Rutgers University
Math 152

Midterm 1 Practice Problems Solutions

1. Evaluate the following antiderivatives.

(a) $\displaystyle\int \frac{dx}{x^2+10x+29}$

Solution. We can use the reference antiderivative

$$\int \frac{du}{1+u^2} = \tan^{-1}(u) + C$$

after completing the square in the denominator and using a substitution. Completing the square gives

$$x^2+10x+29 = (x^2+10x+25) - 25 + 29 = (x+5)^2 + 4.$$

The integral becomes

$$\begin{aligned}
\int \frac{dx}{x^2+10x+29} &= \int \frac{dx}{(x+5)^2+4} \\
&= \frac{1}{4}\int \frac{dx}{\left(\frac{x+5}{2}\right)^2+1} \\
&= \frac{1}{4}\int \frac{2du}{u^2+1} \qquad \left(u = \frac{x+5}{2}\right) \\
&= \frac{1}{2}\tan^{-1}(u) + C \\
&= \boxed{\frac{1}{2}\tan^{-1}\left(\frac{x+5}{2}\right) + C}.
\end{aligned}$$

(b) $\displaystyle\int \frac{dt}{t\sqrt{1-9\ln(t)^2}}$

Solution. We can use the reference antiderivative

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}(u) + C$$

after using the substitution $u = 3\ln(t)$. This gives $du = \frac{3dt}{t}$, so $\frac{dt}{t} = \frac{du}{3}$. The integral becomes

$$\begin{aligned}
\int \frac{dt}{t\sqrt{1-9\ln(t)^2}} &= \int \frac{du}{3\sqrt{1-u^2}} \\
&= \frac{1}{3}\sin^{-1}(u) + C \\
&= \boxed{\frac{1}{3}\sin^{-1}(3\ln(t)) + C}.
\end{aligned}$$

(c) $\displaystyle\int \frac{\sqrt{1+\sqrt{z}}}{\sqrt{z}}dz$

Solution. Method 1. Substitute $u = 1+\sqrt{z}$, so that $\frac{dz}{\sqrt{z}} = 2du$. This gives

$$\begin{aligned}
\int \frac{\sqrt{1+\sqrt{z}}}{\sqrt{z}}dz &= \int 2\sqrt{u}du \\
&= \frac{4}{3}u^{3/2} + C \\
&= \boxed{\frac{4}{3}(1+\sqrt{z})^{3/2} + C}.
\end{aligned}$$

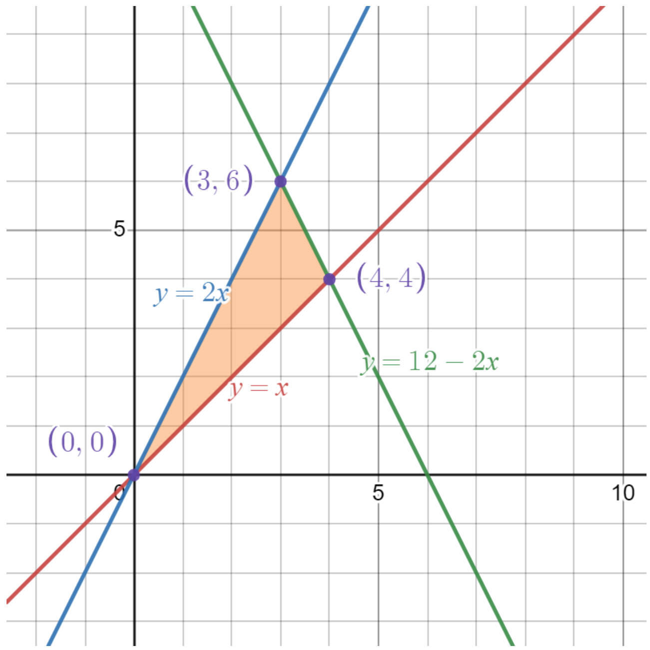
Method 2. Substitute $u = \sqrt{1+\sqrt{z}}$, so that $du = \frac{dz}{4\sqrt{z}\sqrt{1+\sqrt{z}}}$. This gives

$$\begin{aligned}
\int \frac{\sqrt{1+\sqrt{z}}}{\sqrt{z}}dz &= \int (1+\sqrt{z})\frac{dz}{\sqrt{z}\sqrt{1+\sqrt{z}}} \\
&= \int 4u^2 du \\
&= \frac{4}{3}u^3 + C \\
&= \frac{4}{3}\left(\sqrt{1+\sqrt{z}}\right)^3 + C \\
&= \boxed{\frac{4}{3}(1+\sqrt{z})^{3/2} + C}.
\end{aligned}$$

2. Let $\mathcal{R}$ be the region bounded by the lines $y = x$, $y = 2x$ and $y = -2x+12$.

   (a) Sketch the region $\mathcal{R}$. Label the curves, their intersection points and lightly shade the region.

