infty} 1-\sqrt{n} = -\infty$. Therefore,

$$\lim_{n\to\infty} \tan^{-1}\left(1-\sqrt{n}\right) = \text{“}\tan^{-1}(-\infty)\text{”} = -\frac{\pi}{2}.$$

Since the limit exists and is finite, we conclude that $\boxed{\text{the sequence } \{a_n\} \text{ converges to the limit } -\frac{\pi}{2}}$.

(d) $a_n = \dfrac{n+(-1)^n}{n^3+1}$

Solution. If $n \geqslant 1$, we have

$$0 \leqslant n+(-1)^n \leqslant n+1 \text{ and } n^3+1 \geqslant n^3 > 0$$
$$\Rightarrow\ 0 \leqslant \frac{n+(-1)^n}{n^3+1} \leqslant \frac{n+1}{n^3} = \frac{1}{n^2} + \frac{1}{n^3}$$

and $\lim\limits_{n\to\infty} \frac{1}{n^2} + \frac{1}{n^3} = 0$. Therefore by the Sandwich Theorem we have

$$\lim_{n\to\infty} \frac{n+(-1)^n}{n^3+1} = 0.$$

So $\boxed{\text{the sequence } \{a_n\} \text{ converges to the limit } 0}$.

(e) $a_n = \dfrac{4^n - 5^{2n}}{7^n}$

Solution. We have

$$\frac{4^n - 5^{2n}}{7^n} = \left(\frac{4}{7}\right)^n - \left(\frac{25}{7}\right)^n$$

Since a geometric sequence converges to 0 if its common ratio is in $(-1,1)$ and diverges to ∞ if its common ratio is > 1, we have

$$\lim_{n\to\infty} \frac{4^n - 5^{2n}}{7^n} = \text{“}0-\infty\text{”} = -\infty,$$

so the sequence $\boxed{\text{diverges}}$.

(f) $a_n = \cos\left(\dfrac{5}{\sqrt{n}}\right)^n$

Solution. When computing the limit of this sequence, we have an indeterminate power 1^∞. We can resolve the indeterminate form by writing the power in exponential form

$$\lim_{n\to\infty} \cos\left(\frac{5}{\sqrt{n}}\right)^n = \lim_{n\to\infty} e^{n\ln\left(\cos\left(\frac{5}{\sqrt{n}}\right)\right)}.$$

We now compute the limit of the exponent using L'Hôpital's Rule twice:

$$
\begin{aligned}
\lim_{n\to\infty} n \ln\left(\cos\left(\frac{5}{\sqrt{n}}\right)\right) &= \lim_{x\to\infty} \frac{\ln\left(\cos\left(\frac{5}{\sqrt{x}}\right)\right)}{\frac{1}{x}} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty} \frac{\frac{5}{2x^{3/2}} \tan\left(\frac{5}{\sqrt{x}}\right)}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} -\frac{5\tan\left(\frac{5}{\sqrt{x}}\right)}{\frac{2}{\sqrt{x}}} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty} -\frac{-\frac{25}{2x^{3/2}} \sec\left(\frac{5}{\sqrt{x}}\right)^2}{-\frac{1}{x^{3/2}}} \\
&= \lim_{x\to\infty} -\frac{25}{2} \sec\left(\frac{5}{\sqrt{x}}\right)^2 \\
&= -\frac{25}{2} \sec(0)^2 \\
&= -\frac{25}{2}.
\end{aligned}
$$

So

$$\lim_{n\to\infty} e^{n\ln\left(\cos\left(\frac{5}{\sqrt{n}}\right)\right)} = e^{-25/2},$$

and $\boxed{\text{the sequence } \{a_n\} \text{ converges to the limit } e^{-25/2}}$.

(g) $a_n = \left(\dfrac{n+5}{n+3}\right)^{4n}$

Solution. When computing the limit of this sequence, we have an indeterminate power 1^∞. We can resolve the indeterminate form by writing the power in exponential form

$$\lim_{n\to\infty} \left(\frac{n+5}{n+3}\right)^{4n} = \lim_{n\to\infty} e^{4n\ln\left(\frac{n+5}{n+3}\right)}.$$

We now compute the limit of the exponent using L'Hôpital's Rule:

$$
\begin{aligned}
\lim_{n\to\infty} 4n\ln\left(\frac{n+5}{n+3}\right) &= \lim_{x\to\infty} 4\frac{\ln(x+5)-\ln(x+3)}{\frac{1}{x}} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} 4\frac{\frac{1}{x+5} - \frac{1}{x+3}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} -4x^2 \frac{(x+3)-(x+5)}{(x+5)(x+3)} \\
&= \lim_{x\to\infty} \frac{8x^2}{(x+5)(x+3)} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{8}{(1+5/x)(1+3/x)} \\
&= 8.
\end{aligned}
$$

So

$$\lim_{n\to\infty} e^{4n\ln\left(\frac{n+5}{n+3}\right)} = e^8,$$

and $\boxed{\text{the sequence } \{a_n\} \text{ converges to the limit } e^8}$.

(h) $a_n = (2n+1)^{3/n}$.
Solution. This time, the indeterminate power has the form ∞^0. The method stays the same and we write the power in exponential form to obtain

$$\lim_{n\to\infty} (2n+1)^{3/n} = \lim_{n\to\infty} e^{3\ln(2n+1)/n}.$$

We can compute the limit of the exponent using L'Hôpital's Rule as follows:

$$\begin{aligned}\lim_{n\to\infty} \frac{3\ln(2n+1)}{n} &= \lim_{x\to\infty} \frac{3\ln(2x+1)}{x} \\ &\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{3\frac{2}{2x+1}}{1} \\ &= \lim_{x\to\infty} \frac{6}{2x+1} \\ &= 0.\end{aligned}$$

So

$$\lim_{n\to\infty} e^{3\ln(2n+1)/n} = e^0 = 1,$$

and $\boxed{\text{the sequence } \{a_n\} \text{ converges to the limit } 1}$.

(i) $a_n = \sin(n\pi)e^n$

Solution. Observe that $\sin(n\pi) = 0$ if n is an integer. Therefore $a_n = 0$ for all n, and thus

$$\lim_{n\to\infty} \sin(n\pi)e^n = 0.$$

So $\boxed{\text{the sequence } \{a_n\} \text{ converges to the limit } 0}$.

2. Suppose that a_n is a sequence defined inductively by

$$\begin{cases} a_1 = 2, \\ a_{n+1} = \dfrac{5}{a_n+4} \quad \text{for } n \geqslant 1. \end{cases}$$

(a) Find the first 4 terms of the sequence $\{a_n\}$.

Solution. We have

$$\begin{aligned} a_1 &= \boxed{2}, \\ a_2 &= \frac{5}{a_1+4} = \frac{5}{2+4} = \boxed{\frac{5}{6}}, \\ a_3 &= \frac{5}{a_2+4} = \frac{5}{\frac{5}{6}+4} = \boxed{\frac{35}{29}}, \\ a_4 &= \frac{5}{a_3+4} = \frac{5}{\frac{35}{29}+4} = \boxed{\frac{145}{151}}. \end{aligned}$$

(b) The sequence $\{a_n\}$ converges. Find its limit.

Solution. Call L the limit of the sequence. Taking $n \to \infty$ in the recursion formula gives

$$a_{n+1} = \frac{5}{a_n + 4} \;\Rightarrow\; L = \frac{5}{L+4}.$$

Solving this for L gives the solutions $L = -5, 1$. Since the terms of the sequence are positive, the only possibility is $L = 1$. Hence,

$$\boxed{\lim_{n\to\infty} a_n = 1}.$$

Rutgers University
Math 152

Section 10.2: Infinite Series - Worksheet Solutions

1. Each of the series $\sum_{n=n_0}^{\infty} a_n$ below is either geometric or telescoping. For each series, find a formula for the partial sum $S_N = \sum_{n=n_0}^{N} a_n$, then determine if the series converges or diverges, and compute its sum if it does.

(a) $\sum_{n=4}^{\infty} 2^n 3^{-n}$

Solution. This is a geometric series of common ratio $r = \frac{2}{3}$. The partial sum is

$$S_N = \frac{(\text{first term}) - (\text{term after last})}{1 - (\text{common ratio})} = \frac{\left(\frac{2}{3}\right)^4 - \left(\frac{2}{3}\right)^{N+1}}{1 - \frac{2}{3}} = \boxed{\frac{16}{27} - \frac{2^{N+1}}{3^N}}.$$

Since $|r| < 1$, $\boxed{\text{the series converges}}$. We can compute the sum two ways: either taking the limit of S_N when $N \to \infty$ or using the formula for the sum of a convergent geometric series. Either way, we get

$$\boxed{\sum_{n=4}^{\infty} 2^n 3^{-n} = \frac{16}{27}}.$$

(b) $\sum_{n=0}^{\infty} \left(\frac{4}{2n+1} - \frac{4}{2n+5}\right)$

Solution. This is a telescoping series. We have

$$\begin{aligned} S_N &= \left(\frac{4}{1} - \frac{4}{5}\right) + \left(\frac{4}{3} - \frac{4}{7}\right) + \left(\frac{4}{5} - \frac{4}{9}\right) + \cdots + \left(\frac{4}{2N-1} - \frac{4}{2N+3}\right) + \left(\frac{4}{2N+1} - \frac{4}{2N+5}\right) \\ &= \frac{4}{1} + \frac{4}{3} - \frac{4}{2N+3} - \frac{4}{2N+5} \\ &= \boxed{\frac{16}{3} - \frac{4}{2N+3} - \frac{4}{2N+5}}. \end{aligned}$$

So

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(\frac{16}{3} - \frac{4}{2N+3} - \frac{4}{2N+5}\right) = \frac{16}{3} - 0 - 0 = \frac{16}{3}.$$

Therefore, $\boxed{\text{the series converges}}$ and

$$\boxed{\sum_{n=0}^{\infty} \left(\frac{4}{2n+1} - \frac{4}{2n+5}\right) = \frac{16}{3}}.$$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{1-3\cdot 4^{2n}}{5^{n-1}}$

Solution. We have

$$\frac{1-3\cdot 4^{2n}}{5^{n-1}} = 5\left(\frac{1}{5}\right)^n - 15\left(\frac{16}{5}\right)^n.$$

Thus, using the formula for geometric sums, we get

$$\begin{aligned} S_N &= 5\sum_{n=0}^{N}\left(\frac{1}{5}\right)^n - 15\sum_{n=0}^{N}\left(\frac{16}{5}\right)^n \\ &= 5\frac{1-\frac{1}{5^{N+1}}}{1-\frac{1}{5}} - 15\frac{1-\left(\frac{16}{5}\right)^{N+1}}{1-\frac{16}{5}} \\ &= \boxed{\frac{25}{4}\left(1-\frac{1}{5^{N+1}}\right) + \frac{75}{11}\left(1-\left(\frac{16}{5}\right)^{N+1}\right)}. \end{aligned}$$

We have $\frac{1}{5^{N+1}} \to 0$ and $\left(\frac{16}{5}\right)^{N+1} \to \infty$ when $N \to \infty$. Therefore $S_N \to -\infty$ as $N \to \infty$, so

$$\boxed{\sum_{n=0}^{\infty} \frac{1-3\cdot 4^{2n}}{5^{n-1}} \text{ diverges}}.$$

(d) $\displaystyle\sum_{n=3}^{\infty} \ln\left(\frac{3n+1}{3n+4}\right)$

Solution. After rewriting the general term as

$$\ln\left(\frac{3n+1}{3n+4}\right) = \ln(3n+1) - \ln(3n+4)$$

we see that this a telescoping series. We have

$$\begin{aligned} S_N &= \sum_{n=3}^{N} (\ln(3n+1) - \ln(3n+4)) \\ &= (\ln(10)-\ln(13)) + (\ln(13)-\ln(16)) + \cdots + (\ln(3N-2)-\ln(3N+1)) + (\ln(3N+1)-\ln(3N+4)) \\ &= \boxed{\ln(10)-\ln(3N+4)}. \end{aligned}$$

Since $\ln(3N+4) \to \infty$ when $N \to \infty$, we deduce that $S_N \to -\infty$ when $N \to \infty$. Thus

$$\boxed{\sum_{n=3}^{\infty} \ln\left(\frac{3n+1}{3n+4}\right) \text{ diverges}}.$$

(e) $\displaystyle\sum_{n=1}^{\infty} 5\cdot 3^{1-2n}$

Solution. We can rewrite the general term as

$$5 \cdot 3^{1-2n} = \frac{15}{9^n}.$$

So this is a geometric with common ratio $r = \frac{1}{9}$. The partial sum is

$$S_N = \frac{\frac{15}{81} - \frac{15}{9^{N+1}}}{1 - \frac{1}{9}} = \boxed{\frac{5}{24}\left(1 - \frac{1}{9^N}\right)}.$$

Since $|r| < 1$, $\boxed{\text{the series converges}}$. We can compute the sum two ways: either taking the limit of S_N when $N \to \infty$ or using the formula for the sum of a convergent geometric series. Either way, we get

$$\boxed{\sum_{n=1}^{\infty} 5 \cdot 3^{1-2n} = \frac{5}{24}}.$$

(f) $\displaystyle\sum_{n=1}^{\infty} \left(\tan^{-1}(n+1) - \tan^{-1}(n)\right)$

Solution. This is a telescoping series. We have

$$\begin{aligned} S_N &= (\tan^{-1}(2) - \tan^{-1}(1)) + (\tan^{-1}(3) - \tan^{-1}(2)) + \cdots + (\tan^{-1}(N+1) - \tan^{-1}(N)) \\ &= -\tan^{-1}(1) + \tan^{-1}(N+1) \\ &= -\frac{\pi}{4} + \tan^{-1}(N+1) \end{aligned}$$

Since $\lim_{N\to\infty} \tan^{-1}(N+1) = \frac{\pi}{2}$, we have

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(-\frac{\pi}{4} + \tan^{-1}(N+1)\right) = -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4}.$$

So $\boxed{\text{the series converges}}$ and

$$\boxed{\sum_{n=1}^{\infty} \left(\tan^{-1}(n+1) - \tan^{-1}(n)\right) = \frac{\pi}{4}}.$$

2. Use geometric series to express the repeating decimals below as a fraction of two integers.

(a) $1.5222\cdots = 1.5\overline{2}$

Solution. We have

$$\begin{aligned} 1.5\overline{2} &= 1.5 + 0.02 + 0.002 + \cdots \\ &= \frac{15}{10} + \frac{2}{100} + \frac{2}{1000} + \cdots \\ &= \frac{3}{2} + \sum_{n=2}^{\infty} \frac{2}{10^n} \end{aligned}$$

$$\begin{aligned}
&= \frac{3}{2} + \frac{\frac{2}{100}}{1-\frac{1}{10}} \\
&= \frac{3}{2} + \frac{10}{9}\cdot\frac{1}{50} \\
&= \boxed{\frac{137}{90}}.
\end{aligned}$$

(b) $0.126126\cdots = 0.\overline{126}$

Solution. We have

$$\begin{aligned}
0.\overline{126} &= 0.126 + 0.000126 + \cdots \\
&= \frac{126}{1000} + \frac{126}{1000000} + \cdots \\
&= \sum_{n=1}^{\infty} \frac{126}{1000^n} \\
&= \frac{\frac{126}{1000}}{1-\frac{1}{1000}} \\
&= \frac{1000}{999}\cdot\frac{126}{1000} \\
&= \boxed{\frac{14}{111}}.
\end{aligned}$$

3. For each sequence $\{a_n\}_{n=n_0}^{\infty}$ given below, determine

   (i) whether the **sequence** $\{a_n\}_{n=n_0}^{\infty}$ converges or diverges. If the sequence converges, find its limit.

   (ii) whether the **series** $\sum\limits_{n=n_0}^{\infty} a_n$ converges or diverges. If the series converges, find its sum if possible.

   (a) $\left\{\left(1+\frac{4}{n}\right)^n\right\}_{n=1}^{\infty}$

   Solution. (i) The limit of this sequence is an indeterminate power 1^∞. We can write it in exponential form

$$\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^n = \lim_{n\to\infty} e^{n\ln\left(1+\frac{4}{n}\right)}.$$

We can now compute the limit of the exponent using L'Hôpital's Rule as follows:

$$\begin{aligned}
\lim_{n\to\infty} n\ln\left(1+\frac{4}{n}\right) &= \lim_{x\to\infty}\frac{\ln\left(1+\frac{4}{x}\right)}{\frac{1}{x}} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty}\frac{-\frac{4}{x^2}\cdot\frac{1}{1+\frac{4}{x}}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty}\frac{4}{1+\frac{4}{x}} \\
&= 4.
\end{aligned}$$

Therefore

$$\lim_{n\to\infty} e^{n\ln\left(1+\frac{4}{n}\right)} = e^4,$$

so $\boxed{\text{the sequence } \left\{\left(1+\frac{4}{n}\right)^n\right\}_n \text{ converges to the limit } e^4}$.

(ii) Since the limit of the general term $\left(1+\frac{4}{n}\right)^n$ is not zero, the Term Divergence Test tells us that

$$\boxed{\sum_{n=1}^{\infty}\left(1+\frac{4}{n}\right)^n \text{ diverges}}.$$

(b) $\left\{\sqrt{n+1}-\sqrt{n}\right\}_{n=0}^{\infty}$

Solution. (i) The limit of this sequence is an indeterminate form $\infty - \infty$. We can resolve the indeterminate by multiplying by the conjugate in the numerator and denominator:

$$\begin{aligned}
\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right) &= \lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \\
&= \lim_{n\to\infty}\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} \\
&= \lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} \\
&= 0.
\end{aligned}$$

So $\boxed{\text{the sequence } \left\{\sqrt{n+1}-\sqrt{n}\right\}_n \text{ converges to the limit } 0}$.

(ii) To determine if the series converges or diverges, we can use the fact that this series is telescoping. The partial sums can be expressed as follows

$$\begin{aligned}
S_N &= (\sqrt{1}-\sqrt{0}) + (\sqrt{2}-\sqrt{1}) + \cdots + \left(\sqrt{N+1}-\sqrt{N}\right) \\
&= \sqrt{N+1}.
\end{aligned}$$

Therefore, $S_N \to \infty$ as $N \to \infty$, and

$$\boxed{\sum_{n=0}^{\infty}\left(\sqrt{n+1}-\sqrt{n}\right) \text{ diverges}}.$$

(c) $\{e^{-n}\}_{n=0}^{\infty}$

Solution. (i) This is a geometric sequence of common ratio $r = e^{-1}$, which satisfies $|r| < 1$. So

$$\lim_{n\to\infty} e^{-n} = 0,$$

and $\boxed{\text{the sequence } \left\{e^{-n}\right\}_n \text{ converges to the limit } 0}$.

(ii) Since $|r| = e^{-1} < 1$, the geometric series $\boxed{\sum_{n=0}^{\infty} e^{-n} \text{ converges}}$ and we can evaluate the sum as

$$\boxed{\sum_{n=0}^{\infty} e^{-n} = \frac{1}{1 - e^{-1}}}.$$

(d) $\left\{ \dfrac{e^{5n}}{n^{3/2}} \right\}_{n=1}^{\infty}$

Solution. (i) The limit of this sequence is an indeterminate form $\frac{\infty}{\infty}$. We can use L'Hôpital's Rule to compute the limit:

$$\begin{aligned}
\lim_{n\to\infty} \frac{e^{5n}}{n^{3/2}} &= \lim_{x\to\infty} \frac{e^{5x}}{x^{3/2}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{5e^{5x}}{\frac{3}{2}x^{1/2}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{25e^{5x}}{\frac{3}{4}x^{-1/2}} \\
&= \lim_{x\to\infty} \frac{100}{3} x^{1/2} e^{5x} \\
&= \infty.
\end{aligned}$$

So $\boxed{\text{the sequence } \left\{ \dfrac{e^{5n}}{n^{3/2}} \right\}_n \text{ diverges}}$.

(ii) Since the limit of the general term $\frac{e^{5n}}{n^{3/2}}$ is not zero, the Term Divergence Test tells us that

$$\boxed{\sum_{n=1}^{\infty} \frac{e^{5n}}{n^{3/2}} \text{ diverges}}.$$

4. Let $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2 \cdot 5^{n+1}}$. Find the values of x for which the series converges and find the sum of the series when it converges.

Solution. Observe that $f(x)$ is a geometric series of common ratio $r = \frac{x}{5}$. So it will converge when

$$|r| < 1 \;\Rightarrow\; \left|\frac{x}{5}\right| < 1 \;\Rightarrow\; |x| < 5 \;\Rightarrow\; \boxed{-5 < x < 5}.$$

When $-5 < x < 5$, the sum of the series is

$$\begin{aligned}
f(x) &= \frac{\text{first term}}{1 - (\text{common ratio})} \\
&= \frac{\frac{1}{10}}{1 - \frac{x}{5}} \\
&= \boxed{\frac{1}{10 - 2x}}.
\end{aligned}$$

Rutgers University
Math 152

Section 10.3: The Integral Test - Worksheet Solutions

1. For each sequence $\{a_n\}_{n=n_0}^{\infty}$ given below, determine

   (i) whether the **sequence** $\{a_n\}_{n=n_0}^{\infty}$ converges or diverges. If the sequence converges, find its limit.

   (ii) whether the **series** $\sum_{n=n_0}^{\infty} a_n$ converges or diverges. If the series converges, find its sum if possible.

   Note: the integral test is not possible/necessary for all the series.

   (a) $\{n5^{-n}\}_{n=0}^{\infty}$

   Solution. (i) The limit of this sequence is an indeterminate form $\infty \cdot 0$, which can be resolved by rewriting the expression as a fraction and using L'Hôpital's Rule. This gives

$$\begin{aligned}\lim_{n\to\infty} n5^{-n} &= \lim_{x\to\infty} \frac{x}{5^x}\\ &\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{1}{\ln(5)5^x}\\ &= 0.\end{aligned}$$

   So $\boxed{\text{the sequence } \left\{n5^{-n}\right\}_n \text{ converges to the limit } 0}$.

   (ii) We use the Integral Test. Put $f(x) = x5^{-x}$. Then f is continuous and positive on $[0, \infty)$. We have

$$f'(x) = 5^{-x} - \ln(5)x5^{-x} = -5^{-x}(\ln(5)x - 1),$$

   which is negative when $x > \frac{1}{\ln(5)}$. So f is decreasing on $\left[\frac{1}{\ln(5)}, \infty\right)$. Therefore, the Integral Test applies.

   We now need to determine if the improper integral $\int_0^{\infty} x5^{-x}dx$ converges or diverges, which we can do by evaluating it. Let us start by calculating an antiderivative using an IBP with parts

$$\begin{aligned}u = x &\;\Rightarrow\; du = dx,\\ dv = 5^{-x}dx &\;\Rightarrow\; v = -\frac{5^{-x}}{\ln(5)}.\end{aligned}$$

   This gives

$$\begin{aligned}\int x5^{-x}dx &= -\frac{x5^{-x}}{\ln(5)} - \int -\frac{5^{-x}}{\ln(5)}dx\\ &= -\frac{x5^{-x}}{\ln(5)} + \frac{1}{\ln(5)}\int 5^{-x}dx\\ &= -\frac{x5^{-x}}{\ln(5)} - \frac{5^{-x}}{\ln(5)^2} + C.\end{aligned}$$

With this at hand, we can evaluate the improper integral.

$$\begin{aligned}\int_0^\infty x5^{-x}dx &= \lim_{b\to\infty}\int_0^b x5^{-x}dx\\ &= \lim_{b\to\infty}\left[-\frac{x5^{-x}}{\ln(5)}-\frac{5^{-x}}{\ln(5)^2}\right]_0^b\\ &= \lim_{b\to\infty}\left(\frac{1}{\ln(5)^2}-\frac{b5^{-b}}{\ln(5)}-\frac{5^{-b}}{\ln(5)^2}\right)\\ &= \frac{1}{\ln(5)^2}.\end{aligned}$$

So $\int_0^\infty x5^{-x}dx$ converges. It follows that

$$\boxed{\sum_{n=0}^{\infty} n5^{-n} \text{ converges}}.$$

(b) $\left\{\frac{1}{n\,(1+\ln(n)^2)}\right\}_{n=2}^{\infty}$

Solution. (i) Since $n, \ln(n) \to \infty$ when $n \to \infty$, we have

$$\lim_{n\to\infty}\frac{1}{n\,(1+\ln(n)^2)} = 0.$$

So $\boxed{\text{the sequence } \left\{\frac{1}{n\,(1+\ln(n)^2)}\right\}_n \text{ converges to the limit } 0}$.