   Solution.



(b) Calculate the area of the region using (i) an x-integral and (ii) a y-integral.

Solution. (i) The vertical strip at x in the region is bounded by $y = x$ on the bottom. On the top, the strip is bounded by $y = 2x$ for $0 \leqslant x \leqslant 3$ and $y = 12 - 2x$ for $3 \leqslant x \leqslant 4$. Therefore, the vertical strip at x has length

$$\ell(x) = \begin{cases} 2x - x & \text{if } 0 \leqslant x \leqslant 3, \\ (12 - 2x) - x & \text{if } 3 \leqslant x \leqslant 4. \end{cases}$$

Thus the area is given by

$$\begin{aligned} A &= \int_0^3 (2x - x)dx + \int_3^4 ((12 - 2x) - x)dx \\ &= \int_0^3 xdx + \int_3^4 (12 - 3x)dx \\ &= \left[\frac{x^2}{2}\right]_0^3 + \left[12x - \frac{3x^2}{2}\right]_3^4 \\ &= \boxed{6 \text{ square units}}. \end{aligned}$$

(ii) The lines can be expressed as functions of y as follows:

$$\begin{aligned} &y = x \;\Rightarrow\; x = y, \\ &y = 2x \;\Rightarrow\; x = \frac{y}{2}, \\ &y = -2x + 12 \;\Rightarrow\; x = \frac{12 - y}{2}. \end{aligned}$$

The horizontal strip at y is bounded on the left by $x = \frac{y}{2}$. The line bounding the strip on the right is $x = y$ for $0 \leqslant y \leqslant 4$ and $x = \frac{12-y}{2}$ for $4 \leqslant y \leqslant 6$. Therefore, the horizontal strip at y has length

$$\ell(y) = \begin{cases} 2y - \dfrac{y}{2} & \text{if } 0 \leqslant x \leqslant 4, \\ \dfrac{12-y}{2} - \dfrac{y}{2} & \text{if } 4 \leqslant x \leqslant 6. \end{cases}$$

Thus the area is given by

$$\begin{aligned} A &= \int_0^4 \left(y - \frac{y}{2}\right) dy + \int_4^6 \left(\frac{12-y}{2} - \frac{y}{2}\right) dy \\ &= \int_0^4 \frac{y}{2} dy + \int_4^6 (6-y)\, dy \\ &= \left[\frac{y^2}{4}\right]_0^4 + \left[6y - \frac{y^2}{2}\right]_4^6 \\ &= \boxed{6 \text{ square units}}. \end{aligned}$$

(c) The base of a solid is $\mathcal{R}$, and the cross-sections perpendicular to the x-axis are isosceles right triangles with height on the base. Set-up an expression with integrals that calculates the volume of the solid.

Solution. The area of an isosceles right triangle with base ℓ is $\frac{1}{2}\ell^2$. Therefore, using the length of the vertical strip at x that we found in part (b), we get

$$\boxed{V = \int_0^3 \frac{1}{2}(2x-x)^2 dx + \int_3^4 \frac{1}{2}((12-2x)-x)^2 dx}.$$

(d) The base of a solid is $\mathcal{R}$, and the cross-sections perpendicular to the y-axis are circles with diameter in the base. Set-up an expression with integrals that calculates the volume of the solid.

Solution. The area of a circle with diameter ℓ is $\frac{\pi}{4}\ell^2$. Therefore, using the length of the horizontal strip at y that we found in part (b), we get

$$\boxed{V = \int_0^4 \frac{\pi}{4}\left(y - \frac{y}{2}\right)^2 dy + \int_4^6 \frac{\pi}{4}\left(\frac{12-y}{2} - \frac{y}{2}\right)^2 dy}.$$

(e) We create a solid of revolution by revolving $\mathcal{R}$ about the line $x = -2$. Set up an expression with integrals that calculates the volume of the solid using (i) the method of disks/washers and (ii) the method of cylindrical shells.