(ii) We use the Integral Test. Put $f(x) = \frac{1}{x(1+\ln(x)^2)}$. Then f is positive and continuous on $[2,\infty)$. Furthermore, x and $\ln(x)$ are increasing on $[2,\infty)$, so f is decreasing on $[2,\infty)$ as the reciprocal of an increasing function. Therefore, the assumptions of the Integral Test are met.

We now compute the improper integral.

$$\begin{aligned}\int_2^\infty \frac{dx}{x\,(1+\ln(x)^2)} &= \lim_{b\to\infty}\int_2^b \frac{dx}{x\,(1+\ln(x)^2)}\\ &= \lim_{b\to\infty}\left[\arctan\left(\ln(x)\right)\right]_2^b\\ &= \lim_{b\to\infty}\left(\arctan\left(\ln(b)\right)-\arctan(\ln(2))\right)\\ &= \frac{\pi}{2}-\arctan(\ln(2)).\end{aligned}$$

Therefore, $\int_2^\infty \frac{dx}{x\,(1+\ln(x)^2)}$ converges. Thus,

$$\boxed{\sum_{n=2}^{\infty}\frac{1}{n\,(1+\ln(n)^2)} \text{ converges}}.$$

(c) $\left\{ \dfrac{1}{n^{\log_5(3)}} \right\}_{n=1}^{\infty}$

Solution. (i) Since 3¿1, we have $\log_5(3) > 0$. So

$$\lim_{n\to\infty} \frac{1}{n^{\log_5(3)}} = 0,$$

and $\boxed{\text{the sequence } \left\{ \dfrac{1}{n^{\log_5(3)}} \right\}_n \text{ converges to the limit } 0}$.

(ii) Since 3¡5, we have $\log_5(3) < 1$. Therefore, by the p-series test,

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^{\log_5(3)}} \text{ diverges}}.$$

(d) $\left\{\cos\left(n^{1/n}\right)\right\}_{n=1}^{\infty}$

Solution. (i) We have

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{\ln(n)/n}$$

and

$$\lim_{x\to\infty} \frac{\ln(x)}{x} \overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0.$$

so $\lim\limits_{n\to\infty} n^{1/n} = e^0 = 1$, and by the Continuous Function Theorem,

$$\lim_{n\to\infty} \cos\left(n^{1/n}\right) = \cos(1).$$

Hence, $\boxed{\text{the sequence } \left\{\cos\left(n^{1/n}\right)\right\}_n \text{ converges to the limit } \cos(1)}$.

(ii) Since the limit of the general term is $\cos(1) \neq 0$, the Term Divergence Test implies that

$$\boxed{\sum_{n=1}^{\infty} \cos\left(n^{1/n}\right) \text{ diverges}}.$$

(e) $\left\{ \dfrac{1}{(n^2+9)^{3/2}} \right\}_{n=0}^{\infty}$

Solution. (i) We have

$$\lim_{n\to\infty} \frac{1}{(n^2+9)^{3/2}} = 0$$

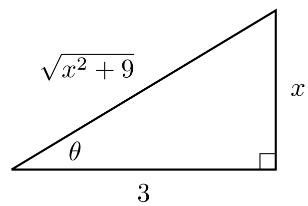
so $\boxed{\text{the sequence } \left\{ \dfrac{1}{(n^2+9)^{3/2}} \right\}_n \text{ converges to the limit } 0}$.

(ii) We use the Integral Test. Put $f(x) = \frac{1}{(x^2+9)^{3/2}}$. Then f is positive, continuous and decreasing (because $y = (x^2+9)^{3/2}$ is increasing) on $[1,\infty)$. So the Integral Test applies.

To compute an antiderivative of f, we can use the trigonometric substitution $x = 3\tan(\theta)$, so that $dx = 3\sec(\theta)^2 d\theta$ and $x^2 + 9 = 9\tan(\theta)^2 + 9 = 9\sec(\theta)^2$. This gives

$$\begin{aligned}\int \frac{dx}{(x^2+9)^{3/2}} &= \int \frac{3\sec(\theta)^2 d\theta}{(9\sec(\theta)^2)^{3/2}} \\ &= \frac{1}{9}\int \cos(\theta) d\theta \\ &= \frac{\sin(\theta)}{9} + C.\end{aligned}$$

To express this result in terms of x, we use the right triangle for this trigonometric substitution, which has base angle θ so that $\tan(\theta) = \frac{x}{3}$ as shown below.



From this we see that $\sin(\theta) = \frac{x}{\sqrt{x^2+9}}$, so we obtain

$$\int \frac{dx}{(x^2+9)^{3/2}} = \frac{x}{9\sqrt{x^2+9}} + C.$$

We can now use this to compute the improper integral.

$$\begin{aligned}\int_0^\infty \frac{dx}{(x^2+9)^{3/2}} &= \lim_{b\to\infty} \int_0^b \frac{dx}{(x^2+9)^{3/2}} \\ &= \lim_{b\to\infty} \left[\frac{x}{9\sqrt{x^2+9}}\right]_0^b \\ &= \lim_{b\to\infty} \frac{b}{9\sqrt{b^2+9}} \cdot \frac{\frac{1}{b}}{\frac{1}{b}} \\ &= \lim_{b\to\infty} \frac{1}{9\sqrt{1+9/b^2}} \\ &= \frac{1}{9}.\end{aligned}$$

So the improper integral $\displaystyle\int_0^\infty \frac{dx}{(x^2+9)^{3/2}}$ converges. It follows that

$$\boxed{\sum_{n-0}^{\infty} \frac{1}{(n^2+9)^{3/2}} \text{ converges}}.$$

(f) $\left\{\sec\left(\frac{\pi}{n}\right) - \sec\left(\frac{\pi}{n+1}\right)\right\}_{n=3}^{\infty}$

Solution. (i) Since $\frac{\pi}{n}, \frac{\pi}{n+1} \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty}\left(\sec\left(\frac{\pi}{n}\right) - \sec\left(\frac{\pi}{n+1}\right)\right) = \sec(0) - \sec(0) = 0.$$

So $\boxed{\text{the sequence } \left\{\sec\left(\frac{\pi}{n}\right) - \sec\left(\frac{\pi}{n+1}\right)\right\}_n \text{ converges to the limit } 0}$.

(ii) To determine if the series converges or diverges, we can use the fact that this series is telescoping. The partial sums can be expressed as follows

$$\begin{aligned} S_N &= \sum_{n=3}^{N} \left(\sec\left(\frac{\pi}{n}\right) - \sec\left(\frac{\pi}{n+1}\right)\right) \\ &= \left((\sec\left(\frac{\pi}{3}\right) - \sec\left(\frac{\pi}{4}\right)\right) + \left(\sec\left(\frac{\pi}{4}\right) - \sec\left(\frac{\pi}{5}\right)\right) + \cdots + \left(\sec\left(\frac{\pi}{N}\right) - \sec\left(\frac{\pi}{N+1}\right)\right) \\ &= \sec\left(\frac{\pi}{3}\right) - \sec\left(\frac{\pi}{N+1}\right) \\ &= 2 - \sec\left(\frac{\pi}{N+1}\right). \end{aligned}$$

Therefore,

$$\sum_{n=3}^{\infty} \left(\sec\left(\frac{\pi}{n}\right) - \sec\left(\frac{\pi}{n+1}\right)\right) = \lim_{N\to\infty} \left(2 - \sec\left(\frac{\pi}{N+1}\right)\right) = 2 - \sec(0) = 2 - 1 = \boxed{1},$$

and in particular

$$\boxed{\sum_{n=3}^{\infty} \left(\sec\left(\frac{\pi}{n}\right) - \sec\left(\frac{\pi}{n+1}\right)\right) \text{ converges}}.$$

(g) $\left\{2^{2n+1}5^{-n}\right\}_{n=0}^{\infty}$

Solution. (i) Observe that we have

$$2^{2n+1}5^{-n} = 2\left(\frac{4}{5}\right)^n.$$

So we have a geometric sequence of common ratio $r = \frac{4}{5}$, which satisfies $|r| < 1$. So

$$\lim_{n\to\infty} 2^{2n+1}5^{-n} = 0,$$

and $\boxed{\text{the sequence } \left\{2^{2n+1}5^{-n}\right\}_n \text{ converges to the limit } 0}$.

(ii) Since $|r| = \frac{4}{5} < 1$, the geometric series $\boxed{\sum_{n=0}^{\infty} 2^{2n+1}5^{-n} \text{ converges}}$ and we can evaluate the sum as

$$\boxed{\sum_{n=0}^{\infty} 2^{2n+1}5^{-n} = \frac{2}{1-\frac{4}{5}} = 10}.$$

(h) $\left\{\left(1+\frac{1}{2n}\right)^n\right\}_{n=1}^{\infty}$

Solution. (i) The limit of this sequence is an indeterminate power 1^∞. We can write it in exponential form

$$\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^n = \lim_{n\to\infty} e^{n\ln\left(1+\frac{1}{2n}\right)}.$$

We can now compute the limit of the exponent using L'Hôpital's Rule as follows:

$$\begin{aligned}\lim_{n\to\infty} n\ln\left(1+\frac{1}{2n}\right) &= \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{2x}\right)}{\frac{1}{x}} \\ &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty}\frac{-\frac{1}{2x^2}\cdot\frac{1}{1+\frac{1}{2x}}}{-\frac{1}{x^2}} \\ &= \lim_{x\to\infty}\frac{1}{2\left(1+\frac{4}{x}\right)} \\ &= \frac{1}{2}.\end{aligned}$$

Therefore

$$\lim_{n\to\infty} e^{n\ln\left(1+\frac{1}{2n}\right)} = e^{1/2},$$

so $\boxed{\text{the sequence } \left\{\left(1+\frac{1}{2n}\right)^n\right\}_n \text{ converges to the limit } e^{1/2}}$.

(ii) Since the limit of the general term $\left(1+\frac{1}{2n}\right)^n$ is not zero, the Term Divergence Test tells us that

$$\boxed{\sum_{n=1}^{\infty}\left(1+\frac{1}{2n}\right)^n \text{ diverges}}.$$

(i) $\left\{\frac{1}{n\ln(n)\ln(\ln(n))}\right\}_{n=4}^{\infty}$

Solution. We have $n\ln(n)\ln(\ln(n))\to\infty$ as $n\to\infty$, so

$$\lim_{n\to\infty}\frac{1}{n\ln(n)\ln(\ln(n))} = 0,$$

and $\boxed{\text{the sequence } \left\{\frac{1}{n\ln(n)\ln(\ln(n))}\right\}_n \text{ converges to the limit } 0}$.

To determine whether the series converges or not, we can use the Integral Test with $f(x) = \frac{1}{x\ln(x)\ln(\ln(x))}$ which is continuous, positive and decreasing (because $y = x\ln(x)\ln(\ln(x))$ is increasing) on $[4, \infty)$. We now compute the improper integral.

$$\begin{aligned}
\int_4^\infty \frac{dx}{x\ln(x)\ln(\ln(x))} &= \lim_{b\to\infty}\int_4^b \frac{dx}{x\ln(x)\ln(\ln(x))} \\
&= \lim_{b\to\infty}\int_{\ln(\ln(4))}^{\ln(\ln(b))} \frac{du}{u} \qquad \left(u = \ln(\ln(x)), du = \frac{dx}{x\ln(x)}\right) \\
&= \lim_{b\to\infty} \left[\ln|u|\right]_{\ln(\ln(4))}^{\ln(\ln(b))} \\
&= \lim_{b\to\infty} \left(\ln(\ln(\ln(b))) - \ln(\ln(\ln(4)))\right) \\
&= \infty.
\end{aligned}$$

So the improper integral $\displaystyle\int_4^\infty \frac{dx}{x\ln(x)\ln(\ln(x))}$ diverges, from which we deduce that

$$\boxed{\sum_{n=4}^\infty \frac{1}{n\ln(n)\ln(\ln(n))} \text{ diverges}}.$$

2. (a) Determine for which values of p the series $\displaystyle\sum_{n=2}^\infty \frac{1}{n\ln(n)^p}$ converges or diverges.

Solution. We use the Integral Test. The function $f(x) = \dfrac{1}{x\ln(x)^p}$ is continuous, positive and decreasing (because $x\ln(x)^p$ is increasing) on $[2,\infty)$. Therefore, the Integral Test applies and we can test for convergence of the series by testing for convergence of the corresponding improper integral.

To compute the integral, we use the substitution $u = \ln(x)$, which gives $du = \frac{dx}{x}$. This gives

$$\begin{aligned}
\int_2^\infty \frac{dx}{x\ln(x)^p} &= \lim_{b\to\infty}\int_2^b \frac{dx}{x\ln(x)^p} \\
&= \lim_{b\to\infty}\int_{\ln(2)}^{\ln(b)} \frac{du}{u^p} \\
&= \int_{\ln(2)}^{\infty} \frac{du}{u^p}.
\end{aligned}$$

This last integral is a type I p-integral, so it converges if $p > 1$ and diverges if $p \leqslant 1$. Therefore,

$$\boxed{\sum_{n=2}^\infty \frac{1}{n\ln(n)^p} \text{ converges if } p > 1, \text{ diverges if } p \leqslant 1}.$$

(b) Determine for which values of p the series $\displaystyle\sum_{n=1}^\infty \frac{\ln(n)}{n^p}$ converges or diverges.

Solution. We use the Integral Test. The function $f(x) = \dfrac{\ln(x)}{x^p}$ is continuous and positive on $[2,\infty)$. We have

$$f'(x) = \frac{1}{x}\cdot\frac{1}{x^p} - \frac{p\ln(x)}{x^{p+1}} = \frac{1 - p\ln(x)}{x^{p+1}}.$$

Observe that $f'(x) < 0$ when $x > e^{1/p}$. So f is decreasing on $[e^{1/p}, \infty)$. Therefore, the Integral Test applies and we can test for convergence of the series by testing for convergence of the corresponding improper integral.

To compute the integral, we will need to distinguish the cases $p = 1$ and $p \neq 1$. When $p = 1$, we have

$$\begin{aligned}\int_1^\infty \frac{\ln(x)}{x} dx &= \lim_{b\to\infty} \int_1^b \frac{\ln(x)}{x} dx \\ &= \lim_{b\to\infty} \left[\frac{\ln(x)^2}{2}\right]_1^b \\ &= \lim_{b\to\infty} \frac{\ln(b)^2}{2} \\ &= \infty,\end{aligned}$$

so we conclude that the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^1}$ diverges.

When $p \neq 1$, we can use an IBP to compute an antiderivative. We will choose the parts

$$\begin{aligned}u = \ln(x) \;&\Rightarrow\; du = \frac{dx}{x}, \\ dv = x^{-p} dx \;&\Rightarrow\; v = \frac{x^{1-p}}{1-p}.\end{aligned}$$

This gives

$$\begin{aligned}\int \frac{\ln(x)}{x^p} dx &= \frac{\ln(x)x^{1-p}}{1-p} - \int \frac{x^{-p}}{1-p} dx \\ &= \frac{\ln(x)x^{1-p}}{1-p} - \frac{x^{1-p}}{(1-p)^2} + C \\ &= \frac{x^{1-p}\left((1-p)\ln(x) - 1\right)}{(1-p)^2} + C.\end{aligned}$$

So the improper integral is

$$\begin{aligned}\int_1^\infty \frac{\ln(x)}{x^p} dx &= \lim_{b\to\infty} \int_1^b \frac{\ln(x)}{x^p} dx \\ &= \lim_{b\to\infty} \left[\frac{x^{1-p}\left((1-p)\ln(x) - 1\right)}{(1-p)^2}\right]_1^b \\ &= \lim_{b\to\infty} \left(\frac{b^{1-p}\left((1-p)\ln(b) - 1\right)}{(1-p)^2} + \frac{1}{(1-p)^2}\right).\end{aligned}$$

In the case $1 - p > 0$, that is $p < 1$, we have

$$\lim_{b\to\infty} b^{1-p}\left((1-p)\ln(b) - 1\right) = \infty,$$

so the improper integral diverges. In the case $1 - p < 0$, that is $p > 1$, we have by L'Hôpital's Rule

$$\lim_{b\to\infty} b^{1-p}\left((1-p)\ln(b) - 1\right) = \lim_{b\to\infty} \frac{(1-p)\ln(b) - 1}{b^{p-1}} = \lim_{b\to\infty} \frac{\frac{(1-p)}{b}}{(p-1)b^{p-2}} = \lim_{b\to\infty} \frac{-1}{b^{p-1}} = 0,$$

so the improper integral converges.

In conclusion,

$$\boxed{\sum_{n=1}^{\infty} \frac{\ln(n)}{n^p} \text{ converges if } p > 1, \text{ diverges if } p \leqslant 1}.$$

Rutgers University
Math 152

Section 10.4: Comparison Tests - Worksheet Solutions

Determine if the series below converge or diverge. Make sure to clearly label and justify the use of any convergence test used. **Note:** some of these problems require convergence tests from previous sections.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{(5\sqrt{n}-2)^3}{3n^2-2n+4}$

   Solution. We use the LCT with $\displaystyle\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{n^{3/2}}{n^2} = \sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$, which diverges as a p-series with $p = \frac{1}{2} \leqslant 1$. We have

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{a_n}{b_n} \\ &= \lim_{n\to\infty} \frac{\frac{(5\sqrt{n}-2)^3}{3n^2-2n+4}}{\frac{1}{n^{1/2}}} \\ &= \lim_{n\to\infty} \frac{(5\sqrt{n}-2)^3 n^{1/2}}{3n^2-2n+4} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \\ &= \lim_{n\to\infty} \frac{(5-\frac{2}{n^{1/2}})^3}{3-\frac{2}{n}+\frac{4}{n^2}} \\ &= \frac{125}{3}. \end{aligned}$$

   Since $0 < L < \infty$ and $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$ diverges, we conclude that $\boxed{\displaystyle\sum_{n=2}^{\infty} \frac{(5\sqrt{n}-2)^3}{3n^2-2n+4} \text{ diverges}}$.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n5^n}$

   Solution. We use the DCT. Observe that for any $n \geqslant 1$, we have

$$0 < \frac{1}{n} \leqslant 1 \ \Rightarrow\ 0 < \frac{3^n}{n5^n} \leqslant \frac{3^n}{5^n}.$$

   Furthermore, $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n$ converges since it is a geometric series with common ration $r = \frac{3}{5}$ satisfying $|r| < 1$. Therefore, $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n5^n} \text{ converges}}$.

   Remark. The LCT with $b_n = \frac{3^n}{5^n}$ would have worked equally well here since

   - $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n} = 0$, and

- $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{5^n}$ converges.

3. $\displaystyle\sum_{n=0}^{\infty} \frac{2^{2n}}{3^n + 11n^2}$

Solution. We use the LCT with $\displaystyle\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{2^{2n}}{3^n} = \sum_{n=2}^{\infty} \left(\frac{4}{3}\right)^n$, which diverges as a geometric series with common ratio $r = \frac{4}{3}$ satisfying $|r| \geqslant 1$. We have

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{a_n}{b_n} \\ &= \lim_{n\to\infty} \frac{\frac{2^{2n}}{3^n+11n^2}}{\frac{2^{2n}}{3^n}} \\ &= \lim_{n\to\infty} \frac{3^n}{3^n + 11n^2} \cdot \frac{\frac{1}{3^n}}{\frac{1}{3^n}} \\ &= \lim_{n\to\infty} \frac{1}{1 - \frac{11n^2}{3^n}}. \end{aligned}$$

Using L'Hôpital's Rule twice, we see that

$$\lim_{x\to\infty} \frac{11x^2}{3^x} \overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{22x}{\ln(3)3^x} \overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{22}{\ln(3)^2 3^x} = 0.$$

Thus,

$$L = \lim_{n\to\infty} \frac{1}{1 - \frac{11n^2}{3^n}} = \frac{1}{1-0} = 1.$$

Since $0 < L < \infty$ and $\displaystyle\sum_{n=0}^{\infty} \frac{2^{2n}}{3^n}$ diverges, we conclude that $\boxed{\displaystyle\sum_{n=0}^{\infty} \frac{2^{2n}}{3^n + 11n^2} \text{ diverges}}$.

Remark. We cannot use the DCT with $b_n = \frac{2^{2n}}{3^n}$ here since

$$0 < 3^n < 3^n + 11n^2 \ \Rightarrow\ 0 < \frac{2^{2n}}{3^n + 11n^2} < \frac{2^{2n}}{3^n}.$$

And knowing that the series of the "bigger terms" diverges does not tell us anything about the series of the "smaller terms".

4. $\displaystyle\sum_{n=3}^{\infty} \frac{\ln(n)^2}{\sqrt{n}}$

Solution. We use the DCT. Observe that for $n \geqslant 3$, we have

$$0 < \ln(3)^2 < \ln(n)^2 \ \Rightarrow\ 0 < \frac{\ln(3)^2}{\sqrt{n}} < \frac{\ln(n)^2}{\sqrt{n}}.$$

Furthermore, $\displaystyle\sum_{n=3}^{\infty} \frac{\ln(3)^2}{\sqrt{n}} = \ln(3)^2 \sum_{n=3}^{\infty} \frac{1}{\sqrt{n}}$ diverges as a p-series with $p = \frac{1}{2} \leqslant 1$. Therefore, $\boxed{\displaystyle\sum_{n=3}^{\infty} \frac{\ln(n)^2}{\sqrt{n}} \text{ diverges}}$.

Remark. We could have also used the LCT with $b_n = \frac{1}{\sqrt{n}}$ here, observing that

- $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \ln(n)^2 = \infty$, and
- $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

5. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln(n)^2}$

Solution. Intuitively, we expect this series to diverge because $\ln(n)$ grows slower than any power of n, so the $\sqrt{n}$ in the denominator dictates the behavior. But because the $\ln(n)^2$ in the denominator is making the fraction smaller, any attempt to compare this series with $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ will be inconclusive. So instead, we will try to compare with another p-series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^p}$. Because we expect divergence, we will need to pick $p \leqslant 1$. Because we need the denominator n^p to grow faster than $\sqrt{n}\ln(n)^2$, we will pick $p > \frac{1}{2}$. A suitable value of p would therefore be $p = \frac{3}{4}$ (but you can repeat the reasoning below with any value of p between $\frac{1}{2}$ and 1).

We use the LCT with $\displaystyle\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n^{3/4}}$, which diverges as a p-series with $p = \frac{3}{4} \leqslant 1$.. We have

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{a_n}{b_n} \\
&= \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}\ln(n)^2}}{\frac{1}{n^{3/4}}} \\
&= \lim_{x\to\infty} \frac{x^{1/4}}{\ln(x)^2} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{x^{-3/4}}{\frac{8\ln(x)}{x}} \\
&= \lim_{x\to\infty} \frac{x^{1/4}}{8\ln(x)} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{x^{-3/4}}{\frac{32}{x}} \\
&= \lim_{x\to\infty} \frac{x^{1/4}}{32} \\
&= \infty
\end{aligned}$$

Since $L = \infty$ and $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{3/4}}$ diverges, we conclude that $\boxed{\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln(n)^2} \text{ diverges}}$.

6. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)^2}$

Solution. We could try to use a comparison with a p-series smartly chosen as in the previous problem, but we would not find a suitable exponent for a conclusive test. Instead, the Integral Test will come to the rescue here.

The function $f(x) = \dfrac{1}{x\ln(x)^2}$ is continuous, positive and decreasing (because $x\ln(x)^2$ is increasing) on $[2, \infty)$. Therefore, the Integral Test applies and we can test for convergence of the series by testing for convergence of the corresponding improper integral.

To compute the integral, we use the substitution $u = \ln(x)$, which gives $du = \frac{dx}{x}$. This gives

$$\begin{aligned}\int_2^\infty \frac{dx}{x\ln(x)^2} &= \lim_{b\to\infty}\int_2^b \frac{dx}{x\ln(x)^2}\\ &= \lim_{b\to\infty}\int_{\ln(2)}^{\ln(b)} \frac{du}{u^2}\\ &= \int_{\ln(2)}^{\infty}\frac{du}{u^2}.\end{aligned}$$

This last integral is a type I p-integral with $p = 2 > 1$, so it converges. Therefore, $\boxed{\sum_{n=2}^{\infty}\frac{1}{n\ln(n)^2} \text{ converges}}$.

7. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{n}{n+3}\right)^n$

Solution. The limit of the general term is an indeterminate form 1^∞. If this indetermination resolves into something not equal to zero, the Term Divergence Test will immediately tell us that the series diverges. So let us try to compute the limit of the general term.