Solution. (i) Since the axis of revolution is vertical, we form a washer by revolving a horizontal strip in the region, so our integral will be a y-integral. The washers have inner radius $r_{\text{in}}(y) = \frac{y}{2} - (-2) = \frac{y}{2} + 2$. The outer radius is $r_{\text{out}}(y) = y - (-2) = y + 2$ if $0 \leqslant y \leqslant 4$, and $r_{\text{out}}(y) = \frac{12-y}{2} - (-2) = 8 - \frac{y}{2}$ if $4 \leqslant y \leqslant 6$. Therefore

$$\boxed{V = \int_0^4 \pi\left((y+2)^2 - \left(\frac{y}{2}+2\right)^2\right) dy + \int_4^6 \pi\left(\left(8 - \frac{y}{2}\right)^2 - \left(\frac{y}{2}+2\right)^2\right) dy}.$$

(ii) Since the axis of revolution is vertical, we form a cylindrical shell by revolving a vertical strip in the region, so our integral will be an x-integral. The shells have radius $r(x) = x - (-2) = x + 2$. The shell height is $h(x) = 2x - x = x$ when $0 \leqslant x \leqslant 4$, and $h(x) = (12 - 2x) - x = 12 - 3x$ when $3 \leqslant x \leqslant 4$. Therefore

$$\boxed{V = \int_0^3 2\pi(x+2)x\,dx + \int_3^4 2\pi(x+2)(12-3x)\,dx}.$$

(f) We create a solid of revolution by revolving $\mathcal{R}$ about the line $y = 8$. Set up an expression with integrals that calculates the volume of the solid using (i) the method of disks/washers and (ii) the method of cylindrical shells.

Solution. (i) Since the axis of revolution is horizontal, we form a washer by revolving a vertical strip in the region, so our integral will be an x-integral. The washers have outer radius $r_{\text{out}}(x) = 8 - x$. The inner radius is $r_{\text{in}}(x) = 8 - 2x$ for $0 \leqslant x \leqslant 3$, and $r_{\text{in}}(x) = 8 - (12 - 2x) = 2x - 4$ for $3 \leqslant x \leqslant 4$. Therefore

$$\boxed{V = \int_0^3 \pi\left((8-x)^2 - (8-2x)^2\right)dx + \int_3^4 \pi\left((8-x)^2 - (2x-4)^2\right)dx}.$$

(ii) Since the axis of revolution is horizontal, we form a cylindrical shell by revolving a horizontal strip in the region, so our integral will be a y-integral. The shells have radius $r(y) = 8 - y$. The shell height is $h(y) = y - \frac{y}{2} = \frac{y}{2}$ if $0 \leqslant y \leqslant 4$, and $h(y) = \frac{12-y}{2} - \frac{y}{2} = 6 - y$ if $4 \leqslant y \leqslant 6$. Therefore

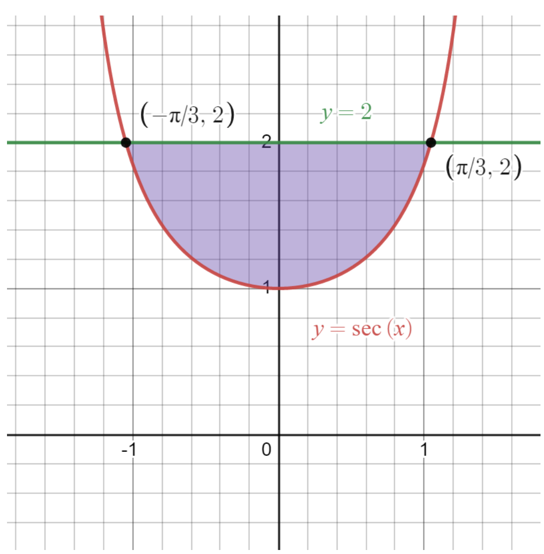
$$\boxed{V = \int_0^4 2\pi(8-y)\frac{y}{2}\,dy + \int_4^6 2\pi(8-y)(6-y)\,dy}.$$

3. Consider the region $\mathcal{R}$ traversed by the y-axis and bounded by the line $y = 2$ and the graph of $y = \sec(x)$.

(a) Sketch the region. Clearly label the curves and their intersection points.

Solution. The curves intersect when

$$\sec(x) = 2 \;\Rightarrow\; x = -\frac{\pi}{3}, \frac{\pi}{3}.$$



(b) Calculate the area of the region.

Solution. The length of the vertical strip at x is $\ell(x) = 2 - \sec(x)$. Therefore, the area of the region is

$$\begin{aligned} A &= \int_{-\pi/3}^{\pi/3} (2 - \sec(x))dx \\ &= 2\int_{0}^{\pi/3} (2 - \sec(x))dx \quad \text{(by symmetry)} \\ &= 2\left[2x - \ln|\sec(x) + \tan(x)|\right]_0^{\pi/3} \\ &= \boxed{2\left(\frac{2\pi}{3} - \ln\left(2 + \sqrt{3}\right)\right) \text{ square units}}. \end{aligned}$$

(c) Calculate the volume of a solid with base $\mathcal{R}$ if the cross-sections perpendicular to the x-axis are (i) squares and (ii) rectangles of perimeter 8.