We can start by writing the power in exponential form

$$\lim_{n\to\infty}\left(\frac{n}{n+3}\right)^n = \lim_{n\to\infty} e^{n\ln\left(\frac{n}{n+3}\right)}.$$

We now compute the limit of the exponent using L'Hôpital's Rule:

$$\begin{aligned}\lim_{n\to\infty} n\ln\left(\frac{n}{n+3}\right) &= \lim_{x\to\infty}\frac{\ln(x)-\ln(x+3)}{\frac{1}{x}}\\ &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \frac{\frac{1}{x}-\frac{1}{x+3}}{-\frac{1}{x^2}}\\ &= \lim_{x\to\infty} -x^2\frac{(x+3)-x}{x(x+3)}\\ &= \lim_{x\to\infty} -\frac{3x}{x+3}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}\\ &= \lim_{x\to\infty} -\frac{3}{1+3/x}\\ &= -3.\end{aligned}$$

So

$$\lim_{n\to\infty} e^{n\ln\left(\frac{n}{n+3}\right)} = e^{-3} \neq 0.$$

By the Term Divergence Test, it follows that $\boxed{\sum_{n=0}^{\infty}\left(\frac{n}{n+3}\right)^n \text{ diverges}}$.

8. $\displaystyle\sum_{n=1}^{\infty}\frac{7-3\cos\left(n^2\right)}{n^5+3}$

Solution. We use the DCT. Observe that $-1 \leqslant \cos(n^2) \leqslant 1$, so $4 \leqslant 7-3\cos\left(n^2\right) \leqslant 10$. Also, $n^5+3 < n^5$. It follows that

$$0 < \frac{7-3\cos\left(n^2\right)}{n^5+3} < \frac{10}{n^5}.$$

Furthermore, $\displaystyle\sum_{n=1}^{\infty}\frac{10}{n^5} = 10\sum_{n=1}^{\infty}\frac{1}{n^5}$ converges as a p-series with $p = 5 > 1$. Therefore, $\boxed{\displaystyle\sum_{n=1}^{\infty}\frac{7-3\cos\left(n^2\right)}{n^5+3} \text{ converges}}$.

9. $\displaystyle\sum_{n=2}^{\infty} n\sin\left(\frac{5}{n^3}\right)$

Solution. We use the LCT with $\displaystyle\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} n\frac{1}{n^3} = \sum_{n=2}^{\infty}\frac{1}{n^2}$, which converges as a p-series with $p = 2 > 1$.

We have

$$\begin{aligned} L &= \lim_{n\to\infty}\frac{a_n}{b_n} \\ &= \lim_{n\to\infty}\frac{n\sin\left(\frac{5}{n^3}\right)}{\frac{1}{n^2}} \\ &= \lim_{x\to\infty}\frac{\sin\left(\frac{5}{x^3}\right)}{\frac{1}{x^3}} \\ &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty}\frac{-\frac{15}{x^4}\cos\left(\frac{5}{x^3}\right)}{-\frac{3}{x^4}} \\ &= \lim_{x\to\infty} 5\cos\left(\frac{5}{x^3}\right) \\ &= 5. \end{aligned}$$

Since $0 < L < \infty$ and $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2}$ converges, we conclude that $\boxed{\displaystyle\sum_{n=2}^{\infty} n\sin\left(\frac{5}{n^3}\right) \text{ converges}}$.

Rutgers University
Math 152

Section 10.5: Absolute Convergence, Ratio & Root Tests - Worksheet Solutions

1. Determine if the series below converge or diverge. Make sure to clearly label and justify the use of any convergence test used. **Note:** some of these problems require convergence tests from previous sections.

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{3^{2n}}$

Solution. We use the Root Test. We have

$$\begin{aligned}\rho &= \lim_{n\to\infty} \sqrt[n]{\left|(-1)^n \frac{n^n}{3^{2n}}\right|} \\ &= \lim_{n\to\infty} \frac{n}{9} \\ &= \infty.\end{aligned}$$

Since $\rho > 1$, we conclude that $\boxed{\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{3^{2n}} \text{ diverges}}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{8n^6+7n+11}}{3n^7+8n^5-1}$

Solution. We use the LCT with $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n^{6/3}}{n^7} = \sum_{n=1}^{\infty} \frac{1}{n^5}$, which converges as a p-series with $p = 5 > 1$. We have

$$\begin{aligned}L &= \lim_{n\to\infty} \frac{a_n}{b_n} \\ &= \lim_{n\to\infty} \frac{\frac{\sqrt[3]{8n^6+7n+11}}{3n^7+8n^5-1}}{\frac{n^{6/3}}{n^7}} \\ &= \lim_{n\to\infty} \frac{\sqrt[3]{8n^6+7n+11}}{3n^7+8n^5-1} \cdot \frac{n^7}{n^{6/3}} \\ &= \lim_{n\to\infty} \frac{\sqrt[3]{8+\frac{7}{n^5}+\frac{11}{n^6}}}{3+\frac{8}{n^2}-\frac{1}{n^7}} \\ &= \frac{\sqrt[3]{8+0+0}}{3+0-0} \\ &= \frac{2}{3}.\end{aligned}$$

Since $0 < L < \infty$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$ converges, we conclude that $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{8n^6+7n+11}}{3n^7+8n^5-1} \text{ converges}}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)!}{e^n n!(n+1)!}$

Solution. We use the Ratio Test. We have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \frac{(2(n+1)+1)!}{e^{n+1}(n+1)!(n+2)!} \cdot \frac{e^n n!(n+1)!}{(2n+1)!} \\
&= \lim_{n\to\infty} \frac{e^n}{e^{n+1}} \cdot \frac{(2n+3)!}{(2n+1)!} \cdot \frac{n!(n+1)!}{(n+1)!(n+2)!} \\
&= \lim_{n\to\infty} \frac{(2n+2)(2n+3)}{e(n+1)(n+2)} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \\
&= \lim_{n\to\infty} \frac{(2+\frac{2}{n})(2+\frac{3}{n})}{e(1+\frac{1}{n})(1+\frac{2}{n})} \\
&= \frac{4}{e}.
\end{aligned}$$

Since $e < 4$, we have $\rho > 1$. So $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)!}{e^n n!(n+1)!} \text{ diverges}}$.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(8n)+3}{4^n}$

Solution. We use the DCT. Observe that $-1 \leqslant \cos(8n) \leqslant 1$, so $2 \leqslant \cos(8n)+3 \leqslant 4$. It follows that

$$0 < \frac{\cos(8n)+3}{4^n} < \frac{4}{4^n}.$$

Furthermore, $\displaystyle\sum_{n=1}^{\infty} \frac{4}{4^n}$ converges as a geometric series with common ratio $r = \frac{1}{4}$ satisfying $|r| < 1$. Therefore, $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{\cos(8n)+3}{4^n} \text{ converges}}$.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{\ln(\ln(n))}$

Solution. Observe that

$$\begin{aligned}
\lim_{n\to\infty} \frac{\ln(n)}{\ln(\ln(n))} &= \lim_{x\to\infty} \frac{\ln(x)}{\ln(\ln(x))} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x\ln(x)}} \\
&= \lim_{x\to\infty} \ln(x) \\
&= \infty.
\end{aligned}$$

Since the general term of the series does not approach 0, the Term Divergence Test tells us that $\boxed{\sum_{n=1}^{\infty} \frac{\ln(n)}{\ln(\ln(n))} \text{ diverges}}$.

(f) $\sum_{n=1}^{\infty} 4^n \left(\frac{n-2}{n}\right)^{n^2}$

Solution. We use the Root Test. We have

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left(4^n \left(\frac{n-2}{n}\right)^{n^2}\right)^{1/n} \\ &= \lim_{n\to\infty} 4 \left(\frac{n-2}{n}\right)^{n} \\ &= 4 \lim_{n\to\infty} \left(1 - \frac{2}{n}\right)^{n}.\end{aligned}$$

This limit is an indeterminate power 1^∞. We can start by writing the power in exponential form

$$\lim_{n\to\infty} \left(1 - \frac{2}{n}\right)^{n} = \lim_{n\to\infty} e^{n\ln\left(1-\frac{2}{n}\right)}.$$

We now compute the limit of the exponent using L'Hôpital's Rule:

$$\begin{aligned}\lim_{n\to\infty} n \ln\left(1 - \frac{2}{n}\right) &= \lim_{x\to\infty} \frac{\ln\left(1-\frac{2}{x}\right)}{\frac{1}{x}} \\ &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \frac{\frac{2}{x^2}\cdot\frac{1}{1-\frac{2}{x}}}{-\frac{1}{x^2}} \\ &= \lim_{x\to\infty} -\frac{2}{1-\frac{2}{x}} \\ &= -2.\end{aligned}$$

Therefore

$$\rho = 4 \lim_{n\to\infty} e^{n\left(1-\frac{2}{n}\right)} = 4e^{-2}.$$

Since $e > 2$, we have $e^2 > 4$ and therefore $\rho < 1$. Hence, $\boxed{\sum_{n=1}^{\infty} 4^n \left(\frac{n-2}{n}\right)^{n^2} \text{ converges absolutely}}$.

(g) $\sum_{n=1}^{\infty} (-1)^n \frac{((2n)!)^2}{(4n)!}$

Solution. We use the Ratio Test. We have

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty} \frac{((2n+2)!)^2}{(4n+4)!} \cdot \frac{(4n)!}{((2n)!)^2}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \left(\frac{(2n+2)!}{(2n)!}\right)^2 \cdot \frac{(4n)!}{(4n+4)!} \\
&= \lim_{n\to\infty} \frac{(2n+2)^2(2n+1)^2}{(4n+1)(4n+2)(4n+3)(4n+4)} \cdot \frac{\frac{1}{n^4}}{\frac{1}{n^4}} \\
&= \lim_{n\to\infty} \frac{(2+\frac{2}{n})^2(2+\frac{1}{n})^2}{(4+\frac{1}{n})(4+\frac{2}{n})(4+\frac{3}{n})(4+\frac{4}{n})} \\
&= \frac{2^2 2^2}{4^4} \\
&= \frac{1}{16}.
\end{aligned}
$$

Since $\rho < 1$, we conclude that $\boxed{\sum_{n=1}^{\infty} (-1)^n \frac{((2n)!)^2}{(4n)!} \text{ converges absolutely}}$.

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^n(n+2)!}$

Solution. We use the Ratio Test. We have

$$
\begin{aligned}
\rho &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{3^{n+1}(n+3)!} \cdot \frac{3^n(n+2)!}{n^n} \\
&= \lim_{n\to\infty} \frac{(n+1)(n+1)^n}{3(n+3)n^n} \\
&= \lim_{n\to\infty} \frac{n+1}{3(n+3)} \left(1+\frac{1}{n}\right)^n \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{3(1+\frac{3}{n})} e^{n\ln\left(1+\frac{1}{n}\right)}.
\end{aligned}
$$

We can compute the limit of the exponent using L'Hôpital's Rule:

$$
\begin{aligned}
\lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) &= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \frac{-\frac{1}{x^2}\cdot\frac{1}{1+\frac{1}{x}}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} \\
&= 1.
\end{aligned}
$$

Therefore

$$
\rho = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{3(1+\frac{3}{n})} e^{n\ln\left(1+\frac{1}{n}\right)} = \frac{1+0}{3(1+0)} e^1 = \frac{e}{3}.
$$

Since $e < 3$, we have $\rho < 1$ so $\boxed{\sum_{n=1}^{\infty} \frac{n^n}{3^n(n+2)!} \text{ converges absolutely}}$.

(i) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+5\sin(n)}{3n}\right)^n$

Solution. We use the Root Test. We have

$$\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{2n+5\sin(n)}{3n} = \lim_{n\to\infty} \left(\frac{2}{3} + \frac{5\sin(n)}{3n}\right).$$

We have $-1 \leqslant \sin(n) \leqslant 1$, so

$$-\frac{5}{3n} \leqslant \frac{5\sin(n)}{3n} \leqslant \frac{5}{3n}.$$

Furthermore, $\lim_{n\to\infty} -\frac{5}{3n} = \lim_{n\to\infty} \frac{5}{3n} = 0$. So by the Sandwich Theorem, we have $\lim_{n\to\infty} \frac{5\sin(n)}{3n} = 0$ and

$$\rho = \lim_{n\to\infty} \left(\frac{2}{3} + \frac{5\sin(n)}{3n}\right) = \frac{2}{3}.$$

Since $\rho < 1$, we conclude that $\boxed{\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+5\sin(n)}{3n}\right)^n \text{ converges absolutely}}$.

2. Let a_n be the sequence defined recursively by

$$a_1 = 7, \quad a_{n+1} = a_n \left(\frac{n}{n+3}\right)^n \text{ for } n \geqslant 1.$$

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges or diverges. Make sure to clearly label and justify the use of any convergence test used.

Solution. The recursive relation gives us information about $\dfrac{a_{n+1}}{a_n}$, so the Ratio Test seems like a good option here. We have

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty} \left|\frac{a_n \left(\frac{n}{n+3}\right)^n}{a_n}\right| \\ &= \lim_{n\to\infty} \left(\frac{n}{n+3}\right)^n.\end{aligned}$$

This limit is an indeterminate power 1^{∞}. We can start by writing the power in exponential form

$$\rho = \lim_{n\to\infty} \left(\frac{n}{n+3}\right)^n = \lim_{n\to\infty} e^{n\ln\left(\frac{n}{n+3}\right)}.$$

We now compute the limit of the exponent using L'Hôpital's Rule:

$$\lim_{n\to\infty} n\ln\left(\frac{n}{n+3}\right) = \lim_{x\to\infty} \frac{\ln(x) - \ln(x+3)}{\frac{1}{x}}$$

$$
\begin{aligned}
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \frac{\frac{1}{x} - \frac{1}{x+3}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} -x^2 \frac{(x+3)-x}{x(x+3)} \\
&= \lim_{x\to\infty} -\frac{3x}{x+3} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x\to\infty} -\frac{3}{1+3/x} \\
&= -3.
\end{aligned}
$$

So

$$\rho = \lim_{n\to\infty} e^{n \ln\left(\frac{n}{n+3}\right)} = e^{-3}.$$

Since $\rho < 1$, we conclude that $\boxed{\displaystyle\sum_{n=1}^{\infty} a_n \text{ converges absolutely}}$.

Rutgers University
Math 152

Section 10.6: Alternating Series & Conditional Convergence - Worksheet Solution

1. Determine if the series below converge absolutely, converge conditionally or diverge. Make sure to clearly label and justify the use of any convergence test used. **Note:** some of these problems require convergence tests from previous sections.

   (a) $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{n\log_2(n)}$

   Solution. We can prove the convergence of this series using the AST. The sequence $a_n = \frac{1}{n\log_2(n)}$ is positive when $n \geqslant 3$, decreasing (since $n\log_2(n)$ is increasing) and $\lim_{n\to\infty} \frac{1}{n\log_2(n)} = 0$. So the AST applies and $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{n\log_2(n)}$ converges.

   We need to determine if the convergence is absolute or conditional, that is, we need to determine whether the series
   $$\sum_{n=3}^{\infty} \left|\frac{(-1)^n}{n\log_2(n)}\right| = \sum_{n=3}^{\infty} \frac{1}{n\log_2(n)}$$
   converges or diverges. To this end, we can use the Integral Test. The function $f(x) = \dfrac{1}{x\log_2(x)}$ is continuous, positive and decreasing (because $x\log_2(x)^2$ is increasing) on $[3,\infty)$. Therefore, the Integral Test applies and we can test for convergence of the series by testing for convergence of the corresponding improper integral.

   To compute the integral, we use the substitution $u = \log_2(x)$, which gives $du = \frac{dx}{\ln(2)x}$. This gives
   $$\begin{aligned}\int_3^{\infty} \frac{dx}{x\log_2(x)} &= \lim_{b\to\infty} \int_3^b \frac{dx}{x\log_2(x)} \\ &= \lim_{b\to\infty} \int_{\log_2(3)}^{\log_2(b)} \frac{\ln(2)du}{u} \\ &= \ln(2)\int_{\log_2(3)}^{\infty} \frac{du}{u}.\end{aligned}$$

   This last integral is a type I p-integral with $p = 1$, so it diverges. Therefore, $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\log_2(n)}$ diverges.

   In conclusion, $\boxed{\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{n\log_2(n)} \text{ converges conditionally}}$.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!}$

Solution. We use the Ratio Test. We have

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \qquad = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}$$
$$= \lim_{n\to\infty} \frac{2}{n+1}$$
$$= 0.$$

Since $\rho < 1$, we conclude that $\boxed{\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!} \text{ converges absolutely}}$.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{n\arctan(n)}{\sqrt[3]{8n^6+1}}$

Solution. Note that this series has non-negative terms. We use the LCT with $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n}{\sqrt[3]{n^6}} = \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges as a p-series with $p = 1$. We have

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{a_n}{b_n} \\ &= \lim_{n\to\infty} \frac{\frac{n\arctan(n)}{\sqrt[3]{8n^6+1}}}{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{n^2\arctan(n)}{\sqrt[3]{8n^6+1}}\cdot\frac{\frac{1}{n^2}}{\frac{1}{n^2}} \\ &= \lim_{n\to\infty} \frac{\arctan(n)}{\sqrt[3]{8+\frac{1}{n^6}}} \\ &= \frac{\frac{\pi}{2}}{\sqrt[3]{8}} \\ &= \frac{\pi}{4}. \end{aligned}$$

Since $0 < L < \infty$ and $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we deduce that $\boxed{\sum_{n=0}^{\infty} \frac{n\arctan(n)}{\sqrt[3]{8n^6+1}} \text{ diverges}}$.

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n+\cos(n)}$

Solution. Note that this series has non-negative terms. We use the LCT with $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{1}{3^n}$, which converges as a geometric series with common ratio $r = \frac{1}{3}$, $|r| < 1$. We have

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{\frac{1}{3^n+\cos(n)}}{\frac{1}{3^n}} \\
&= \lim_{n\to\infty} \frac{3^n}{3^n+\cos(n)} \cdot \frac{\frac{1}{3^n}}{\frac{1}{3^n}} \\
&= \lim_{n\to\infty} \frac{1}{1+\frac{\cos(n)}{3^n}}.
\end{aligned}$$

Because $-1 \leqslant \cos(n) \leqslant 1$, we have

$$-\frac{1}{3^n} \leqslant \frac{\cos(n)}{3^n} \leqslant \frac{1}{3^n}$$

and $\lim\limits_{n\to\infty} -\frac{1}{3^n} = \lim\limits_{n\to\infty} \frac{1}{3^n} = 0$. By the Sandwich Theorem, we obtain $\lim\limits_{n\to\infty} \frac{\cos(n)}{3^n} = 0$ and

$$L = \lim_{n\to\infty} \frac{1}{1+\frac{\cos(n)}{3^n}} = \frac{1}{1+0} = 1.$$

Since $0 < L < \infty$ and $\sum\limits_{n=0}^{\infty} b_n = \sum\limits_{n=0}^{\infty} \frac{1}{3^n}$ converges, we deduce that $\boxed{\sum\limits_{n=0}^{\infty} \frac{1}{3^n+\cos(n)} \text{ converges absolutely}}$.

(e) $\sum\limits_{n=2}^{\infty} \frac{\sec(\pi n)}{\sqrt{n}}$

Solution. Note that $\sec(\pi n) = (-1)^n$ for any integer n. So

$$\sum_{n=2}^{\infty} \frac{\sec(\pi n)}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}},$$

and the series is alternating. Let us use the AST. The sequence $a_n = \frac{1}{\sqrt{n}}$ is positive, decreasing (since $\sqrt{n}$ is increasing) and $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. Therefore, the AST applies and $\sum\limits_{n=2}^{\infty} \frac{\sec(\pi n)}{\sqrt{n}}$ converges.

We need to determine if the convergence is absolute or conditional, so we consider the series

$$\sum_{n=2}^{\infty} \left| \frac{\sec(\pi n)}{\sqrt{n}} \right| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}.$$

This series is a p-series with $p = \frac{1}{2} \leqslant 1$, so it diverges. In conclusion, $\boxed{\sum\limits_{n=2}^{\infty} \frac{\sec(\pi n)}{\sqrt{n}} \text{ converges conditionally}}$.

(f) $\sum\limits_{n=2}^{\infty} (-1)^n \ln\left(\frac{n+1}{n}\right)$

Solution. Let us use the AST with $a_n = \ln\left(\frac{n+1}{n}\right)$. The sequence a_n has positive terms since $\frac{n+1}{n} > 1$, so $\ln\left(\frac{n+1}{n}\right) > \ln(1) = 0$. The sequence a_n is decreasing since

$$\frac{d}{dx} \ln\left(\frac{x+1}{x}\right) = \frac{d}{dx}\left(\ln(x+1) - \ln(x)\right) = \frac{1}{x+1} - \frac{1}{x} = -\frac{1}{x(x+1)} < 0 \quad \text{for} \quad x > 0.$$

Finally, observe that

$$\lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right) = \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right) = \ln(1+0) = \ln(1) = 0.$$

Therefore, the AST applies and $\displaystyle\sum_{n=2}^{\infty} (-1)^n \ln\left(\frac{n+1}{n}\right)$ converges.

We need to determine if the convergence is absolute or conditional, so we consider the series

$$\sum_{n=2}^{\infty} \left|(-1)^n \ln\left(\frac{n+1}{n}\right)\right| = \sum_{n=2}^{\infty} \ln\left(\frac{n+1}{n}\right) = \sum_{n=2}^{\infty} (\ln(n+1) - \ln(n)).$$

This series looks telescoping, and inspecting the partial sums, we see that

$$\begin{aligned} S_N &= \sum_{n=2}^{N} (\ln(n+1) - \ln(n)) \\ &= (\ln(3) - \ln(2)) + (\ln(4) - \ln(3)) + \cdots + (\ln(N) - \ln(N-1)) + (\ln(N+1) - \ln(N)) \\ &= \ln(N+1) - \ln(2). \end{aligned}$$

Therefore,

$$\sum_{n=2}^{\infty} (\ln(n+1) - \ln(n)) = \lim_{N\to\infty} S_N = \lim_{N\to\infty} (\ln(N+1) - \ln(2)) = \infty.$$

So $\displaystyle\sum_{n=2}^{\infty} (\ln(n+1) - \ln(n))$ diverges, and $\boxed{\displaystyle\sum_{n=2}^{\infty} (-1)^n \ln\left(\frac{n+1}{n}\right) \text{ converges conditionally}}$.

(g) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{e^{\sqrt{n}}}$

Solution. This series is a bit tricky because it is not geometric. Indeed, the exponent of e is $\sqrt{n}$, and not n. The Root Test is also inconclusive since

$$\lim_{n\to\infty} \left(\frac{1}{e^{\sqrt{n}}}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{e^{\sqrt{n}/n}} = \lim_{n\to\infty} \frac{1}{e^{1/\sqrt{n}}} = \frac{1}{e^0} = 1.$$

Still, we expect the series to converge because $e^{\sqrt{n}}$ grows faster than any power of n (even though it grows slower than the exponential e^n). This hints that we might be able to prove convergence by comparing with a convergent p-series.

So let us use the LCT with $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which converges as a p-series with $p = \frac{3}{2} > 1$. We have

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{a_n}{b_n} \\ &= \lim_{n\to\infty} \frac{\frac{1}{e^{\sqrt{n}}}}{\frac{1}{n^{3/2}}} \\ &= \lim_{x\to\infty} \frac{x^{3/2}}{e^{\sqrt{x}}} \end{aligned}$$

$$
\begin{aligned}
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \frac{\frac{3}{2}x^{1/2}}{\frac{1}{2}x^{-1/2}e^{\sqrt{x}}} \\
&= \lim_{x\to\infty} \frac{3x}{e^{\sqrt{x}}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \frac{3}{\frac{1}{2}x^{-1/2}e^{\sqrt{x}}} \\
&= \lim_{x\to\infty} \frac{6\sqrt{x}}{e^{\sqrt{x}}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \frac{3x^{-1/2}}{\frac{1}{2}x^{-1/2}e^{\sqrt{x}}} \\
&= \lim_{x\to\infty} \frac{6}{e^{\sqrt{x}}} \\
&= 0.
\end{aligned}
$$

Since $L = 0$ and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, we conclude that $\boxed{\sum_{n=0}^{\infty} \frac{1}{e^{\sqrt{n}}} \text{ converges absolutely}}$.

(h) $\sum_{n=0}^{\infty} (-1)^n \frac{n}{2n+1}$

Solution. This series is alternating, but the AST does not apply. Indeed, we have

$$\lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2}.$$

It follows that $\lim_{n\to\infty} (-1)^n \frac{n}{2n+1}$ does not exists. Therefore, the Term Divergence Test tells us that $\boxed{\sum_{n=0}^{\infty} (-1)^n \frac{n}{2n+1} \text{ diverges}}$.

(i) $\sum_{n=3}^{\infty} \cos\left(\frac{\pi}{n}\right)^{n^2}$

Solution. Given the exponent n^2, the Root Test is tempting here, but it would turn out to be inconclusive (try it). Let us try to directly compute the limit of the general term to see if the Term Divergence Test would apply. The limit $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right)^{n^2}$ is an indeterminate power 1^{∞}. Let us write it as

$$\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right)^{n^2} = \lim_{n\to\infty} e^{n^2 \ln\left(\cos\left(\frac{\pi}{n}\right)\right)}$$

and compute the limit of the exponent using L'Hôpital's Rule. We have

$$
\begin{aligned}
\lim_{n\to\infty} n^2 \ln\left(\cos\left(\frac{\pi}{n}\right)\right) &= \lim_{x\to\infty} \frac{\ln\left(\cos\left(\frac{\pi}{x}\right)\right)}{\frac{1}{x^2}} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty} \frac{\frac{\pi}{x^2}\tan\left(\frac{\pi}{x}\right)}{-\frac{2}{x^3}} \\
&= \lim_{x\to\infty} -\frac{\pi\tan\left(\frac{\pi}{x}\right)}{\frac{2}{x}}
\end{aligned}
$$

$$\begin{aligned}
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty} -\frac{-\frac{\pi^2}{x^2}\sec\left(\frac{\pi}{x}\right)^2}{-\frac{2}{x^2}} \\
&= \lim_{x\to\infty} -\frac{\pi^2}{2}\sec\left(\frac{\pi}{x}\right)^2 \\
&= -\frac{\pi^2}{2}\sec(0)^2 \\
&= -\frac{\pi^2}{2}.
\end{aligned}$$

So

$$\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right)^{n^2} = \lim_{n\to\infty} e^{n^2\ln\left(\cos\left(\frac{\pi}{n}\right)\right)} = e^{-\pi^2/2}.$$

Since this limit is not equal to zero, the Term Divergence Test tells us that $\boxed{\sum_{n=3}^{\infty} \cos\left(\frac{\pi}{n}\right)^{n^2} \text{ diverges}}$.

2. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{7n+4}}$.

(a) Show that this series meets the conditions of the Alternating Series Estimation Theorem.

Solution. The sequence $a_n = \frac{1}{\sqrt[3]{7n+4}}$ satisfies the following conditions.

- $\frac{1}{\sqrt[3]{7n+4}} > 0$ for any $n \geqslant 1$.
- The sequence $a_n = \frac{1}{\sqrt[3]{7n+4}}$ is decreasing since $\sqrt[3]{7n+4}$ is increasing.
- $\lim_{n\to\infty} \frac{1}{\sqrt[3]{7n+4}} = 0$.

Therefore, $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{7n+4}} \text{ meets the conditions of the Alternating Series Estimation Theorem}}$.

(b) Find the smallest integer N for which the partial sum $S_N = \displaystyle\sum_{n=1}^{N} \frac{(-1)^n}{\sqrt[3]{7n+4}}$ approximates the sum of the series with an error of at most 0.1.

Solution. The Alternating Series Estimation Theorem tells us that the best estimate for the error is $|S - S_N| \leqslant a_{N+1}$. Therefore, we will want $a_{N+1} \leqslant 0.1$. This gives

$$\begin{aligned}
&\frac{1}{\sqrt[3]{7(N+1)+4}} \leqslant 0.1 \\
&\Rightarrow \sqrt[3]{7N+11} \geqslant 10 \\
&\Rightarrow 7N + 11 \geqslant 1000 \\
&\Rightarrow 7N \geqslant 989 \\
&\Rightarrow N \geqslant \frac{989}{7} \simeq 141.3.
\end{aligned}$$

Therefore, the smallest value of N giving us the desired error is $\boxed{N = 142}$.

3. Consider the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{3n-7}+9}$.

(a) Show that this series meets the conditions of the Alternating Series Estimation Theorem.

Solution. The sequence $a_n = \frac{1}{2^{3n-7}+9}$ satisfies the following conditions.

- $\frac{1}{2^{3n-7}+9} > 0$ for any $n \geqslant 0$.
- The sequence $a_n = \frac{1}{2^{3n-7}+9}$ is decreasing since $2^{3n-7}+11$ is increasing.
- $\lim\limits_{n\to\infty} \frac{1}{2^{3n-7}+9} = 0$.

Therefore, $\boxed{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{3n-7}+9} \text{ meets the conditions of the Alternating Series Estimation Theorem}}$.

(b) Find the smallest integer N for which the partial sum $S_N = \displaystyle\sum_{n=0}^{N} \frac{(-1)^{n+1}}{2^{3n-7}+9}$ approximates the sum of the series with an error of at most 10^{-3}.

Solution. The Alternating Series Estimation Theorem tells us that the best estimate for the error is $|S - S_N| \leqslant a_{N+1}$. Therefore, we will want $a_{N+1} \leqslant 10^{-3}$. This gives

$$\begin{aligned}
&\frac{1}{2^{3(N+1)-7}+9} \leqslant 10^{-3} \\
&\Rightarrow\ 2^{3N-4}+9 \geqslant 1000 \\
&\Rightarrow 2^{3N-4} \geqslant 991 \\
&\Rightarrow 3N-4 \geqslant 10 \quad (2^9 = 512, 2^{10} = 1024) \\
&\Rightarrow N \geqslant \frac{14}{3} \simeq 4.7.
\end{aligned}$$

Therefore, the smallest value of N giving us the desired error is $\boxed{N = 5}$.

Rutgers University
Math 152

Section 10.7: Power Series - Worksheet Solutions

1. Find the radius and interval of convergence of the power series below. Specify for which values of x in the interval of convergence the series converges absolutely and for which it converges conditionally.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{\sqrt[3]{n}5^n}$.

Solution. We use the Ratio Test to find the radius of convergence. We have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{\sqrt[3]{n+1}\ 5^{n+1}} \cdot \frac{\sqrt[3]{n}\ 5^n}{(x-3)^n} \right| \\
&= \lim_{n\to\infty} \frac{|x-3|\sqrt[3]{n}}{5\sqrt[3]{n+1}} \cdot \frac{\frac{1}{\sqrt[3]{n}}}{\frac{1}{\sqrt[3]{n}}} \\
&= \lim_{n\to\infty} \frac{|x-3|}{5\sqrt[3]{1+\frac{1}{n}}} \\
&= \frac{|x-3|}{5}.
\end{aligned}$$

The power series converges absolutely when $\rho < 1$, that is

$$\frac{|x-3|}{5} < 1 \ \Rightarrow\ |x-3| < 5 \ \Rightarrow\ -5 < x-3 < 5 \ \Rightarrow -2 < x < 8.$$

Therefore the radius of convergence is $\boxed{R = 5}$.

To find the interval of convergence, we need to determine if the power series converges at the endpoints $x = -2, 8$.

- At $x = -2$, the power series becomes

$$\sum_{n=1}^{\infty} \frac{(-2-3)^n}{\sqrt[3]{n}5^n} = \sum_{n=1}^{\infty} \frac{(-5)^n}{\sqrt[3]{n}5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}.$$

This series converges by the AST since $a_n = \frac{1}{\sqrt[3]{n}}$ is positive, decreasing and converges to 0. However, the series does not converge absolutely since

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt[3]{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}},$$

which is a divergent p-series with $p = \frac{1}{3} \leqslant 1$. Therefore, the power series converges conditionally at $x = -2$.

- At $x = 8$, the power series becomes

$$\sum_{n=1}^{\infty} \frac{(8-3)^n}{\sqrt[3]{n}5^n} = \sum_{n=1}^{\infty} \frac{5^n}{\sqrt[3]{n}5^n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}.$$

  As we noted above, this is a divergent p-series with $p = \frac{1}{3}$.

In conclusion, the IOC is $\boxed{[-2, 8)}$. Furthermore, the power series $\boxed{\text{converges absolutely on } (-2, 8)}$ and $\boxed{\text{converges conditionally at } x = -2}$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x-9)^{3n}}{8^n (n+1)}$.

Solution. We use the Ratio Test to find the radius of convergence. We have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-9)^{3(n+1)}}{8^{n+1}(n+2)} \cdot \frac{8^n(n+1)}{(-1)^n(x-9)^{3n}} \right| \\
&= \lim_{n\to\infty} \frac{|x-9|^3(n+1)}{8(n+2)} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{|x-9|^3\left(1+\frac{1}{n}\right)}{8\left(1+\frac{2}{n}\right)} \\
&= \frac{|x-9|^3}{8}.
\end{aligned}$$

The power series converges absolutely when $\rho < 1$, that is

$$\frac{|x-9|^3}{8} < 1 \;\Rightarrow\; |x-9| < 2 \;\Rightarrow\; -2 < x-9 < 2 \;\Rightarrow 7 < x < 11.$$

Therefore the radius of convergence is $\boxed{R = 9}$.

To find the interval of convergence, we need to determine if the power series converges at the endpoints $x = 7, 11$.

- At $x = 7$, the power series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n(7-9)^{3n}}{8^n(n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n(-2)^{3n}}{8^n(n+1)} = \sum_{n=0}^{\infty} \frac{8^n}{8^n(n+1)} = \sum_{n=0}^{\infty} \frac{1}{n+1}.$$

  This series is a divergent p-series with $p = 1$. Therefore, the power series converges diverges at $x = 7$.

- At $x = 8$, the power series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n(11-9)^{3n}}{8^n(n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{3n}}{8^n(n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n 8^n}{8^n(n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}.$$

  This series converges by the AST since $a_n = \frac{1}{n+1}$ is positive, decreasing and converges to 0. However, the series does not converge absolutely since

$$\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{n+1} \right| = \sum_{n=0}^{\infty} \frac{1}{n+1},$$

which is a divergent p-series with $p = 1$ as noted above. Therefore, the power series converges conditionally at $x = 11$.

In conclusion, the IOC is $\boxed{(9, 11]}$. Furthermore, the power series $\boxed{\text{converges absolutely on } (9, 11)}$ and $\boxed{\text{converges conditionally at } x = 11}$.

(c) $\displaystyle\sum_{n=0}^{\infty} n3^n(2x+1)^n$.

Solution. We use the Ratio Test to find the radius of convergence. We have

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(n+1)3^{n+1}(2x+1)^{n+1}}{n3^n(2x+1)^n}\right| \\ &= \lim_{n\to\infty} 3\left(1+\frac{1}{n}\right)|2x+1| \\ &= 3|2x+1|.\end{aligned}$$

The power series converges absolutely when $\rho < 1$, that is

$$3|2x+1| < 1 \;\Rightarrow\; \left|x+\frac{1}{2}\right| < \frac{1}{6} \;\Rightarrow\; -\frac{1}{6} < x + \frac{1}{2} < \frac{1}{6} \;\Rightarrow -\frac{2}{3} < x < -\frac{1}{3}.$$

Therefore the radius of convergence is $\boxed{R = \frac{1}{6}}$.

To find the interval of convergence, we need to determine if the power series converges at the endpoints $x = -\frac{2}{3}, -\frac{1}{3}$.

- At $x = -\frac{2}{3}$, the power series becomes

$$\sum_{n=0}^{\infty} n3^n\left(2\left(-\frac{2}{3}\right)+1\right)^n = \sum_{n=0}^{\infty}\sum_{n=0}^{\infty} n3^n\left(-\frac{1}{3}\right)^n = \sum_{n=0}^{\infty}(-1)^n n.$$

  Since $\lim_{n\to\infty}(-1)^n n$ does not exist (and in particular is not equal to 0), this series diverges by the Term Divergence Test. Therefore, the power series diverges at $x = -\frac{2}{3}$.

- At $x = -\frac{1}{3}$, the power series becomes

$$\sum_{n=0}^{\infty} n3^n\left(2\left(-\frac{1}{3}\right)+1\right)^n = \sum_{n=0}^{\infty}\sum_{n=0}^{\infty} n3^n\left(\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} n.$$

  Since $\lim_{n\to\infty} n = \infty$ (and in particular is not equal to 0), this series diverges by the Term Divergence Test. Therefore, the power series diverges at $x = -\frac{1}{3}$.

In conclusion, the IOC is $\boxed{\left(-\frac{2}{3}, -\frac{1}{3}\right)}$. Furthermore, the power series $\boxed{\text{converges absolutely on } \left(-\frac{2}{3}, -\frac{1}{3}\right)}$ and $\boxed{\text{never converges conditionally}}$.

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{n^n(x+2)^n}{6^n}$.

Solution. We use the Root Test to find the radius of convergence. We have

$$\begin{aligned}\rho &= \lim_{n\to\infty} |a_n|^{1/n} \\ &= \lim_{n\to\infty} \left|\frac{n^n(x+2)^n}{6^n}\right|^{1/n} \\ &= \lim_{n\to\infty} \frac{n|x+2|}{6} \\ &= \begin{cases} 0 & \text{if } x=-2, \\ \infty & \text{if } x\neq -2. \end{cases}\end{aligned}$$

Therefore, the series converges conditionally for $x=-2$ and diverges otherwise. So the radius of convergence is $\boxed{R=0}$ and the IOC is $\{-2\}$. The power series $\boxed{\text{converges absolutely at } x=-2}$ and $\boxed{\text{never converges conditionally}}$.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(x-4)^{2n}}{36^n\sqrt{n}}$.

Solution. We use the Ratio Test to find the radius of convergence. We have

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty} \left|\frac{(-1)^{n+1}(x-4)^{2(n+1)}}{36^{n+1}\sqrt{n+1}} \cdot \frac{36^n\sqrt{n}}{(-1)^n(x-4)^{2n}}\right| \\ &= \lim_{n\to\infty} \frac{|x-4|^2\sqrt{n}}{36\sqrt{n+1}} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} \\ &= \lim_{n\to\infty} \frac{|x-4|^2}{36\sqrt{1+\frac{1}{n}}} \\ &= \frac{|x-4|^2}{36}.\end{aligned}$$

The power series converges absolutely when $\rho<1$, that is

$$\frac{|x-4|^2}{36} < 1 \;\Rightarrow\; |x-4|<6 \;\Rightarrow\; -6<x-4<6 \;\Rightarrow -2<x<10.$$

Therefore the radius of convergence is $\boxed{R=6}$.

To find the interval of convergence, we need to determine if the power series converges at the endpoints $x=-2,10$.

- At $x=-2$, the power series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n(-2-4)^{2n}}{36^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n(-6)^{2n}}{36^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n 36^n}{36^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$

This series converges by the AST since $a_n = \frac{1}{\sqrt{n}}$ is positive, decreasing and converges to 0. However, the series does not converge absolutely since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}},$$

which is a divergent p-series with $p = \frac{1}{2} \leqslant 1$. Therefore, the power series converges conditionally at $x = -2$.

- At $x = 10$, the power series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n (10-4)^{2n}}{36^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n 6^{2n}}{36^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n 36^n}{36^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$

As noted above, this series converges conditionally. Therefore, the power series converges conditionally at $x = 10$.

In conclusion, the IOC is $\boxed{[-2, 10]}$. Furthermore, the power series $\boxed{\text{converges absolutely on } (-2, 10)}$ and $\boxed{\text{converges conditionally at } x = -2, 10}$.

(f) $\displaystyle\sum_{n=0}^{\infty} \frac{(3x+2)^n}{n^2+4}$.

Solution. We use the Ratio Test to find the radius of convergence. We have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n\to\infty} \left| \frac{(3x+2)^{n+1}}{(n+1)^2+4} \cdot \frac{n^2+4}{(3x+2)^n} \right| \\
&= \lim_{n\to\infty} \frac{|3x+2|(n^2+4)}{n^2+2n+5} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \\
&= \lim_{n\to\infty} \frac{|3x+2|\left(1+\frac{4}{n^2}\right)}{1+\frac{2}{n}+\frac{5}{n^2}} \\
&= |3x+2|.
\end{aligned}$$

The power series converges absolutely when $\rho < 1$, that is

$$|3x+2| < 1 \;\Rightarrow\; \left| x + \frac{2}{3} \right| < \frac{1}{3} \;\Rightarrow\; -\frac{1}{3} < x + \frac{2}{3} < \frac{1}{3} \;\Rightarrow -1 < x < -\frac{1}{3}.$$

Therefore the radius of convergence is $\boxed{R = \frac{1}{3}}$.

To find the interval of convergence, we need to determine if the power series converges at the endpoints $x = -1, -\dfrac{1}{3}$.

- At $x = -1$, the power series becomes

$$\sum_{n=0}^{\infty} \frac{(3(-1)+2)^n}{n^2+4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+4}.$$

We can show that this series converges absolutely using the DCT. We have

$$0 \leqslant \left|\frac{(-1)^n}{n^2+4}\right| \leqslant \frac{1}{n^2},$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges as a p-series with $p = 2 > 1$. Therefore, the power series converges absolutely at $x = -1$.

- At $x = -\frac{1}{3}$, the power series becomes

$$\sum_{n=0}^{\infty} \frac{\left(3\left(-\frac{1}{3}\right)+2\right)^n}{n^2+4} = \sum_{n=0}^{\infty} \frac{1}{n^2+4}.$$

As noted above, this series converges absolutely. Therefore, the power series converges absolutely at $x = -\frac{1}{3}$.

In conclusion, the IOC is $\boxed{\left[-1, -\frac{1}{3}\right]}$. Furthermore, the power series $\boxed{\text{converges absolutely on } \left[-1, -\frac{1}{3}\right]}$ and $\boxed{\text{never converges conditionally}}$.

2. Find the radius of convergence of the following power series.

(a) $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} x^{2n}$.

Solution. We use the Ratio Test. We have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \left|\frac{((n+1)!)^2 x^{2(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{(n!)^2 x^{2n}}\right| \\
&= \lim_{n\to\infty} |x|^2 \left(\frac{(n+1)!}{n!}\right)^2 \frac{(2n)!}{(2n+2)!} \\
&= \lim_{n\to\infty} |x|^2 \frac{(n+1)^2}{(2n+1)(2n+2)} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \\
&= \lim_{n\to\infty} |x|^2 \frac{\left(1+\frac{1}{n}\right)^2}{\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)} \\
&= \frac{|x|^2}{4}.
\end{aligned}$$

The power series converges absolutely when $\rho < 1$, that is $|x|^2 < 4$, or $|x| < 2$. Therefore, the radius of convergence is $\boxed{R = 2}$.

(b) $\sum_{n=1}^{\infty} \left(1 - \frac{3}{n}\right)^{n^2} (x+5)^n$.

Solution. We use the Root Test. We have

$$\begin{aligned}\rho &= \lim_{n\to\infty} |a_n|^{1/n} \\ &= \lim_{n\to\infty} \left| \left(1-\frac{3}{n}\right)^{n^2} (x+5)^n \right|^{1/n} \\ &= \lim_{n\to\infty} |x+5| \left(1-\frac{3}{n}\right)^{n} \\ &= |x+5| \lim_{n\to\infty} \left(1-\frac{3}{n}\right)^{n}.\end{aligned}$$

This last limit is an indeterminate exponent 1^∞. To compute it, we can write the expression in base e

$$\lim_{n\to\infty} \left(1-\frac{3}{n}\right)^{n} = \lim_{n\to\infty} e^{n\ln\left(1-\frac{3}{n}\right)}$$

and compute the limit of the exponent using L'Hôpital's Rule. This gives

$$\begin{aligned}\lim_{n\to\infty} n\ln\left(1-\frac{3}{n}\right) &= \lim_{x\to\infty} \frac{\ln\left(1-\frac{3}{x}\right)}{\frac{1}{x}} \\ &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty} \frac{-\frac{3}{x^2}\cdot\frac{1}{1+\frac{3}{x}}}{-\frac{1}{x^2}} \\ &= \lim_{x\to\infty} -\frac{3}{1-\frac{3}{x}} \\ &= -3.\end{aligned}$$

Therefore

$$\rho = |x+5| \lim_{n\to\infty} e^{n\ln\left(1-\frac{3}{n}\right)} = |x+5|e^{-3}.$$

The power series converges absolutely when $\rho < 1$, that is $|x+5|e^{-3} < 1$, or $|x+5| < e^3$. Therefore, the radius of convergence is $\boxed{R = e^3}$.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{n^n} x^n.$

Solution. We use the Ratio Test. We have

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty} \left|\frac{(n+1)!x^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{n!x^n}\right| \\ &= \lim_{n\to\infty} |x| \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} \\ &= \lim_{n\to\infty} |x|(n+1)\frac{n^n}{(n+1)^{n+1}} \\ &= \lim_{n\to\infty} |x| \frac{n^n}{(n+1)^n} \\ &= |x| \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n.\end{aligned}$$

This last limit is an indeterminate exponent 1^{∞}. To compute it, we can write the expression in base e

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n}=\lim_{n\to\infty}e^{n\ln\left(\frac{n}{n+1}\right)}$$

and compute the limit of the exponent using L'Hôpital's Rule. This gives

$$\begin{aligned}\lim_{n\to\infty}n\ln\left(\frac{n}{n+1}\right)&=\lim_{x\to\infty}\frac{\ln(x)-\ln(x+1)}{\frac{1}{x}}\\&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}}\lim_{x\to\infty}\frac{\frac{1}{x}-\frac{1}{x+1}}{-\frac{1}{x^2}}\\&=\lim_{x\to\infty}-\frac{x^2}{x(x+1)}\\&=\lim_{x\to\infty}-\frac{1}{1+1/x}\\&=-1.\end{aligned}$$

Therefore

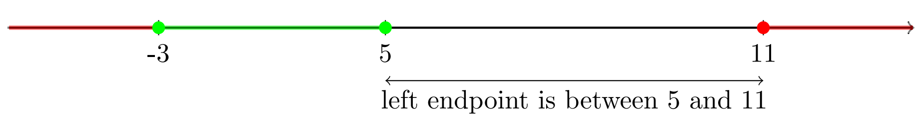
$$\rho=|x|\lim_{n\to\infty}e^{n\ln\left(\frac{n}{n+1}\right)}=|x|e^{-1}.$$

The power series converges absolutely when $\rho<1$, that is $|x|e^{-1}<1$, or $|x|<e$. Therefore, the radius of convergence is $\boxed{R=e}$.

3. Suppose that a power series converges absolutely at $x=5$, converges conditionally at $x=-3$ and diverges at $x=11$. What can you say, if anything, about the convergence or divergence of the power series at the following values of x?

   (a) $x=-4$. (b) $x=2$. (c) $x=15$. (d) $x=7$.

   Solution. We know that a power series converges absolutely in the interior of its interval of convergence. Therefore, we know that $x=-3$ is an endpoint of the interval of convergence. Since $x=5$ is in the interval of convergence, we know that $x=-3$ is the left endpoint of the interval of convergence and the right endpoint of the interval of convergence is at least 5. Since the power series diverges at $x=11$, we know that the right endpoint of the interval of convergence must be less than 11. These observations are summarized on the figure below, where red indicates divergence and green indicates convergence.



   We conclude that

   (a) $\boxed{\text{the power series diverges at } x=-4}$.

   (b) $\boxed{\text{the power series converges at } x=2}$.

   (c) $\boxed{\text{the power series diverges at } x=15}$.

   (d) $\boxed{\text{We cannot conclude anything about the behavior of the power series at } x=7}$.

4. Let $f(x) = \dfrac{3}{2+7x}$. Use the power series representation of $\dfrac{1}{1-x}$ and power series operations to find a power series representation of $f(x)$ centered at $a = 0$. What are the radius and interval of convergence of the resulting power series?

Solution. A bit of algebra will help us make $\frac{3}{2+7x}$ look like $\frac{1}{1-x}$ up to a suitable substitution. Namely, if we factor 3 from the numerator and 2 from the denominator, we get

$$\frac{3}{2+7x} = \frac{3}{2} \cdot \frac{1}{1+\frac{7x}{2}} = \frac{3}{2} \cdot \frac{1}{1-\left(-\frac{7x}{2}\right)} = \frac{3}{2} \cdot \frac{1}{1-u},$$

with $u = -\frac{7x}{2}$. For $|u| < 1$, we know that $\frac{1}{1-u}$ is the sum of a geometric series of common ratio u and first term 1, that is:

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n.$$

So

$$\frac{3}{2+7x} = \frac{3}{2} \cdot \frac{1}{1-\left(-\frac{7x}{2}\right)} = \frac{3}{2}\sum_{n=0}^{\infty} \left(-\frac{7x}{2}\right)^n = \boxed{\sum_{n=0}^{\infty} \frac{3(-7)^n x^n}{2^{n+1}}}.$$

This geometric series converges when the common ratio $r = -\frac{7x}{2}$ satisfies $|r| < 1$. This gives

$$\left|-\frac{7x}{2}\right| < 1 \ \Rightarrow\ |x| < \frac{2}{7} \ \Rightarrow\ -\frac{2}{7} < x < \frac{2}{7}.$$

Therefore, the radius of convergence is $\boxed{R = \frac{2}{7}}$ and the interval of convergence is $\boxed{\left(-\frac{2}{7}, \frac{2}{7}\right)}$.

5. Consider the power series $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(x+1)^n}{3^n(n+1)}$.

(a) Find the radius and interval of convergence of f.

Solution. We use the Ratio Test to find the radius of convergence. We have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \left|\frac{(x+1)^{n+1}}{3^{n+1}(n+2)} \cdot \frac{3^n(n+1)}{(x+1)^n}\right| \\
&= \lim_{n\to\infty} \frac{|x+1|(n+1)}{3(n+2)} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{|x+1|\left(1+\frac{1}{n}\right)}{3\left(1+\frac{2}{n}\right)} \\
&= \frac{|x+1|}{3}.
\end{aligned}$$

The power series converges absolutely when $\rho < 1$, that is

$$\frac{|x+1|}{3} < 1 \ \Rightarrow\ |x+1| < 3 \ \Rightarrow\ -3 < x+1 < 3 \ \Rightarrow -4 < x < 2.$$

Therefore the radius of convergence is $\boxed{R = 3}$.

To find the interval of convergence, we need to determine if the power series converges at the endpoints $x = -4, 2$.

- At $x = -4$, the power series becomes

$$\sum_{n=0}^{\infty} \frac{(-4+1)^n}{3^n(n+1)} = \sum_{n=0}^{\infty} \frac{(-3)^n}{3^n(n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}.$$

  This series converges by the AST since $a_n = \frac{1}{n+1}$ is positive, decreasing and converges to 0. Therefore, the power series converges at $x = -4$.

- At $x = 2$, the power series becomes

$$\sum_{n=0}^{\infty} \frac{(2+1)^n}{3^n(n+1)} = \sum_{n=0}^{\infty} \frac{3^n}{3^n(n+1)} = \sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n}.$$

  This is a divergent p-series with $p = 1$. Therefore, the power series converges diverges at $x = 2$.

In conclusion, the IOC is $\boxed{[-4, 2)}$.

(b) Find a power series representation of $f'(x)$ centered at $a = -1$. What are its radius and interval of convergence?

Solution. Differentiating term-by-term gives

$$\begin{aligned} f'(x) &= \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{(x+1)^n}{3^n(n+1)}\right) \\ &= \sum_{n=0}^{\infty} \frac{n(x+1)^{n-1}}{3^n(n+1)} \\ &= \boxed{\sum_{n=1}^{\infty} \frac{n(x+1)^{n-1}}{3^n(n+1)}}. \end{aligned}$$

We know that the radius of convergence does not change when differentiating term-by-term, so $\boxed{R = 3}$. The interval, however, may change and we need to test the endpoints to determine it.

- At $x = -4$, the power series becomes

$$\sum_{n=1}^{\infty} \frac{n(-4+1)^{n-1}}{3^n(n+1)} = \sum_{n=1}^{\infty} \frac{n(-3)^{n-1}}{3^n(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{3(n+1)}.$$

  This series diverges by the Term Divergence Test since

$$\lim_{n\to\infty} \frac{n}{3(n+1)} = \lim_{n\to\infty} \frac{1}{3(1+1/n)} = \frac{1}{3},$$

  and therefore $\lim\limits_{n\to\infty} \frac{(-1)^{n-1}n}{3(n+1)}$ does not exist. Thus, the power series diverges at $x = -4$.

- At $x = 2$, the power series becomes

$$\sum_{n=1}^{\infty} \frac{n(2+1)^{n-1}}{3^n(n+1)} = \sum_{n=1}^{\infty} \frac{n3^{n-1}}{3^n(n+1)} = \sum_{n=1}^{\infty} \frac{n}{3(n+1)}.$$

This series diverges by the Term Divergence Test since

$$\lim_{n\to\infty} \frac{n}{3(n+1)} = \lim_{n\to\infty} \frac{1}{3(1+1/n)} = \frac{1}{3}.$$

Thus, the power series diverges at $x = 2$.

In conclusion, the IOC is $\boxed{(-4, 2)}$.

(c) Let $g(x)$ be the antiderivative of $f(x)$ such that $g(-1) = -8$. Find a power series representation of $g(x)$ centered at $a = -1$. What are its radius and interval of convergence?

Solution. Integrating term-by-term gives

$$\begin{aligned} g(x) &= \sum_{n=0}^{\infty} \int \frac{(x+1)^n}{3^n(n+1)} dx \\ &= \sum_{n=0}^{\infty} \frac{(x+1)^{n+1}}{3^n(n+1)^2} + C. \end{aligned}$$

To find C, we use $g(-1) = -8$, which gives

$$\sum_{n=0}^{\infty} \frac{(-1+1)^{n+1}}{3^n(n+1)^2} + C = -8 \Rightarrow C = -8.$$

Therefore,

$$\boxed{g(x) = -8 + \sum_{n=0}^{\infty} \frac{(x+1)^{n+1}}{3^n(n+1)^2}}.$$

We know that the radius of convergence does not change when differentiating term-by-term, so $\boxed{R = 3}$. The interval, however, may change and we need to test the endpoints to determine it.

- At $x = -4$, the power series becomes

$$\sum_{n=0}^{\infty} \frac{(-4+1)^{n+1}}{3^n(n+1)^2} = \sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{3^n(n+1)^2} = \sum_{n=0}^{\infty} \frac{3(-1)^n}{(n+1)^2}.$$

  This series converges absolutely since

$$\sum_{n=0}^{\infty} \left| \frac{3(-1)^n}{(n+1)^2} \right| = 3\sum_{n=1}^{\infty} \frac{1}{n^2},$$

  which is a convergent p-series with $p = 2 > 1$. So the power series converges at $x = -4$.

- At $x = 2$, the power series becomes

$$\sum_{n=0}^{\infty} \frac{(2+1)^{n+1}}{3^n(n+1)^2} = \sum_{n=0}^{\infty} \frac{3^{n+1}}{3^n(n+1)^2} = \sum_{n=0}^{\infty} \frac{3}{(n+1)^2} = 3\sum_{n=1}^{\infty} \frac{1}{n^2},$$

  which is a convergent p-series with $p = 2 > 1$. So the power series converges at $x = 2$.

In conclusion, the IOC is $\boxed{[-4, 2]}$.

6. (a) Use term-by-term differentiation to find a power series representation of $\frac{1}{(1-x)^2}$. What is its raidus of convergence?

Solution. We have the power series representation

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

with radius of convergence $R = 1$. Differentiating term-by-term gives

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{n=0}^{\infty} \frac{d}{dx} x^n$$
$$\Rightarrow \boxed{\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}}.$$

The radius of convergence does not change when differentiating term-by-term, so $\boxed{R = 1}$.

(b) Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{5^n}$.

Solution. Observe that

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{5^n} = \sum_{n=1}^{\infty} n\left(-\frac{1}{5}\right)^n = -\frac{1}{5}\sum_{n=1}^{\infty} n\left(-\frac{1}{5}\right)^{n-1}.$$

Using the power series representation from the previous part with $x = -\frac{1}{5}$, we get

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{5^n} = -\frac{1}{5} \cdot \frac{1}{\left(1-\left(-\frac{1}{5}\right)\right)^2}$$
$$= \boxed{-\frac{5}{36}}.$$

Rutgers University
Math 152

Section 10.8: Taylor and Maclaurin Series - Worksheet Solutions

1. Find the Taylor polynomials for the following functions at the order and center indicated.

   (a) $f(x) = 2\cos\left(\frac{\pi}{3} - 5x\right)$, $T_4(x)$ at $a = 0$.

   Solution. We have

$$\begin{aligned}
f(x) = 2\cos\left(\frac{\pi}{3} - 5x\right) &\Rightarrow c_0 = f(0) = 1, \\
f'(x) = 10\sin\left(\frac{\pi}{3} - 5x\right) &\Rightarrow c_1 = f'(0) = 5\sqrt{3}, \\
f''(x) = -50\cos\left(\frac{\pi}{3} - 5x\right) &\Rightarrow c_2 = \frac{f''(0)}{2!} = -\frac{25}{2}, \\
f^{(3)}(x) = -250\sin\left(\frac{\pi}{3} - 5x\right) &\Rightarrow c_3 = \frac{f^{(3)}(0)}{3!} = -\frac{125\sqrt{3}}{6}, \\
f^{(4)}(x) = 1250\cos\left(\frac{\pi}{3} - 5x\right) &\Rightarrow c_4 = \frac{f^{(4)}(0)}{4!} = \frac{625}{24}.
\end{aligned}$$

   Thus

$$\boxed{T_4(x) = 1 + 5\sqrt{3}x - \frac{25x^2}{2} - \frac{125\sqrt{3}x^3}{6} + \frac{625x^4}{24}}.$$

   (b) $f(x) = \sqrt[3]{4 + 2x}$, $T_3(x)$ at $a = 2$.

   Solution. We have

$$\begin{aligned}
f(x) = \sqrt[3]{4 + 2x} &\Rightarrow c_0 = f(2) = 2, \\
f'(x) = \frac{2}{3}(4 + 2x)^{-2/3} &\Rightarrow c_1 = f'(2) = \frac{1}{6}, \\
f''(x) = -\frac{8}{9}(4 + 2x)^{-5/3} &\Rightarrow c_2 = \frac{f''(2)}{2!} = -\frac{1}{72}, \\
f^{(3)}(x) = \frac{80}{27}(4 + 2x)^{-8/3} &\Rightarrow c_3 = \frac{f^{(3)}(2)}{3!} = \frac{5}{2592}.
\end{aligned}$$

   Thus

$$\boxed{T_3(x) = 2 + \frac{1}{6}(x - 2) - \frac{1}{72}(x - 2)^2 + \frac{5}{2592}(x - 2)^3}.$$

(c) $f(x) = 2^{3-x}$, $T_4(x)$ at $a = 1$.

Solution. We have

$$\begin{aligned}
f(x) = 2^{3-x} &\Rightarrow c_0 = f(1) = 4, \\
f'(x) = -\ln(2)2^{3-x} &\Rightarrow c_1 = f'(1) = -4\ln(2), \\
f''(x) = \ln(2)^2 2^{3-x} &\Rightarrow c_2 = \frac{f''(1)}{2!} = 2\ln(2)^2, \\
f^{(3)}(x) = -\ln(2)^3 2^{3-x} &\Rightarrow c_3 = \frac{f^{(3)}(1)}{3!} = -\frac{2\ln(2)^3}{3}, \\
f^{(4)}(x) = \ln(2)^4 2^{3-x} &\Rightarrow c_4 = \frac{f^{(4)}(1)}{4!} = \frac{\ln(2)^4}{6}.
\end{aligned}$$

Thus

$$\boxed{T_4(x) = 4 - 4\ln(2)(x-1) + 2\ln(2)^2(x-1)^2 - \frac{2\ln(2)^3}{3}(x-1)^3 + \frac{\ln(2)^4}{6}(x-1)^4}.$$

(d) $f(x) = \ln(\cos(x))$, $T_3(x)$ at $a = \frac{\pi}{4}$.

Solution. We have

$$\begin{aligned}
f(x) = \ln(\cos(x)) &\Rightarrow c_0 = f\left(\frac{\pi}{4}\right) = -\frac{\ln(2)}{2}, \\
f'(x) = -\frac{\cos(x)}{\sin(x)} = -\tan(x) &\Rightarrow c_1 = f'\left(\frac{\pi}{4}\right) = -1, \\
f''(x) = -\sec(x)^2 &\Rightarrow c_2 = \frac{f''\left(\frac{\pi}{4}\right)}{2!} = -1, \\
f^{(3)}(x) = -2\sec(x)^2\tan(x) &\Rightarrow c_3 = \frac{f^{(3)}\left(\frac{\pi}{4}\right)}{3!} = -\frac{2}{3}.
\end{aligned}$$

Thus

$$\boxed{T_3(x) = -\frac{\ln(2)}{2} - \left(x - \frac{\pi}{4}\right) - \left(x - \frac{\pi}{4}\right)^2 - \frac{2}{3}\left(x - \frac{\pi}{4}\right)^3}.$$

(e) $f(x) = \frac{6}{5-3x}$, $T_4(x)$ at $a = 1$.

Solution. We can use the formula for the sum of a geometric series to find this Taylor polynomial. We start by finding the Taylor series, and we then keep the terms of degrees 0 through 4 to obtain the Taylor polynomial. We have

$$\begin{aligned}
\frac{6}{5-3x} &= \frac{6}{5-3(x-1)-3} \\
&= \frac{6}{2-3(x-1)} \\
&= \frac{3}{1-\frac{3(x-1)}{2}} \\
&= \sum_{n=0}^{\infty} 3\left(\frac{3(x-1)}{2}\right)^n \\
&= 3+\frac{9(x-1)}{2}+\frac{27(x-1)^2}{4}+\frac{81(x-1)^3}{8}+\frac{243(x-1)^4}{16}+\cdots.
\end{aligned}$$

Thus

$$\boxed{T_4(x) = 3+\frac{9(x-1)}{2}+\frac{27(x-1)^2}{4}+\frac{81(x-1)^3}{8}+\frac{243(x-1)^4}{16}}.$$

(f) $f(x) = \ln(5+x)$, $T_3(x)$ at $a=-4$.

Solution. We have

$$\begin{aligned}
f(x) = \ln(5+x) &\Rightarrow c_0 = f(-4) = 0, \\
f'(x) = \frac{1}{5+x} &\Rightarrow c_1 = f'(-4) = 1, \\
f''(x) = -\frac{1}{(5+x)^2} &\Rightarrow c_2 = \frac{f''(-4)}{2!} = -\frac{1}{2}, \\
f^{(3)}(x) = \frac{2}{(5+x)^3} &\Rightarrow c_3 = \frac{f^{(3)}(-4)}{3!} = \frac{1}{3}.
\end{aligned}$$

Thus

$$\boxed{T_3(x) = (x+4) - \frac{(x+4)^2}{2} - \frac{(x+4)^3}{3}}.$$

2. In 1.(b), you found the third degree Taylor polynomial of $f(x) = \sqrt[3]{4+2x}$ centered at $a=2$. Use this Taylor polynomial to estimate $\sqrt[3]{8.6}$.

Solution. We first need to find the input x to plug into $f(x)$ in order to get $\sqrt[3]{8.6}$. We want $f(x) = \sqrt[3]{4+2x} = \sqrt[3]{8.6}$, so we will need $4+2x = 8.6$, that is $x = 2.3$ Therefore, the estimate we get is

$$\sqrt[3]{8.6} = f(2.3) \simeq T_3(2.3) = 2+\frac{1}{6}(2.3-2)-\frac{1}{72}(2.3-2)^2+\frac{5}{2592}(2.3-2)^3 \simeq \boxed{2.0488}.$$

3. Consider the function $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{9^n(n+1)}(x-4)^{2n+1}$.

(a) Find the radius and interval of convergence of f.

Solution. We use the Ratio Test. We have

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+3}}{9^{n+1}(n+2)}\cdot\frac{9^n(n+1)}{(-1)^n(x-4)^{2n+1}}\right| \\ &= \lim_{n\to\infty}\frac{|x-4|^2(n+1)}{9(n+2)}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} \\ &= \lim_{n\to\infty}\frac{|x-4|^2\left(1+\frac{1}{n}\right)}{9\left(1+\frac{2}{n}\right)} \\ &= \frac{|x-4|^2}{9}.\end{aligned}$$

The series converges absolutely when $\frac{|x-4|^2}{9}<1$, that is $1<x<7$, and diverges if $x<1$ or $x>7$. We now test the endpoints.

At $x=1$, we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{9^n(n+1)}(1-4)^{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{9^n(n+1)}(-3)^{2n+1}=-3\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}.$$

At $x=7$, we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{9^n(n+1)}(7-4)^{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{9^n(n+1)}3^{2n+1}=3\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}.$$

Both series converge by the AST since $a_n=\frac{1}{n+1}$ is positive, decreasing and converges to 0.

In conclusion, the radius of convergence is $\boxed{R=3}$ and the interval of convergence is $\boxed{[2,7]}$.

(b) Find $f^{(7)}(4)$, $f^{(8)}(4)$ and $f^{(9)}(4)$.

Solution. Since the given series must be the Taylor series of f at $a=4$, the coefficient of $(x-4)^7$ in the series is $\dfrac{f^{(7)}(4)}{7!}$. The term in $(x-4)^7$ is obtained in the series when $2n+1=7$, that is for $n=3$. So looking at the resulting coefficient gives

$$\frac{f^{(7)}(4)}{7!}=\frac{(-1)^3}{9^3(3+1)}.$$

So

$$\boxed{f^{(7)}(4)=-\frac{7!}{4\cdot 9^3}}.$$

Similarly, the coefficient of $(x-4)^8$ in the series is $\dfrac{f^{(8)}(4)}{8!}$. The term in $(x-4)^8$ is obtained in the series when $2n+1=8$. Since this equation has no solution with n being an integer, we deduce that there is no term in $(x-4)^8$ appearing in the series. Therefore

$$\boxed{f^{(8)}(4)=0}.$$

The coefficient of $(x-4)^9$ in the series is $\frac{f^{(9)}(4)}{9!}$. The term in $(x-4)^9$ is obtained in the series when $2n+1=9$, that is for $n=4$. So looking at the resulting coefficient gives

$$\frac{f^{(9)}(4)}{9!} = \frac{(-1)^4}{9^4(4+1)}.$$

So

$$\boxed{f^{(9)}(4) = \frac{9!}{5 \cdot 9^4}}.$$

4. Use the reference Maclaurin series to calculate the Maclaurin series of the following functions.

(a) $f(x) = x^7 \cos(4x^5)$.

Solution. We know

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

So we get

$$\begin{aligned} x^7 \cos(4x^5) &= x^7 \sum_{n=0}^{\infty} \frac{(-1)^n (4x^5)^{2n}}{(2n)!} \\ &= x^7 \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{10n}}{(2n)!} \\ &= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{10n+7}}{(2n)!}}. \end{aligned}$$

(b) $f(x) = e^{-x^3} - 1 + x^3$.

Solution. From

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

we get

$$e^{-x^3} = \sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!} = 1 - x^3 + \frac{x^6}{2} - \frac{x^9}{6} + \cdots.$$

We then see that adding $-1 + x^3$ to e^{-x^3} will cancel out the first two terms of this Maclaurin series, giving

$$\begin{aligned} e^{-x^3} - 1 + x^3 &= \left(1 - x^3 + \frac{x^6}{2} - \frac{x^9}{6} + \cdots\right) - 1 + x^3 \\ &= \frac{x^6}{2} - \frac{x^9}{6} + \cdots \\ &= \boxed{\sum_{n=2}^{\infty} \frac{(-1)^n x^{3n}}{n!}}. \end{aligned}$$

(c) $f(x) = \sin(2x) - 2\tan^{-1}(x)$

Solution. We know that

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots ,$$

so

$$\sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!} = 2x - \frac{8x^3}{6} + \frac{32x^5}{120} + \cdots .$$

On the other hand, we have

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots ,$$

so

$$2\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{2(-1)^n x^{2n+1}}{2n+1} = 2x - \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots .$$

When we subtract these two Maclaurin series, the terms $2x$ will cancel out. We can group together the remaining terms of same degree to obtain

$$\begin{aligned} \sin(2x) - 2\tan^{-1}(x) &= \left(2x - \frac{8x^3}{6} + \frac{32x^5}{120} + \cdots\right) - \left(2x - \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots\right) \\ &= -\left(\frac{8}{6} - \frac{2}{3}\right) x^3 + \left(\frac{32}{120} - \frac{2}{5}\right) x^5 + \cdots \\ &= \boxed{\sum_{n=1}^{\infty} (-1)^n \left(\frac{2^{2n+1}}{(2n+1)!} - \frac{2}{2n+1}\right) x^{2n+1}}. \end{aligned}$$

Rutgers University
Math 152

Section 10.9: Convergence of Taylor Series - Worksheet Solutions

1. Use the Remainder Estimation Theorem to estimate the error made when approximating $f(x) = \sqrt{1+3x}$ by its 2^{nd} degree Maclaurin polynomial $T_2(x)$ on the interval $[0, 0.1]$.

   Solution. The Remainder Estimation Theorem gives us the error bound

$$|R_2(x)| \leqslant \frac{M\,|x|^3}{3!} \leqslant \frac{M(0.1)^3}{6}.$$

   In this estimate, M is any number such that $M \geqslant \left|f^{(3)}(x)\right|$ for x in the interval $[0, 0.1]$. We have

$$\begin{aligned} f'(x) &= \frac{3}{2\sqrt{1+3x}}, \\ f''(x) &= -\frac{9}{4(1+3x)^{3/2}}, \\ f^{(3)}(x) &= \frac{81}{8(1+3x)^{5/2}}. \end{aligned}$$

   Since the smallest value of $1+3x$ is 1 on the interval $[0, 0.1]$, we have $\left|f^{(3)}(x)\right| \leqslant \frac{81}{8}$. So we can choose $M = \frac{81}{8}$ for our error estimation. We obtain the error

$$\frac{81(0.1)^3}{8 \cdot 6} = \boxed{\frac{27}{16000}}.$$

2. Consider the function $f(x) = \dfrac{1}{1+2x}$.

   (a) Find the Maclaurin series of f using geometric series. What are its radius and interval of convergence?

   Solution. We recognize that $f(x)$ is the sum of a geometric series with first term 1 and common ratio $-2x$, so

$$f(x) = \sum_{n=0}^{\infty} (-2x)^n = \boxed{\sum_{n=0}^{\infty} (-2)^n x^n}.$$

   We know that a geometric series converges when its common ratio r satisfies $-1 < r < 1$. So the Maclaurin series of f converges when $-1 < -2x < 1$, which gives $-\frac{1}{2} < x < \frac{1}{2}$. Therefore, the ROC is $\boxed{R = \frac{1}{2}}$ and the IOC is $\boxed{\left(-\frac{1}{2}, \frac{1}{2}\right)}$.

(b) Find the Maclaurin series of f using its definition. (Hint: compute the first few derivatives of f and identify a pattern to find a formula for $f^{(n)}(x)$.)

Solution. We have

$$\begin{aligned}
f'(x) &= -\frac{2}{(1+2x)^2}, \\
f''(x) &= \frac{2^2 \cdot 2}{(1+2x)^3}, \\
f^{(3)}(x) &= -\frac{2^3 \cdot 2 \cdot 3}{(1+2x)^3}, \\
f^{(4)}(x) &= \frac{2^4 \cdot 2 \cdot 3 \cdot 4}{(1+2x)^4}, \\
f^{(5)}(x) &= -\frac{2^5 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(1+2x)^5}.
\end{aligned}$$

From this, we can see the following pattern for the higher derivatives of f:

$$f^{(n)}(x) = (-1)^n \frac{2^n n!}{(1+2x)^n}.$$

Therefore, the coefficients of the Maclaurin series of f are given by

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^n 2^n n!}{n!} = (-2)^n.$$

We deduce that the Maclaurin series of f is given by

$$\boxed{T(x) = \sum_{n=0}^{\infty} (-2)^n x^n}.$$

(c) Find is the smallest integer N for which the Maclaurin polynomial $T_N(x)$ of $f(x)$ approximates $f(x)$ with an error of at most 10^{-4} on the interval $[-0.1, 0.1]$.

Solution. Let us start by finding an expression for the error made when approximating $f(x)$ with $T_N(x)$ on the interval $[-0.1, 0.1]$. The Remainder Estimation Theorem gives us the error bound

$$|R_N(x)| \leqslant \frac{M\,|x|^{N+1}}{(N+1)!} \leqslant \frac{M(0.1)^{N+1}}{(N+1)!}.$$

In this estimate, M is any number such that $M \geqslant \left|f^{(N+1)}(x)\right|$ for x in the interval $[-0.1, 0.1]$. Using the formula for the higher derivatives of f that we found in the previous question, we have

$$\left|f^{(N+1)}(x)\right| = \left|(-1)^{N+1}\frac{2^{N+1}(N+1)!}{(1+2x)^{N+1}}\right| = \frac{2^{N+1}(N+1)!}{|1+2x|^{N+1}}.$$

This expression is largest when the denominator is smallest. In the interval $[-0.1, 0.1]$, this occurs when $x = -0.1$. So we can take for M the quantity

$$M = \frac{2^{N+1}(N+1)!}{(1+2(-0.1))^{N+1}} = \frac{2^{N+1}(N+1)!}{0.8^{N+1}} = 2.5^{N+1}(N+1)!.$$

So the error becomes

$$\frac{M(0.1)^{N+1}}{(N+1)!} = \frac{2.5^{N+1}(N+1)!(0.1)^{N+1}}{(N+1)!} = 0.25^{N+1} = \frac{1}{4^{N+1}}.$$

Now we want this error to be less than 10^{-4}, that is

$$\frac{1}{4^{N+1}} < 10^{-4}.$$

Taking reciprocals, this is equivalent to $4^{N+1} > 10000$. Computing the first few powers of 4, we see that this condition is met when the exponent of 4 is at least 7. So the smallest value of N giving the desired error is when $N+1=7$, that is $\boxed{N=6}$.

3. Find the Maclaurin series of the following functions.

(a) $f(x) = x^4 \ln(1-5x^3)$.

Solution. From

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

we get

$$\begin{aligned} x^4 \ln(1-5x^3) &= x^4 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-5x^3)^n}{n} \\ &= x^4 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-5)^n x^{3n}}{n} \\ &= \boxed{\sum_{n=1}^{\infty} -\frac{5^n x^{3n+4}}{n}}. \end{aligned}$$

(b) $f(x) = e^{-x^2/2} - \cos(x)$.

Solution. We know

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots,$$

so

$$\begin{aligned} e^{-x^2/2} &= \sum_{n=0}^{\infty} \frac{\left(-\frac{x^2}{2}\right)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!} \\ &= 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \cdots \end{aligned}$$

We also know that

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots .$$

When we subtract these two Maclaurin series, the first two terms will cancel out. Combining the remaining terms gives

$$\begin{aligned} e^{-x^2/2} - \cos(x) &= \left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \cdots\right) - \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right) \\ &= \left(\frac{1}{8} - \frac{1}{24}\right) x^4 - \left(\frac{1}{48} - \frac{1}{720}\right) x^6 + \cdots \\ &= \boxed{\sum_{n=2}^{\infty} (-1)^n \left(\frac{1}{2^n n!} - \frac{1}{(2n)!}\right) x^{2n}}. \end{aligned}$$

4. Find the first three non-zero terms of the Maclaurin series of $\sin(2x)e^{3x}$.

Solution. We have

$$\sin(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} = 2x - \frac{4x^3}{3} + \cdots ,$$

and

$$e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = 1 + 3x + \frac{9x^2}{2} + \cdots .$$

So

$$\sin(2x)e^{3x} = \left(2x - \frac{4x^3}{3} + \cdots\right)\left(1 + 3x + \frac{9x^2}{2} + \cdots\right).$$

We can now distribute the terms and collect terms of same degree. Let us detail degree by degree what we obtain.

- We do not get any terms in degree 0.
- We get one term in degree 1, from multiplying the $2x$ in the left factor with the 1 in the right factor. This gives us $2x$ as our first non-zero term.
- We get one term in degree 2, from multiplying the $2x$ in the left factor with the $3x$ in the right factor. This gives us $6x^2$ as our second non-zero term.
- We get two terms in degree 3, the first one from multiplying the $2x$ in the left factor with the $\frac{9x^2}{2}$ in the right factor, and the second one from multiplying the $-\dfrac{4x^3}{3}$ in the left factor with the 1 in the right factor. This gives us $9x^3 - \frac{4x^3}{3} = \frac{23x^3}{3}$ as our third non-zero term.

Therefore, the first three non-zero terms of the Maclaurin of $\sin(2x)e^{3x}$ are

$$\boxed{2x + 6x^2 + \frac{23x^3}{3}}.$$

Rutgers University
Math 152

Section 10.10: Applications of Taylor Series - Worksheet Solutions

1. Use Maclaurin series to compute the following limits.

   (a) $\displaystyle\lim_{x\to 0} \frac{e^{-2x^2} - \cos(2x)}{x^2 \ln(1+5x) - 5x^3}$.

   Solution. We find the first few terms of the Maclaurin series of the numerator and denominator.

$$\begin{aligned} e^{-2x^2} - \cos(2x) &= \left(1 - 2x^2 + \frac{(-2x^2)^2}{2!} + \frac{(-2x^2)^3}{3!} \cdots\right) - \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots\right) \\ &= \frac{4}{3}x^4 - \frac{56}{45}x^6 + \cdots \end{aligned}$$

$$\begin{aligned} x^2 \ln(1+5x) - 5x^3 &= x^2\left(5x - \frac{(5x)^2}{2} + \frac{(5x)^3}{3} + \cdots\right) - 5x^3 \\ &= -\frac{25x^4}{2} + \frac{125x^5}{3} + \cdots \end{aligned}$$

   Therefore,

$$\begin{aligned} \lim_{x\to 0} \frac{e^{-2x^2} - \cos(2x)}{x^2 \ln(1+5x) - 5x^3} &= \lim_{x\to 0} \frac{\frac{4}{3}x^4 - \frac{56}{45}x^6 + \cdots}{-\frac{25x^4}{2} + \frac{125x^5}{3} + \cdots} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}} \\ &= \lim_{x\to 0} \frac{\frac{4}{3} - \frac{56}{45}x^2 + \cdots}{-\frac{25}{2} + \frac{125x}{3} + \cdots} \\ &= \frac{\frac{4}{3}}{-\frac{25}{2}} \\ &= \boxed{-\frac{8}{75}}. \end{aligned}$$

   (b) $\displaystyle\lim_{x\to\infty} x^3 \left(\tan^{-1}\left(\frac{4}{x}\right) - 2\sin\left(\frac{2}{x}\right)\right)$.

   Solution. We have

$$\begin{aligned} \tan^{-1}\left(\frac{4}{x}\right) - 2\sin\left(\frac{2}{x}\right) &= \left(\frac{4}{x} - \frac{1}{3}\left(\frac{4}{x}\right)^3 + \frac{1}{5}\left(\frac{4}{x}\right)^5 + \cdots\right) - 2\left(\frac{2}{x} - \frac{1}{3!}\left(\frac{2}{x}\right)^3 + \frac{1}{5!}\left(\frac{2}{x}\right)^5 + \cdots\right) \\ &= -\frac{56}{3x^3} + \frac{3064}{15x^5} + \cdots. \end{aligned}$$

   So

$$\lim_{x\to\infty} x^3 \left(\tan^{-1}\left(\frac{4}{x}\right) - 2\sin\left(\frac{2}{x}\right)\right) = \lim_{x\to\infty} x^3\left(-\frac{56}{3x^3} + \frac{3064}{15x^5} + \cdots\right)$$

$$= \lim_{x\to\infty} -\frac{56}{3} + \frac{3064}{15x^2} + \cdots$$
$$= \boxed{-\frac{56}{3}}.$$

(c) $\displaystyle\lim_{x\to 0} \frac{\sin(x^6)}{\cos(x^3) - 1}$.

Solution. We have

$$\cos(x^3) - 1 = \left(1 - \frac{(x^3)^2}{2!} + \frac{(x^3)^4}{4!} + \cdots\right) - 1$$
$$= -\frac{x^6}{2} + \frac{x^{12}}{24} + \cdots$$

$$\sin(x^6) = x^6 - \frac{(x^6)^3}{3!} + \cdots$$
$$= x^6 - \frac{x^{18}}{6} + \cdots$$

Thus

$$\lim_{x\to 0} \frac{\sin(x^6)}{\cos(x^3) - 1} = \lim_{x\to 0} \frac{x^6 - \frac{x^{18}}{6} + \cdots}{-\frac{x^6}{2} + \frac{x^{12}}{24} + \cdots} \cdot \frac{\frac{1}{x^6}}{\frac{1}{x^6}}$$
$$= \lim_{x\to 0} \frac{1 - \frac{x^{12}}{6} + \cdots}{-\frac{1}{2} + \frac{x^6}{24} + \cdots}$$
$$= \boxed{-2}.$$

(d) $\displaystyle\lim_{x\to\infty} x^2 \left(5\ln\left(1 + \frac{3}{x}\right) - 3\ln\left(1 + \frac{5}{x}\right)\right)$.

Solution. We have

$$5\ln\left(1 + \frac{3}{x}\right) - 3\ln\left(1 + \frac{5}{x}\right) = 5\left(\frac{3}{x} - \frac{1}{2}\left(\frac{3}{x}\right)^2 + \frac{1}{3}\left(\frac{3}{x}\right)^3 + \cdots\right) - 3\left(\frac{5}{x} - \frac{1}{2}\left(\frac{5}{x}\right)^2 + \frac{1}{3}\left(\frac{5}{x}\right)^3 + \cdots\right)$$
$$= \frac{15}{x^2} - \frac{80}{x^3} + \cdots$$

So

$$\lim_{x\to\infty} x^2 \left(5\ln\left(1 + \frac{3}{x}\right) - 3\ln\left(1 + \frac{5}{x}\right)\right) = \lim_{x\to\infty} x^2 \left(\frac{15}{x^2} - \frac{80}{x^3} + \cdots\right)$$
$$= \lim_{x\to\infty} 15 - \frac{80}{x} + \cdots$$
$$= \boxed{15}.$$

2. Use Maclaurin series to write each integral below as the sum of an infinite series of numbers (your series should not contain x).

(a) $\displaystyle\int_0^{1/2} \cos(5x^2)dx.$

Solution. The Maclaurin series for the integrand is

$$\cos(5x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (5x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n} x^{4n}}{(2n)!}.$$

Integrating term-by-term gives

$$\begin{aligned}
\int_0^{1/2} \cos(x^2)dx &= \sum_{n=0}^{\infty} \int_0^{1/2} \frac{(-1)^n 5^{2n} x^{4n}}{(2n)!} dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n}}{(2n)!} \int_0^{1/2} x^{4n} dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n}}{(2n)!} \left[\frac{x^{4n+1}}{4n+1} \right]_0^{1/2} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n}}{(2n)!} \left(\frac{1}{(4n+1)2^{4n+1}} - \frac{0^{4n+1}}{4n+1} \right) \\
&= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n}}{(2n)!(4n+1)2^{4n+1}}}.
\end{aligned}$$

(b) $\displaystyle\int_0^1 x^3 e^{-4x^3} dx.$

Solution. The Maclaurin series for the integrand is

$$x^3 e^{-4x^3} = x^3 \sum_{n=0}^{\infty} \frac{(-4x^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-4)^n x^{3n+3}}{n!}.$$

Integrating term-by-term gives

$$\begin{aligned}
\int_0^1 x^3 e^{-4x^3} dx &= \sum_{n=0}^{\infty} \int_0^1 \frac{(-4)^n x^{3n+3}}{n!} dx \\
&= \sum_{n=0}^{\infty} \frac{(-4)^n}{n!} \int_0^1 x^{3n+3} dx \\
&= \sum_{n=0}^{\infty} \frac{(-4)^n}{n!} \left[\frac{x^{3n+4}}{3n+4} \right]_0^1 \\
&= \sum_{n=0}^{\infty} \frac{(-4)^n}{n!} \left(\frac{1^{3n+4}}{3n+4} - \frac{0^{3n+4}}{3n+4} \right) \\
&= \boxed{\sum_{n=0}^{\infty} \frac{(-4)^n}{n!(3n+4)}}.
\end{aligned}$$

(c) $\displaystyle\int_0^{1/3} x^7 \sin(2x^5)dx.$

Solution. The Maclaurin series for the integrand is

$$x^7 \sin(2x^5) = x^7 \sum_{n=0}^{\infty} \frac{(-1)^n (2x^5)^{2n+1}}{(2n+1)!} = x^7 \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{10n+5}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{10n+12}}{(2n+1)!}.$$

Integrating term-by-term gives

$$\begin{aligned}
\int_0^{1/3} x^7 \sin(2x^5)dx &= \sum_{n=0}^{\infty} \int_0^{1/3} \frac{(-1)^n 2^{2n+1} x^{10n+12}}{(2n+1)!} dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} \int_0^{1/3} x^{10n+12} dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} \left[\frac{x^{10n+13}}{10n+13} \right]_0^{1/3} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} \left(\frac{1}{(10n+13)3^{10n+13}} - \frac{0^{10n+13}}{10n+13} \right) \\
&= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!(10n+13)3^{10n+13}}}.
\end{aligned}$$

3. (a) Use Maclaurin series to write the integral $I = \displaystyle\int_0^1 e^{-x^2} dx$ as the sum of an infinite series of numbers.

Solution. The Maclaurin series for the integrand is

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}.$$

Integrating term-by-term gives

$$\begin{aligned}
\int_0^1 e^{-x^2} dx &= \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n x^{2n}}{n!} dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^{2n} dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[\frac{x^{2n+1}}{2n+1} \right]_0^1 \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{1^{2n+1}}{2n+1} - \frac{0^{2n+1}}{2n+1} \right) \\
&= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)}}.
\end{aligned}$$

(b) Use the Alternating Series Estimation Theorem to find how many terms of the series found in (a) need to be summed in order to obtain an approximation of I with an error of less than 10^{-5}.

Solution. The alternating series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)}$$

meets the assumptions of the Alternating Series Estimation Theorem since $a_n = \frac{1}{n!(2n+1)}$ is positive, decreasing and converges to 0. Therefore, the error made approximating I with the partial $S_N = \displaystyle\sum_{n=0}^{N} \frac{(-1)^n}{n!(2n+1)}$ is

$$|I - S_N| \leqslant a_{N+1} = \frac{1}{(N+1)!(2N+3)}.$$

Let us find the smallest value of N giving us an error of less than 10^{-5}. We want $\frac{1}{(N+1)!(2N+3)} < 10^{-5}$, or $(N+1)!(2N+3) > 10^5$. Solving this with a calculator gives the smallest value as $N = 6$. The corresponding partial sum

$$S_6 = \sum_{n=0}^{6} \frac{(-1)^n}{n!(2n+1)}$$

has $\boxed{\text{7 terms}}$.

Rutgers University
Math 152

Sections 11.1, 11.2: Parametric Curves - Worksheet

1. Find an equation of the tangent line to the given parametric curve at the point defined by the given value of t.

(a) $\begin{cases} x = 5t^2 - 7 \\ y = t^4 - 3t \end{cases}$, $t = -1$.

Solution. We have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t-3}{10t}.$$

So the slope of the tangent line to the curve at $t = -1$ is

$$\frac{dy}{dx}_{|t=-1} = \frac{-4-3}{-10} = \frac{7}{10}.$$

Also, the tangent line to the curve at $t = -1$ passes through $(x(-1), y(-1)) = (-2, 4)$. It follows that it has equation

$$\boxed{y = \frac{7}{10}(x+2) + 4}.$$

(b) $\begin{cases} x = e^{4t} - e^t + 2 \\ y = t - 3e^{2t} \end{cases}$, $t = 0$.

Solution. We have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1-6e^{2t}}{4e^{4t}-e^t}.$$

So the slope of the tangent line to the curve at $t = 0$ is

$$\frac{dy}{dx}_{|t=0} = \frac{1-6}{4-1} = -\frac{5}{3}.$$

Also, the tangent line to the curve at $t = 0$ passes through $(x(0), y(0)) = (2, -3)$. It follows that it has equation

$$\boxed{y = -\frac{5}{3}(x-2) - 3}.$$

(c) $\begin{cases} x = \sec(3t) \\ y = \cot(2t - \pi) \end{cases}$, $t = \frac{\pi}{12}$.

Solution. We have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2\csc^2(2t-\pi)}{3\sec(3t)\tan(3t)}.$$

So the slope of the tangent line to the curve at $t = \frac{\pi}{12}$ is

$$\frac{dy}{dx}_{|t=\frac{\pi}{12}} = \frac{-2\csc^2\left(-\frac{5\pi}{6}\right)}{3\sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right)} = -\frac{8}{3\sqrt{2}}.$$

Also, the tangent line to the curve at $t = \frac{\pi}{12}$ passes through $\left(x\left(\frac{\pi}{12}\right), y\left(\frac{\pi}{12}\right)\right) = (\sqrt{2}, -\sqrt{3})$. It follows that it has equation

$$\boxed{y = \frac{8}{3\sqrt{2}}(x - \sqrt{2}) - \sqrt{3}}.$$

2. Find all points on the following parametric curves where the tangent line is (i) horizontal, and (ii) vertical.

   (a) $\begin{cases} x = \sin(2t) + 1 \\ y = \cos(t) \end{cases}, 0 \leqslant t < 2\pi.$

   Solution. We have

   $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\sin(t)}{2\cos(2t)}.$$

   (i) The tangent line is horizontal when $\frac{dy}{dx} = 0$, that is when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$. For $0 \leqslant t < 2\pi$, we have

   $$\begin{aligned} &\frac{dy}{dt} = 0 \\ &\Rightarrow \sin(t) = 0 \\ &\Rightarrow t = 0, \pi. \end{aligned}$$

   Observe that $\frac{dx}{dt} \neq 0$ for these values of t. To find the corresponding points on the curve, we plug-in these values of t in $(x(t), y(t))$ and we get the points $\boxed{(1,1), (1,-1)}$.

   (ii) The tangent line is vertical when "$\frac{dy}{dx} = \infty$", that is when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$. For $0 \leqslant t < 2\pi$, we have

   $$\begin{aligned} &\frac{dx}{dt} = 0 \\ &\Rightarrow \cos(2t) = 0 \\ &\Rightarrow t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}. \end{aligned}$$

   Observe that $\frac{dy}{dt} \neq 0$ for these values of t. To find the corresponding points on the curve, we plug-in these values of t in $(x(t), y(t))$ and we get the points $\boxed{\left(2, \frac{\sqrt{2}}{2}\right), \left(2, -\frac{\sqrt{2}}{2}\right), \left(0, \frac{\sqrt{2}}{2}\right), \left(0, -\frac{\sqrt{2}}{2}\right)}$.

   (b) $\begin{cases} x = 3t - t^3 \\ y = t^2 + 4t + 3 \end{cases}$

   Solution. We have

   $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t+4}{3-3t^2}.$$

(i) The tangent line is horizontal when $\frac{dy}{dx} = 0$, that is when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$. We have

$$\begin{aligned} &\frac{dy}{dt} = 0 \\ &\Rightarrow\ 2t + 4 = 0 \\ &\Rightarrow\ t = -2. \end{aligned}$$

Observe that $\frac{dx}{dt} \neq 0$ for this values of t. To find the corresponding point on the curve, we plug-in $t = -2$ in $(x(t), y(t))$ and we get the point $\boxed{(2, -1)}$.

(ii) The tangent line is vertical when "$\frac{dy}{dx} = \infty$", that is when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$. We have

$$\begin{aligned} &\frac{dx}{dt} = 0 \\ &\Rightarrow\ 3 - 3t^2 = 0 \\ &\Rightarrow\ t = -1, 1. \end{aligned}$$

Observe that $\frac{dy}{dt} \neq 0$ for these values of t. To find the corresponding points on the curve, we plug-in these values of t in $(x(t), y(t))$ and we get the points $\boxed{(-2, 0), (2, 8)}$.

(c) $\begin{cases} x = 4t - e^{2t} \\ y = t^2 - 18\ln|t| \end{cases}$

Solution. We have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t - \frac{18}{t}}{4 - 2e^{2t}}.$$

(i) The tangent line is horizontal when $\frac{dy}{dx} = 0$, that is when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$. We have

$$\begin{aligned} &\frac{dy}{dt} = 0 \\ &\Rightarrow\ 2t - \frac{18}{t} = 0 \\ &\Rightarrow\ t^2 = 9 \\ &\Rightarrow\ t = -3, 3. \end{aligned}$$

Observe that $\frac{dx}{dt} \neq 0$ for these values of t. To find the corresponding point on the curve, we plug-in these values of t in $(x(t), y(t))$ and we get the points $\boxed{\left(-12 - e^{-6}, 9 - 18\ln(3)\right), \left(12 - e^{6}, 9 - 18\ln(3)\right)}$.

(ii) The tangent line is vertical when "$\frac{dy}{dx} = \infty$", that is when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$. We have

$$\begin{aligned} &\frac{dx}{dt} = 0 \\ &\Rightarrow\ 4 - 2e^{2t} = 0 \\ &\Rightarrow\ t = \frac{1}{2}\ln(2). \end{aligned}$$

Observe that $\frac{dy}{dt} \neq 0$ for this value of t. To find the corresponding points on the curve, we plug-in this value of t in $(x(t), y(t))$ and we get the point $\boxed{\left(2\ln(2) - 2, \frac{\ln(2)^2}{4} - 18\ln\left(\frac{\ln(2)}{2}\right)\right)}$.

3. Consider the ellipse of equation $x^2 + 4y^2 = 4$.

(a) Find a parametrization of the ellipse.

Solution. The equation of the ellipse can be written as

$$\left(\frac{x}{2}\right)^2 + y^2 = 1,$$

from which we see that a possible parametrization is

$$\boxed{\begin{cases} x = 2\cos(t) \\ y = \sin(t) \end{cases}, 0 \leqslant t < 2\pi},$$

where we have chosen the parameter interval to ensure that the ellipse is traced exactly one.

(b) Find the area enclosed by the ellipse.

Solution. By symmetry, it suffices to find the area of the region inside the ellipse in the first quadrant and multiply it by 4. This region is bounded by the parametric curve $x = 2\cos(t), y = \sin(t)$ from $x = 0$ (which corresponds to $t = \frac{\pi}{2}$) to $x = 2$ (which corresponds to $t = 0$). Therefore, the area is given by

$$\begin{aligned}
A &= 4\int_0^2 y dx \\
&= 4\int_{\pi/2}^0 y(t)x'(t)dt \\
&= 4\int_{\pi/2}^0 \sin(t)(-2\sin(t))dt \\
&= 8\int_0^{\pi/2} \sin(t)^2 dt \\
&= 8\int_0^{\pi/2} \frac{1-\cos(2t)}{2} dt \\
&= 4\left[t - \frac{\sin(2t)}{2}\right]_0^{\pi/2} \\
&= \boxed{2\pi \text{ square units}}.
\end{aligned}$$

(c) Find the area of the surface obtained by revolving the top-half of the ellipse about the x-axis.

Solution. The top-half of the ellipse is parametrized by $x = 2\cos(t), y = \sin(t)$ for $0 \leqslant t \leqslant \pi$. Note that by symmetry, it suffices to find the area obtained by revolving the curve for $0 \leqslant t \leqslant \frac{\pi}{2}$ and multiply by 2. Therefore, the surface area is

$$\begin{aligned}
A &= 2\int_0^{\pi/2} 2\pi y(t)\sqrt{x'(t)^2 + y'(t)^2} dt \\
&= 4\pi\int_0^{\pi/2} \sin(t)\sqrt{4\sin(t)^2 + \cos(t)^2} dt.
\end{aligned}$$

This integral can be computed by expressing the inside of the square root in terms of $\cos(t)$ only (with the help of the Pythagorean identity) and then using a substitution. This gives

$$\begin{aligned} A &= 4\pi \int_0^{\pi/2} \sin(t)\sqrt{4(1-\cos(t)^2)+\cos(t)^2}dt \\ &= 4\pi \int_0^{\pi/2} \sin(t)\sqrt{4-3\cos(t)^2}dt \\ &= 8\pi \int_0^{\pi/2} \sin(t)\sqrt{1-\left(\frac{\sqrt{3}\cos(t)}{2}\right)^2}dt \\ &= \frac{16\pi}{\sqrt{3}} \int_0^{\sqrt{3}/2} \sqrt{1-u^2}du \quad \left(u = dfrac\sqrt{3}\cos(t)2\right). \end{aligned}$$

This integral can be computed with a trigonometric substitution $u = \sin(\theta)$, so that $du = \cos(\theta)d\theta$ and $\sqrt{1-u^2} = \cos(\theta)$. The bounds become

$$\begin{aligned} u &= 0 \ \Rightarrow \ \theta = \sin^{-1}(0) = 0, \\ u &= \frac{\sqrt{3}}{2} \ \Rightarrow \ \theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}. \end{aligned}$$

So

$$\begin{aligned} A &= \frac{16\pi}{\sqrt{3}} \int_0^{\pi/3} \cos(\theta)\cos(\theta)d\theta \\ &= \frac{16\pi}{\sqrt{3}} \int_0^{\pi/3} \cos(\theta)^2 d\theta \\ &= \frac{16\pi}{\sqrt{3}} \int_0^{\pi/3} \frac{1+\cos(2\theta)}{2}d\theta \\ &= \frac{8\pi}{\sqrt{3}} \left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{\pi/3} \\ &= \boxed{\frac{8\pi}{\sqrt{3}}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right) \text{ square units}}. \end{aligned}$$

4. For each of the following parametric curves: (i) find the arc length, (ii) set-up (but do not evaluate) an integral that computes the area of the surface obtained by revolving the curve about the x-axis and (iii) set-up (but do not evaluate) an integral that computes the area of the surface obtained by revolving the curve about the y-axis.

   (a) $\begin{cases} x = e^{4t} \\ y = e^{5t} \end{cases}, \ 0 \leqslant t \leqslant 1.$

   Solution. (i) The arc length is given by

$$L = \int_0^1 \sqrt{x'(t)^2 + y'(t)^2}dt$$

$$= \int_0^1 \sqrt{16e^{8t} + 25e^{10t}} dt$$
$$= \int_0^1 e^{4t}\sqrt{16 + 25e^{2t}} dt$$

We can calculate that integral by substituting $u = 16 + 25e^{2t}$, which gives $du = 50e^{2t}dt$. The extraneous factor e^{2t} in the integrand can be expressed in terms of u as $e^{2t} = \frac{u-16}{25}$. Finally, the bounds become

$$x = 0 \;\Rightarrow\; u = 16 + 25 = 41,$$
$$x = 1 \;\Rightarrow\; u = 16 + 25e.$$

We obtain

$$\begin{aligned} L &= \frac{1}{50}\int_{41}^{16+25e} \frac{u-16}{25}\sqrt{u} du \\ &= \frac{1}{175}\int_{41}^{16+25e} \left(u^{3/2} - 16u^{1/2}\right) du \\ &= \frac{1}{175}\left[\frac{2}{5}u^{5/2} - \frac{32}{3}u^{3/2}\right]_{41}^{16+25e} \\ &= \boxed{\frac{1}{175}\left(\frac{2}{5}(16+25e)^{5/2} - \frac{32}{3}(16+25e)^{3/2} - \frac{2}{5}41^{5/2} - \frac{32}{3}41^{3/2}\right) \text{ units}}. \end{aligned}$$

(ii) Revolution about the x-axis: $\boxed{A = \int_0^1 2\pi e^{5t}\sqrt{16e^{8t} + 25e^{10t}} dt}$.

(ii) Revolution about the y-axis: $\boxed{A = \int_0^1 2\pi e^{4t}\sqrt{16e^{8t} + 25e^{10t}} dt}$.

(b) $\begin{cases} x = \ln(t) \\ y = \sin^{-1}(t) \end{cases}$, $\frac{1}{2} \leqslant t \leqslant \frac{1}{\sqrt{2}}$.

Solution. (i) The arc length is given by

$$\begin{aligned} L &= \int_{1/2}^{1/\sqrt{2}} \sqrt{x'(t)^2 + y'(t)^2} dt \\ &= \int_{1/2}^{1/\sqrt{2}} \sqrt{\frac{1}{t^2} + \frac{1}{1-t^2}} dt \\ &= \int_{1/2}^{1/\sqrt{2}} \sqrt{\frac{1-t^2+t^2}{t^2(1-t^2)}} dt \\ &= \int_{1/2}^{1/\sqrt{2}} \sqrt{\frac{1}{t^2(1-t^2)}} dt \\ &= \int_{1/2}^{1/\sqrt{2}} \frac{1}{t\sqrt{1-t^2}} dt. \end{aligned}$$

This integral can be computed with the trigonometric substitution $t = \sin(\theta)$, which gives $dt = \cos(\theta)d\theta$ and $\sqrt{1-t^2} = \sqrt{1-\sin(\theta)^2} = \cos(\theta)$. The bounds become

$$x = \frac{1}{2} \ \Rightarrow\ \theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6},$$
$$x = \frac{1}{\sqrt{2}} \ \Rightarrow\ \theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}.$$

So

$$\begin{aligned} L &= \int_{\pi/6}^{\pi/4} \frac{1}{\sin(\theta)\cos(\theta)}\cos(\theta)d\theta \\ &= \int_{\pi/6}^{\pi/4} \csc(\theta)d\theta \\ &= [-\ln|\csc(\theta)+\cot(\theta)]_{\pi/6}^{\pi/4} \\ &= \boxed{\ln\left(2+\sqrt{3}\right) - \ln(\sqrt{2}+1) \text{ units}}. \end{aligned}$$

(ii) Revolution about the x-axis: $\boxed{A = \int_{1/2}^{1/\sqrt{2}} 2\pi\sin^{-1}(t)\sqrt{\frac{1}{t^2}+\frac{1}{1-t^2}}dt}$.

(ii) Revolution about the y-axis: $\boxed{A = \int_{1/2}^{1/\sqrt{2}} 2\pi\ln(t)\sqrt{\frac{1}{t^2}+\frac{1}{1-t^2}}dt}$.

(c) $\begin{cases} x = t^3 - t \\ y = \sqrt{3}t^2 \end{cases}$, $0 \leqslant t \leqslant 1$.

Solution. (i) The arc length is given by

$$\begin{aligned} L &= \int_0^1 \sqrt{x'(t)^2 + y'(t)^2}dt \\ &= \int_0^1 \sqrt{(3t^2-1)^2 + \left(2\sqrt{3}t\right)^2}dt \\ &= \int_0^1 \sqrt{9t^4 - 6t^2 + 1 + 12t^2}dt \\ &= \int_0^1 \sqrt{9t^4 + 6t^2 + 1}dt \\ &= \int_0^1 \sqrt{(3t+1)^2}dt \\ &= \int_0^1 (3t+1)dt \\ &= \left[\frac{3t^2}{2} + t\right]_0^1 \\ &= \boxed{\frac{5}{2} \text{ units}}. \end{aligned}$$

(ii) Revolution about the x-axis: $\boxed{A = \int_0^1 2\pi\sqrt{3}t^2\sqrt{(3t^2-1)^2+\left(2\sqrt{3}t\right)^2}\,dt}$.

(ii) Revolution about the y-axis: $\boxed{A = \int_0^1 2\pi\left(t^3-t\right)\sqrt{(3t^2-1)^2+\left(2\sqrt{3}t\right)^2}\,dt}$.

Rutgers University
Math 152

Sections 11.3, 11.4: Polar Coordinates - Worksheet Solutions

1. Convert the following Cartesian equations to polar.

(a) $y = 11$.

Solution. Since $y = r\sin(\theta)$, the equation becomes $r\sin(\theta) = 11$ or $\boxed{r = 11\csc(\theta)}$.

(b) $x + y = 0$.

Solution. The graph of this equation is a line through the origin of slope -1. So it has polar equation $\theta = \theta_0$, where θ_0 is a constant such that $\tan(\theta_0) = -1$. This gives $\theta_0 = \frac{3\pi}{4}$, so the polar equation is $\boxed{\theta = \frac{3\pi}{4}}$.

(c) $(x-3)^2 + y^2 = 9$.

Solution. Replacing $x = r\cos(\theta)$ and $y = r\sin(\theta)$ gives

$$\begin{aligned}
&(r\cos(\theta) - 3)^2 + (r\sin(\theta))^2 = 9 \\
&r^2\cos(\theta)^2 - 6r\cos(\theta) + 9 + r^2\sin(\theta)^2 = 9 \\
&r^2 - 6r\cos(\theta) = 0 \\
&r(r - 6\cos(\theta)) = 0 \\
&r = 0 \text{ or } r = 6\cos(\theta).
\end{aligned}$$

Note that $r = 0$ (which is the polar equation of the origin) is already included in the graph of $r = 6\cos(\theta)$. So we need only keep the equation $\boxed{r = 6\cos(\theta)}$.

(d) $y = 7 + 2x$.

Solution. Replacing $x = r\cos(\theta)$ and $y = r\sin(\theta)$ gives

$$\begin{aligned}
&r\sin(\theta) = 7 + 2r\cos(\theta) \\
&r\sin(\theta) - 2r\cos(\theta) = 7 \\
&r(\sin(\theta) - 2\cos(\theta)) = 7 \\
&\boxed{r = \frac{7}{\sin(\theta) - 2\cos(\theta)}}.
\end{aligned}$$

(e) $x^2 + y^2 + xy = 2$.

Solution. Replacing $x = r\cos(\theta)$ and $y = r\sin(\theta)$ gives

$$\begin{aligned}
&(r\cos(\theta))^2 + (r\sin(\theta))^2 + r\cos(\theta) r\sin(\theta) = 2 \\
&r^2 + r^2\cos(\theta)\sin(\theta) = 2 \\
&r^2(1 + \cos(\theta)\sin(\theta)) = 2 \\
&r^2 = \frac{2}{1 + \sin(\theta)\cos(\theta)} \\
&r = \pm\sqrt{\frac{2}{1 + \sin(\theta)\cos(\theta)}}
\end{aligned}$$

It may seem like we need two polar equations for this curve, but keeping only one of the two will be enough. Indeed, we can see (using either the Cartesian equation and replacing (x, y) by $(-x, -y)$, or the polar equation and replacing (r, θ) by $(r, \theta + \pi)$) that this curve is symmetric about the origin. Furthermore the two polar equations that we have found trace graphs that are symmetric of each other about the origin. Since the curve they trace is symmetric about the origin, their graphs give the same curve. It follows that a polar equation for this curve is $\boxed{r = \sqrt{\frac{2}{1 + \sin(\theta)\cos(\theta)}}}$.

(f) $y^2 = 3x^2$.

Solution. This Cartesian equation is equivalent to $y = \sqrt{3}x$ or $y = -\sqrt{3}x$. These equations represent lines through the origin of respective slopes $\sqrt{3}$ and $-\sqrt{3}$. So their polar equations are of the form $\theta = \theta_0$, with θ_0 a constant whose tangent is equal to the slope. It follows that the polar equation is $\boxed{\theta = \frac{\pi}{3} \text{ or } \theta = -\frac{\pi}{3}}$.

2. Convert the following polar equations to Cartesian. Then describe the graph.

(a) $r = -7\sec(\theta)$.

Solution. We can write the equation in the form $r = -\frac{7}{\cos(\theta)}$, or $r\cos(\theta) = -7$, so $\boxed{x = -7}$ is the Cartesian equation. The graph is a vertical line.

(b) $r = \dfrac{5}{3\sin(\theta) - 4\cos(\theta)}$.

Solution. We write the equation as

$$r(3\sin(\theta) - 4\cos(\theta)) = 5 \;\Rightarrow\; 3r\sin(\theta) - 4r\cos(\theta) = 5 \;\Rightarrow\; \boxed{3y - 4x = 5}.$$

The corresponding graph is a line of slope $\frac{4}{3}$ and y-intercept $\frac{5}{3}$.

(c) $\theta = \dfrac{\pi}{6}$.

Solution. The graph of this polar equation is a line through the origin of slope $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$, so it has Cartesian equation $\boxed{y = \dfrac{x}{\sqrt{3}}}$.

(d) $r = 3\cot(\theta)\csc(\theta)$.

Solution. Let us start by expressing the trigonometric functions in terms of sine and cosine. We have

$$r = 3\frac{\cos(\theta)}{\sin(\theta)} \cdot \frac{1}{\sin(\theta)} \quad \Rightarrow \quad r\sin(\theta)^2 = 3\cos(\theta).$$

To find the Cartesian equation, we can multiply this last polar equation by r to get $r^2\sin(\theta)^2 = 3r\cos(\theta)$, or $\boxed{y^2 = 3x}$. The graph is a sideways parabola opening towards the right.

(e) $r = 5\sin(\theta)$.

Solution. We can multiply the equation by r to get $r^2 = 5r\sin(\theta)$, or $\boxed{x^2 + y^2 = 5y}$. This equation can be rearranged as $x^2 + y^2 - 5y = 0$, or $x^2 + \left(y - \frac{5}{2}\right)^2 = \frac{25}{4}$. From this we see that the graph is a circle of center $\left(0, \frac{5}{2}\right)$ and radius $\frac{5}{2}$.

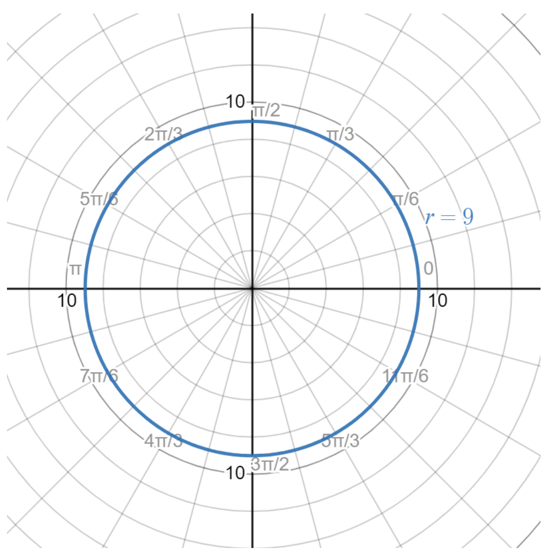
(f) $r = 2\cos(\theta) + 6\sin(\theta)$.

Solution. We can again multiply by r to get $r^2 = 2r\cos(\theta) + 6r\sin(\theta)$, or $\boxed{x^2 + y^2 = 2x + 6y}$. Writing the equation as $(x-1)^2 + (y-3)^2 = 10$, we see that the graph is a circle of center $(1, 3)$ and radius $\sqrt{10}$.

3. For each the following polar curves, identify the symmetries and sketch the graph.

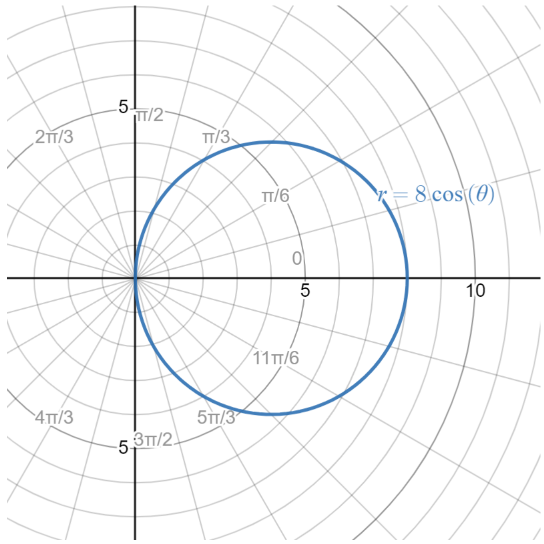
(a) $r = 9$.

Solution. We recognize the polar equation of a circle of radius 9. In particular the graph is symmetric about the origin, the x-axis and the y-axis (or about any line through the origin, which we can see by noting that r does not vary with θ).



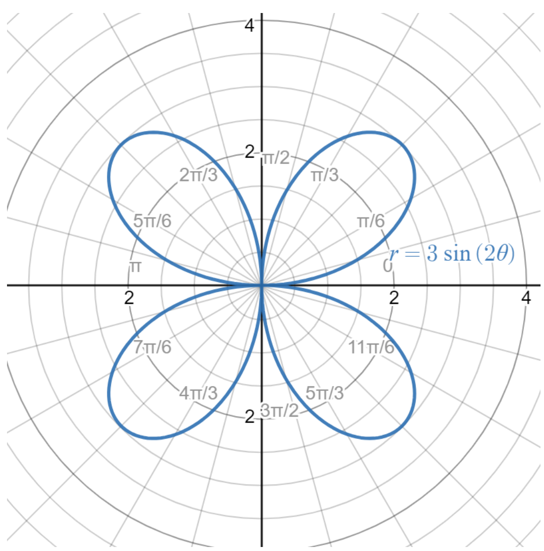
(b) $r = 8\cos(\theta)$.

Solution. If (r, θ) is on the graph, then $8\cos(-\theta) = 8\cos(\theta) = r$, so $(r, -\theta)$ is also on the graph. We deduce that the graph is symmetric about the x-axis. It is not symmetric about the y-axis or the origin since $\cos(\pi + \theta) = \cos(\pi - \theta) = -\cos(\theta)$. The sketch reveals that the graph is a circle of radius 4 centered at $(4, 0)$.



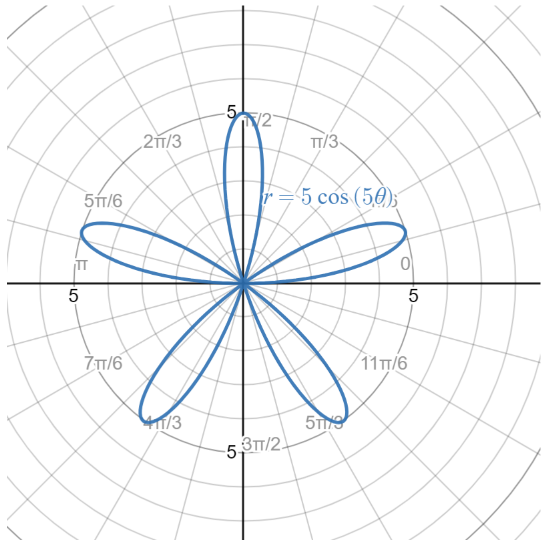
(c) $r = 3\sin(2\theta)$.

Solution. If (r, θ) is on the graph, then $3\sin(-2\theta)) = -3\sin(2\theta) = -r$, so $(-r, -\theta) = (r, \pi - \theta)$ is also on the graph, revealing a symmetry about the y-axis. Likewise, $3\sin(2(\pi - \theta)) = -3\sin(2\theta) = -r$, so $(-r, \pi - \theta) = (r, -\theta)$ is also on the graph, indicating a symmetry about the x-axis. Since the graph is symmetric about both the x-axis and the y-axis, it is also symmetric about the origin. Sketching the graph gives a four-leaved rose.



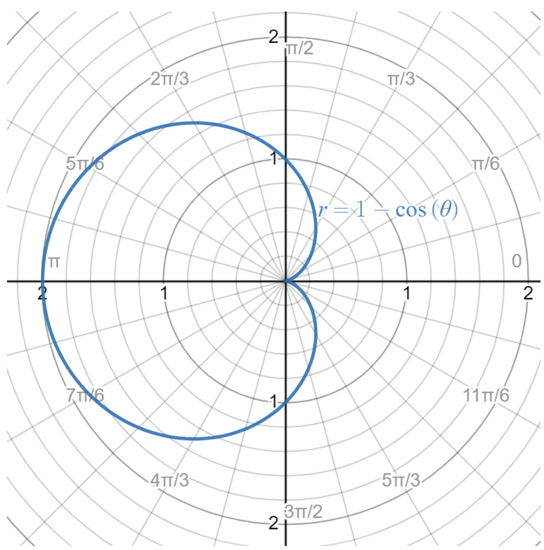
(d) $r = 5\cos(5\theta)$.

Solution. If (r, θ) is on the graph, then $5\cos(5(-\theta)) = 5\cos(5\theta) = r$, so $(r, -\theta)$ is also on the graph, indicating a symmetry about the x-axis. However, $5\cos(5(\pi - \theta)) = -5\cos(5\theta) = -r$ and $5\cos(5(\pi + \theta)) = -5\cos(5\theta) = -r$, so there is no symmetry about the y-axis or the origin. Sketching the graph gives a five-leaved rose.



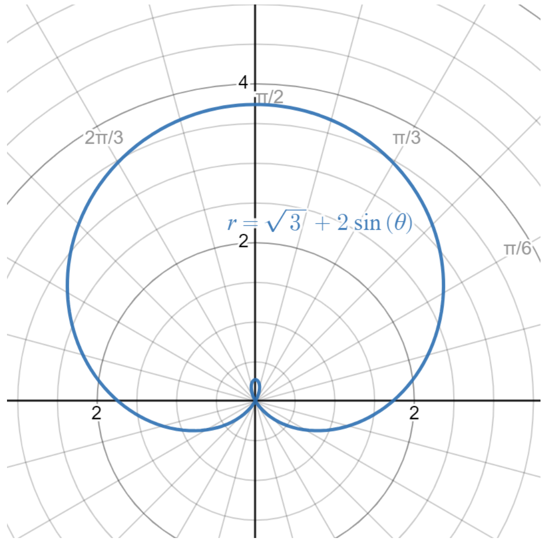
(e) $r = 1 - \cos(\theta)$.

Solution. If (r, θ) is on the graph, then $1 - \cos(-\theta) = 1 - \cos(\theta)$ so $(-r, \theta)$ is also on the graph, indicating a symmetry about the x-axis. However, $1 - \cos(\pi - \theta) = 1 - \cos(\pi + \theta) = 1 + \cos(\theta)$, so the graph is not symmetric about the y-axis or the origin. Sketching the graph gives a cardioid.



(f) $r = \sqrt{3} + 2\sin(\theta)$.

Solution. If (r, θ) is on the graph, then $\sqrt{3} + 2\sin(\pi - \theta) = \sqrt{3} + 2\sin(\theta) = r$, so $(r, \pi - \theta)$ is also on the graph, indicating a symmetry about the y-axis. However, $\sqrt{3} + 2\sin(-\theta) = \sqrt{3} + 2\sin(\pi + \theta) = \sqrt{3} - 2\sin(\theta)$, so the graph is not symmetric about the x-axis or the origin. Sketching the graph gives a looped limaçon.



4. Find an equation of the tangent line to the following polar curves at the given value of θ.

(a) $r = \cos(3\theta)$, $\theta = \dfrac{\pi}{4}$.

Solution. The curve can be parametrized by $x = r\cos(\theta) = \cos(\theta)\cos(3\theta)$ and $y = r\sin(\theta) = \sin(\theta)\cos(3\theta)$. Therefore

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos(\theta)\cos(3\theta) - 3\sin(\theta)\sin(3\theta)}{-\sin(\theta)\cos(3\theta) - 3\cos(\theta)\sin(3\theta)}.$$

So the slope of the tangent line to the curve at $\theta = \frac{\pi}{4}$ is

$$\frac{dy}{dx}_{|\theta=\pi/4} = \frac{\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{3\pi}{4}\right) - 3\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{3\pi}{4}\right)}{-\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{3\pi}{4}\right) - 3\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{3\pi}{4}\right)} = -1.$$

Furthermore, the tangent line passes through the point $\left(x\left(\frac{\pi}{4}\right), y\left(\frac{\pi}{4}\right)\right) = \left(-\frac{1}{2}, -\frac{1}{2}\right)$. So it has equation

$$\boxed{y = -1\left(x + \frac{1}{2}\right) - \frac{1}{2}}.$$

(b) $r = 1 + 2\sin(\theta)$, $\theta = \dfrac{\pi}{6}$.

Solution. The curve can be parametrized by $x = r\cos(\theta) = \cos(\theta)(1 + 2\sin(\theta))$ and $y = r\sin(\theta) = \sin(\theta)(1 + 2\sin(\theta))$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\ &= \frac{\cos(\theta)(1 + 2\sin(\theta)) + 2\sin(\theta)\cos(\theta)}{-\sin(\theta)(1 + 2\sin(\theta)) + 2\cos(\theta)\cos(\theta)}.\end{aligned}$$

So the slope of the tangent line to the curve at $\theta = \frac{\pi}{6}$ is

$$\begin{aligned}\frac{dy}{dx}_{|\theta=\pi/6} &= \frac{\cos\left(\frac{\pi}{6}\right)\left(1 + 2\sin\left(\frac{\pi}{6}\right)\right) + 2\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right)}{-\sin\left(\frac{\pi}{6}\right)\left(1 + 2\sin\left(\frac{\pi}{6}\right)\right) + 2\cos\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right)} \\ &= 3\sqrt{3}.\end{aligned}$$

Furthermore, the tangent line passes through the point $\left(x\left(\frac{\pi}{6}\right), y\left(\frac{\pi}{6}\right)\right) = \left(\sqrt{3}, 1\right)$. So it has equation

$$\boxed{y = 3\sqrt{3}\left(x - \sqrt{3}\right) + 1}.$$

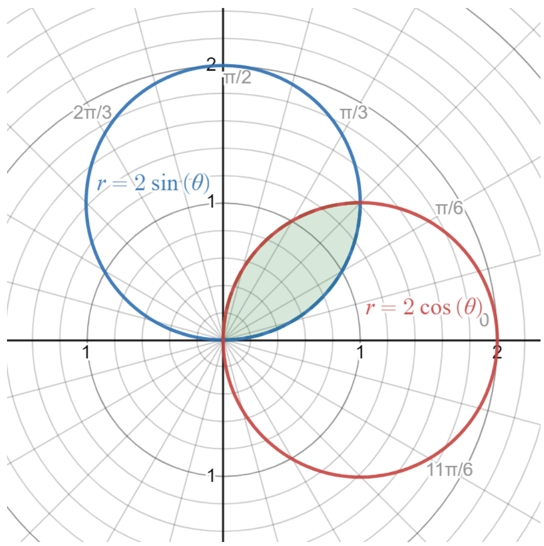
Rutgers University
Math 152

Sections 11.5: Areas and Lengths in Polar Coordinates - Worksheet Solutions

1. Find the areas of the given regions.

   (a) The region shared by the circles $r = 2\sin(\theta)$ and $r = 2\cos(\theta)$.

   Solution. The circles intersect at the origin and when $\theta = \frac{\pi}{4}$, see the figure below.

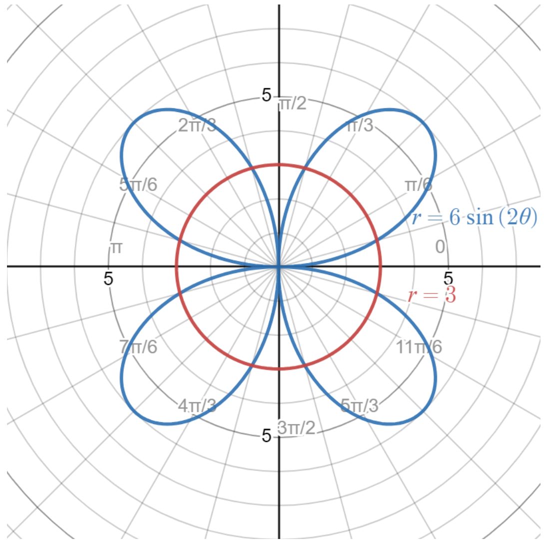


Note that the region is not radially simple. For $0 \leqslant \theta \leqslant \frac{\pi}{4}$, the ray at θ in the region is bounded by the origin and $r = 2\sin(\theta)$. For $\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{2}$, the ray at θ in the region is bounded by the origin and $r = 2\cos(\theta)$. Note that this divides the region into two regions of equal area. Therefore, the area is given by

$$\begin{aligned} A &= \int_0^{\pi/4} \frac{1}{2}(2\sin(\theta))^2 d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}(2\cos(\theta))^2 d\theta \\ &= 4\int_0^{\pi/4} \sin(\theta)^2 d\theta \\ &= 4\int_0^{\pi/4} \frac{1-\sin(2\theta)}{2} d\theta \\ &= 2\left[\theta - \frac{\cos(2\theta)}{2}\right]_0^{\pi/4} \\ &= \boxed{\frac{\pi}{2} + 1 \text{ square units}}. \end{aligned}$$

(b) The region contained inside the leaves of the rose $r = 6\sin(2\theta)$ and outside the circle $r = 3$.

Solution. The curves are sketched below.



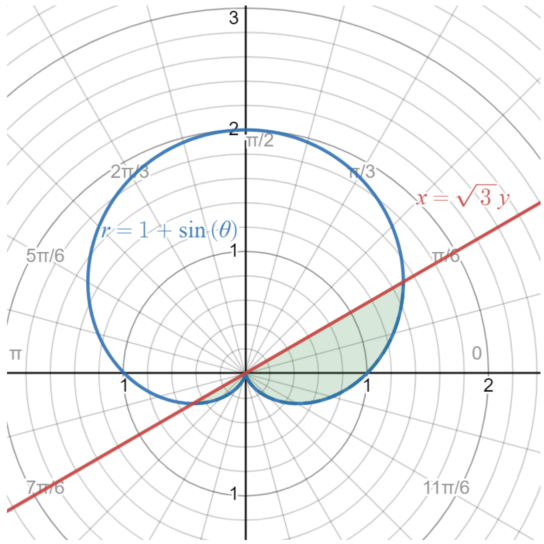
Right off the bat, we observe that by symmetry, it suffices to compute the area of the part of the region in the leaf located in the first quadrant and multiply it by 4. Note that the curves intersect when $6\sin(2\theta) = 3$ or $\sin(2\theta) = \frac{1}{2}$, which gives $\theta = \frac{\pi}{12}, \frac{5\pi}{12}$ as solutions in the first quadrant.

The ray at θ in the region is bounded by $r = 6\sin(2\theta)$ and $r = 3$. Therefore, the area is

$$\begin{aligned}
A &= 4\int_{\pi/12}^{5\pi/12} \frac{1}{2}\left((6\sin(2\theta)^2 - 3^2)\right) d\theta \\
&= 2\int_{\pi/12}^{5\pi/12} \left(36\sin(2\theta)^2 - 9\right) d\theta \\
&= 2\int_{\pi/12}^{5\pi/12} (18(1-\cos(4\theta)) - 9)\, d\theta \\
&= 18\int_{\pi/12}^{5\pi/12} (1 - 2\cos(4\theta))\, d\theta \\
&= 18\left[\theta - \frac{\sin(4\theta)}{2}\right]_{\pi/12}^{5\pi/12} \\
&= \boxed{6\pi + 9\sqrt{3} \text{ square units}}.
\end{aligned}$$

(c) The region inside the cardioid $r = 1 + \sin(\theta)$ and below the line $x = \sqrt{3}y$.

Solution. The line $x = \sqrt{3}y$ is a line through the origin of slope $\frac{1}{\sqrt{3}}$, so it has polar equation $\theta = \frac{\pi}{6}$ or $\theta = -\frac{5\pi}{6}$. The region is sketched below.

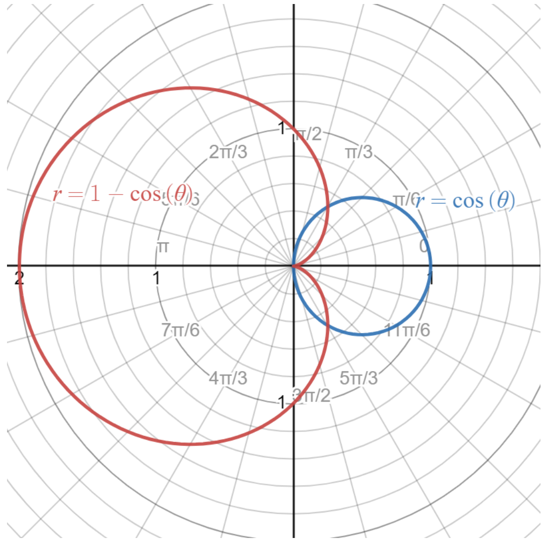


For $-\frac{5\pi}{6} \leqslant \theta \leqslant \frac{\pi}{6}$, the ray at θ in the region is bounded by $r = 1+\sin(\theta)$. Therefore, the area is

$$\begin{aligned}
A &= \int_{-5\pi/6}^{\pi/6} \frac{1}{2}(1+\sin(\theta))^2 d\theta \\
&= \frac{1}{2}\int_{-5\pi/6}^{\pi/6} (1+2\sin(\theta)+\sin(\theta)^2) d\theta \\
&= \frac{1}{2}\int_{-5\pi/6}^{\pi/6} \left(1+2\sin(\theta)+\frac{1-\cos(2\theta)}{2}\right) d\theta \\
&= \frac{1}{4}\int_{-5\pi/6}^{\pi/6} \left(3+4\sin(\theta)-\cos(2\theta)\right) d\theta \\
&= \frac{1}{4}\left[3\theta - 4\cos(\theta) - \frac{\sin(2\theta)}{2}\right]_{-5\pi/6}^{\pi/6} \\
&= \boxed{\frac{6\pi-9\sqrt{3}}{8} \text{ square units}}.
\end{aligned}$$

(d) The region inside the circle $r = \cos(\theta)$ and outside the cardioid $r = 1-\cos(\theta)$.

Solution. Observe that the curves intersect when $\cos(\theta) = 1-\cos(\theta)$, that is $\cos(\theta) = \frac{1}{2}$, which gives $\theta = -\frac{\pi}{3}, \frac{\pi}{3}$. The region is sketched below.

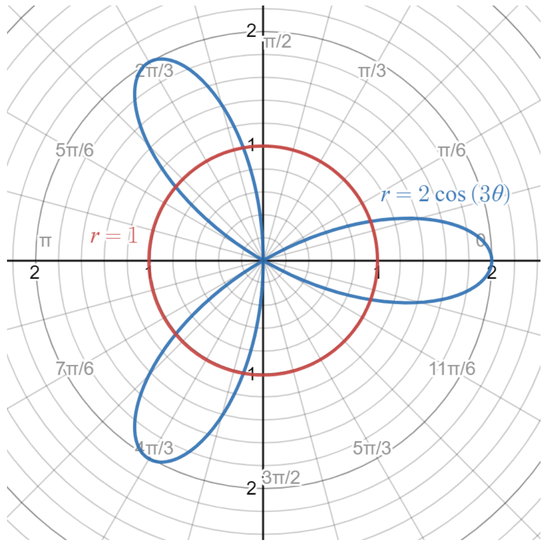


For $-\frac{\pi}{3} \leqslant \theta \leqslant \frac{\pi}{3}$, the ray at θ in the region is bounded between $r = 1 - \cos(\theta)$ and $r = \cos(\theta)$. Also, using the symmetry of the region about the x-axis, it suffices to compute the area for $0 \leqslant \theta \leqslant \frac{\pi}{3}$ and double it. Therefore, the area is

$$\begin{aligned} A &= 2\int_0^{\pi/3} \frac{1}{2}\left(\cos(\theta)^2 - \left(1 - \cos(\theta)^2\right)\right) d\theta \\ &= \int_0^{\pi/3} \left(\cos(\theta)^2 - 1 + 2\cos(\theta) - \cos(\theta)^2\right) d\theta \\ &= \int_0^{\pi/3} (2\cos(\theta) - 1)\, d\theta \\ &= [2\sin(\theta) - \theta]_0^{\pi/3} \\ &= \boxed{\sqrt{3} - \frac{\pi}{3} \text{ square units}}. \end{aligned}$$

(e) The region shared by one leaf of the rose $r = 2\cos(3\theta)$ and the circle $r = 1$.

Solution. The curves are sketched below.

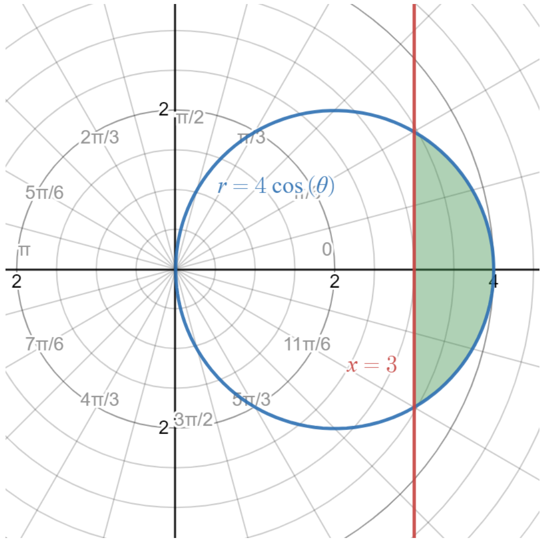


By symmetry, it suffices to find the area of the region in the first quadrant shared by one leaf of the rose and the circle and double it. In the first quadrant, the curves intersect when $2\cos(3\theta) = 1$, that is $\cos(3\theta) = \frac{1}{2}$, which gives $\theta = \frac{\pi}{9}$. Furthermore, the leaf crosses the origin when $\cos(3\theta) = 0$, which gives $\theta = \frac{\pi}{6}$.

For $0 \leqslant \theta \leqslant \frac{\pi}{9}$, the ray at θ is the region is bounded by $r = 1$. For $\frac{\pi}{9} \leqslant \theta \leqslant \frac{\pi}{6}$, the ray at θ is the region is bounded by $r = 2\cos(3\theta)$. Therefore the area is

$$\begin{aligned}
A &= 2\left(\int_0^{\pi/9} \frac{1}{2} 1^2 d\theta + \int_{\pi/9}^{\pi/6} \frac{1}{2}(2\cos(3\theta))^2 d\theta\right) \\
&= \int_0^{\pi/9} d\theta + 4\int_{\pi/9}^{\pi/6} \cos(3\theta)^2 d\theta \\
&= \frac{\pi}{9} + 4\int_{\pi/9}^{\pi/6} \frac{1+\cos(6\theta)}{2} d\theta \\
&= \frac{\pi}{9} + 2\left[\theta + \frac{\sin(6\theta)}{6}\right]_{\pi/9}^{\pi/6} \\
&= \frac{\pi}{9} + 2\left(\frac{\pi}{18} - \frac{\sqrt{3}}{12}\right) \\
&= \boxed{\frac{2\pi}{9} - \frac{\sqrt{3}}{6} \text{ square units}}.
\end{aligned}$$

2. Consider the region $\mathcal{R}$ contained in the circle $r = 4\cos(\theta)$ to the right of the line $x = 3$.



(a) Find the area of the region $\mathcal{R}$ using integration with respect to x.

Solution. The circle has Cartesian equation $(x-2)^2+y^2=4$. If we solve for y, we get the two solutions $y=\pm\sqrt{4-(x-2)^2}$. The equation $y=\sqrt{4-(x-2)^2}$ corresponds to the top-half semi-circle, while the equation $y=-\sqrt{4-(x-2)^2}$ corresponds to the bottom-half semi-circle. For $3\leqslant x\leqslant 4$, the vertical strip at x in the region is bounded by the top-half semi-circle and the bottom-half semi circle, so it has length $\ell(x)=\sqrt{4-(x-2)^2}-(-\sqrt{4-(x-2)^2})=2\sqrt{4-(x-2)^2}$. Therefore, the area is

$$A=\int_3^4 \ell(x)dx=2\int_3^4\sqrt{4-(x-2)^2}dx.$$

We can compute this integral using a trigonometric substitution. We will want $4-(x-2)^2=4-4\sin(\theta)^2$, so we will substitute $x=2+2\sin(\theta)$. Then we get $dx=2\cos(\theta)d\theta$ and $\sqrt{4-(x-2)^2}=\sqrt{4-4\sin(\theta)^2}=2\cos(\theta)$. The bounds become

$$x=3\ \Rightarrow\ \theta=\sin^{-1}\left(\frac{x-2}{2}\right)=\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6},$$
$$x=4\ \Rightarrow\ \theta=\sin^{-1}\left(\frac{x-2}{2}\right)=\sin^{-1}(1)=\frac{\pi}{2}.$$

So the integral becomes

$$\begin{aligned}A&=2\int_{\pi/6}^{\pi/2}(2\cos(\theta))2\cos(\theta)d\theta\\&=8\int_{\pi/6}^{\pi/2}\cos(\theta)^2d\theta\\&=4\int_{\pi/6}^{\pi/2}(1+\cos(2\theta))d\theta\\&=4\left[\theta+\frac{\sin(2\theta)}{2}\right]_{\pi/6}^{\pi/2}\\&=\boxed{\frac{4\pi}{3}-\sqrt{3}\text{ square units}}.\end{aligned}$$

(b) Find the area of the region $\mathcal{R}$ using integration with respect to y.

Solution. If we solve the Cartesian equation of the circle for x, we get $x = 2 \pm \sqrt{4-y^2}$. The equation corresponding the the right semi-circle, which bounds the region, is $x = 2+\sqrt{4-y^2}$. The horizontal strip at y in the region is bounded on the left by $x = 3$ and on the right by $x = 2+\sqrt{4-y^2}$, so it has length $\ell(y) = 2+\sqrt{4-y^2} - 3 = \sqrt{4-y^2} - 1$. Plugging-in $x = 3$ in the equation of the circle gives $y = \pm\sqrt{3}$ as the boundaries of the region. Note that by symmetry, it suffices to calculate the area above the x-axis, so for $0 \leqslant y \leqslant \sqrt{3}$, and double it. Therefore, the area is

$$A = 2\int_0^{\sqrt{3}} \ell(y)dy = 2\int_0^{\sqrt{3}} \left(\sqrt{4-y^2} - 1\right) dy = 2\int_0^{\sqrt{3}} \sqrt{4-y^2}dy - 2\sqrt{3}.$$

We can again compute this integral with the trigonometric substitution $y = 2\sin(\theta)$ to get

$$\begin{aligned} A &= 2\int_0^{\pi/3} (2\cos(\theta))2\cos(\theta)d\theta - 2\sqrt{3} \\ &= 8\int_0^{\pi/3} \cos(\theta)^2 d\theta - 2\sqrt{3} \\ &= 4\int_0^{\pi/3} (1+\cos(2\theta))d\theta - 2\sqrt{3} \\ &= 4\left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{\pi/3} - 2\sqrt{3} \\ &= 4\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right) - 2\sqrt{3} \\ &= \boxed{\frac{4\pi}{3} - \sqrt{3} \text{ square units}}. \end{aligned}$$

(c) Find the area of the region $\mathcal{R}$ using integration with respect to θ.

Solution. The vertical line $x = 3$ has polar equation $r = 3\sec(\theta)$. The two curves intersect when $3\sec(\theta) = 4\cos(\theta)$, which gives $\cos(\theta)^2 = \frac{3}{4}$ or $\theta = \pm\frac{\pi}{6}$. Note that by symmetry, it will suffice to compute the area of the region above the x-axis, that is for $0 \leqslant \theta \leqslant \frac{\pi}{6}$, and double it. When $0 \leqslant \theta \leqslant \frac{\pi}{6}$, the ray at θ in the region is bounded between $r = 4\cos(\theta)$ and $r = 3\sec(\theta)$. So the area is

$$\begin{aligned} A &= 2\int_0^{\pi/6} \frac{1}{2}\left((4\cos(\theta))^2 - (3\sec(\theta))^2\right) d\theta \\ &= \int_0^{\pi/6} \left(16\cos(\theta)^2 - 9\sec(\theta)^2\right) d\theta \\ &= \int_0^{\pi/6} \left(8 + 8\cos(2\theta) - 9\sec(\theta)^2\right) d\theta \\ &= [8\theta + 4\sin(2\theta) - 9\tan(\theta)]_0^{\pi/6} \\ &= \frac{8\pi}{6} + 4\frac{\sqrt{3}}{2} - 9\frac{\sqrt{3}}{3} \\ &= \boxed{\frac{4\pi}{3} - \sqrt{3} \text{ square units}}. \end{aligned}$$

3. Find the lengths of the given polar curves.

(a) $r = \sqrt{1+\cos(2\theta)}$, $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

Solution. We have

$$\frac{dr}{d\theta} = \frac{-2\sin(2\theta)}{2\sqrt{1+\cos(2\theta)}} = -\frac{\sin(2\theta)}{\sqrt{1+\cos(2\theta)}}.$$

So

$$\begin{aligned}
\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} &= \sqrt{1+\cos(2\theta) + \frac{\sin(2\theta)^2}{1+\cos(2\theta)}} \\
&= \sqrt{\frac{(1+\cos(2\theta))^2 + \sin(2\theta)^2}{1+\cos(2\theta)}} \\
&= \sqrt{\frac{1+2\cos(2\theta)+\cos(2\theta)^2+\sin(2\theta)^2}{1+\cos(2\theta)}} \\
&= \sqrt{\frac{1+2\cos(2\theta)+1}{1+\cos(2\theta)}} \\
&= \sqrt{\frac{2+2\cos(2\theta)}{1+\cos(2\theta)}} \\
&= \sqrt{2}.
\end{aligned}$$

Therefore, the length of the curve is

$$\begin{aligned}
L &= \int_0^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\
&= \int_0^{\pi/2} \sqrt{2} d\theta \\
&= \boxed{\frac{\pi\sqrt{2}}{2} \text{ units}}.
\end{aligned}$$

(b) $r = \dfrac{2}{1-\cos(\theta)}$, $\frac{\pi}{2} \leqslant \theta \leqslant \pi$.

Solution. The trained eye notices that this expression looks like something we'd get from a double-angle formula. Indeed, we know that $\sin(x)^2 = \frac{1-\cos(2x)}{2}$, so taking reciprocals and replacing x by $\frac{\theta}{2}$ gives

$$\frac{2}{1-\cos(\theta)} = \frac{1}{\sin\left(\frac{\theta}{2}\right)^2} = \csc\left(\frac{\theta}{2}\right)^2.$$

Working with the form $r = \csc\left(\frac{\theta}{2}\right)^2$ of the polar equation of the curve will simply computations a little. We have

$$\frac{dr}{d\theta} = 2\csc\left(\frac{\theta}{2}\right)\left(-\frac{1}{2}\csc\left(\frac{\theta}{2}\right)\cot\left(\frac{\theta}{2}\right)\right) = -\csc\left(\frac{\theta}{2}\right)^2\cot\left(\frac{\theta}{2}\right).$$

So

$$\begin{aligned}
\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} &= \sqrt{\csc\left(\frac{\theta}{2}\right)^4+\csc\left(\frac{\theta}{2}\right)^4\cot\left(\frac{\theta}{2}\right)^2} \\
&= \csc\left(\frac{\theta}{2}\right)^2\sqrt{1+\cot\left(\frac{\theta}{2}\right)^2} \\
&= \csc\left(\frac{\theta}{2}\right)^2\sqrt{\csc\left(\frac{\theta}{2}\right)^2} \\
&= \csc\left(\frac{\theta}{2}\right)^3.
\end{aligned}$$

Therefore, the length of the curve is

$$\begin{aligned}
L &= \int_{\pi/2}^{\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta \\
&= \int_{\pi/2}^{\pi}\csc\left(\frac{\theta}{2}\right)^3 d\theta \\
&= 2\int_{\pi/4}^{\pi/2}\csc\left(u\right)^3 du,
\end{aligned}$$

where we have made the substitution $u=\frac{\theta}{2}$ in the last integral. In Section 8.3, we learned that we can compute such integrals with an IBP followed by a trigonometric identity giving a relation that we can solve for the unknown integral. For the IPB, we use the parts

$$\begin{aligned}
u = \csc(u) &\;\Rightarrow\; du = -\csc(u)\cot(u)du, \\
dv = \csc(u)^2 du &\;\Rightarrow\; v = -\cot(u).
\end{aligned}$$

We get

$$\begin{aligned}
\int_{\pi/4}^{\pi/2}\csc\left(u\right)^3 du &= \left[-\csc(u)\cot(u)\right]_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2}(-\csc(u)\cot(u))(-\cot(u))du \\
\int_{\pi/4}^{\pi/2}\csc\left(u\right)^3 du &= \left[-\csc(u)\cot(u)\right]_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2}(-\csc(u)\cot(u))(-\cot(u))du \\
\int_{\pi/4}^{\pi/2}\csc\left(u\right)^3 du &= \sqrt{2} - \int_{\pi/4}^{\pi/2}\csc(u)\cot(u)^2 du \\
\int_{\pi/4}^{\pi/2}\csc\left(u\right)^3 du &= \sqrt{2} - \int_{\pi/4}^{\pi/2}\csc(u)(\csc(u)^2-1)du \\
\int_{\pi/4}^{\pi/2}\csc\left(u\right)^3 du &= \sqrt{2} - \int_{\pi/4}^{\pi/2}\csc(u)^3 du + \int_{\pi/4}^{\pi/2}\csc(u)du \\
\int_{\pi/4}^{\pi/2}\csc\left(u\right)^3 du &= \sqrt{2} - \int_{\pi/4}^{\pi/2}\csc(u)^3 du + \left[-\ln\left|\csc(u)+\cot(u)\right|\right]_{\pi/4}^{\pi/2} \\
\int_{\pi/4}^{\pi/2}\csc\left(u\right)^3 du &= \sqrt{2} - \int_{\pi/4}^{\pi/2}\csc(u)^3 du + \ln\left(\sqrt{2}+1\right).
\end{aligned}$$

Moving the term $-\int_{\pi/4}^{\pi/2} \csc(u)^3 du$ to the left-hand side gives

$$2\int_{\pi/4}^{\pi/2} \csc(u)^3 du = \sqrt{2} - \ln\left(\sqrt{2}+1\right),$$

so

$$L = 2\int_{\pi/4}^{\pi/2} \csc(u)^3 du = \boxed{\sqrt{2} - \ln\left(\sqrt{2}+1\right) \text{ units}}.$$

(c) $r = e^{3\theta}$, $0 \leqslant \theta \leqslant \pi$.

Solution. We have

$$\frac{dr}{d\theta} = 3e^{3\theta}.$$

So

$$\begin{aligned}
L &= \int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \\
&= \int_0^{\pi} \sqrt{\left(e^{3\theta}\right)^2 + \left(3e^{3\theta}\right)^2} d\theta \\
&= \int_0^{\pi} \sqrt{e^{6\theta} + 9e^{6\theta}} d\theta \\
&= \int_0^{\pi} \sqrt{10e^{6\theta}} d\theta \\
&= \int_0^{\pi} \sqrt{10}e^{3\theta} d\theta \\
&= \left[\frac{\sqrt{10}e^{3\theta}}{3}\right]_0^{\pi} \\
&= \boxed{\frac{\sqrt{10}(e^{3\pi}-1)}{3} \text{ units}}.
\end{aligned}$$