Solution. The vertical strip at x is bounded by $y = 2$ on the top and $y = \sec(x)$ on the bottom. Therefore, it has length $\ell(x) = 2 - \sec(x)$.

(i) The area of a square with side length ℓ is ℓ^2. Using the expression for $\ell(x)$ we found above, we can express the area of the cross-section at x as

$$A(x) = \ell(x)^2 = (2 - \sec(x))^2 = 4 - 4\sec(x) + \sec(x)^2.$$

Therefore the volume is

$$\begin{aligned} V &= \int_{-\pi/3}^{\pi/3} A(x)dx \\ &= \int_{-\pi/3}^{\pi/3} (4 - 4\sec(x) + \sec(x)^2)dx \\ &= 2\int_{0}^{\pi/3} (4 - 4\sec(x) + \sec(x)^2)dx \quad \text{(by parity)} \\ &= 2\left[4x - 4\ln|\sec(x) + \tan(x)| + \tan(x)\right]_0^{\pi/3} \\ &= 2\left(4\frac{\pi}{3} - 4\ln\left|\sec\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{3}\right)\right| + \tan\left(\frac{\pi}{3}\right)\right) \\ &= \boxed{2\left(\frac{4\pi}{3} - 4\ln\left(2 + \sqrt{3}\right) + \sqrt{3}\right) \text{ cubic units}}. \end{aligned}$$

(ii) If a rectangle has perimeter 8 and length ℓ, its height h satisfies $2\ell + 2h = 8$, so $h = 4 - \ell$. So the area of such a rectangle is $\ell(4 - \ell)$. Therefore, the area of the cross-section at x is

$$A(x) = \ell(x)(4 - \ell(x)) = (2 - \sec(x))(4 - (2 - \sec(x))) = (2 - \sec(x))(2 + \sec(x)) = 4 - \sec(x)^2.$$

Therefore the volume is

$$V = \int_{-\pi/3}^{\pi/3} A(x)dx$$

$$
\begin{aligned}
&= \int_{-\pi/3}^{\pi/3} (4 - \sec(x)^2) dx \\
&= 2\int_{0}^{\pi/3} (4 - \sec(x)^2) dx \quad \text{(by parity)} \\
&= 2\left[4x - \tan(x)\right]_0^{\pi/3} \\
&= 2\left(4\frac{\pi}{3} - \tan\left(\frac{\pi}{3}\right)\right) \\
&= \boxed{2\left(\frac{4\pi}{3} - \sqrt{3}\right) \text{ cubic units}}.
\end{aligned}
$$

(d) We revolve the region $\mathcal{R}$ to obtain a solid of revolution. Calculate the volume of the solid if the axis of revolution is (i) the x-axis and (ii) the line $y = 2$.

Solution. (i) Revolving the vertical strip at x around the x-axis creates a washer with inner radius $r_{\text{in}}(x) = \sec(x)$ and outer radius $r_{\text{out}}(x) = 2$. Therefore the volume is given by

$$
\begin{aligned}
V &= \int_{-\pi/3}^{\pi/3} \pi \left(r_{\text{out}}(x)^2 - r_{\text{in}}(x)^2\right) dx \\
&= \pi \int_{-\pi/3}^{\pi/3} \left(4 - \sec(x)^2\right) dx \\
&= \boxed{2\pi\left(\frac{4\pi}{3} - \sqrt{3}\right) \text{ cubic units}}.
\end{aligned}
$$

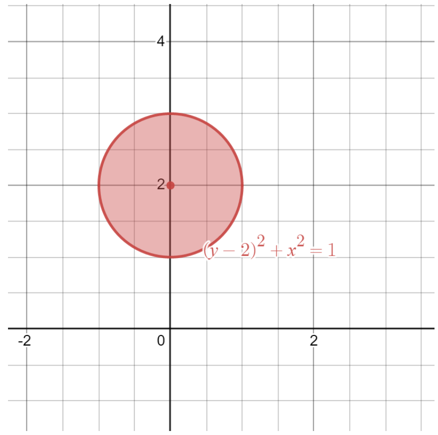
(We can use the integral calculation from (b)(ii).)

(ii) Revolving the vertical strip at x around the line $y = 2$ creates a disk with radius $r(x) = 2 - \sec(x)$. Therefore the volume is given by

$$
\begin{aligned}
V &= \int_{-\pi/3}^{\pi/3} \pi r(x)^2 dx \\
&= \int_{-\pi/3}^{\pi/3} \pi \left(2 - \sec(x)\right)^2 dx \\
&= \boxed{2\pi\left(\frac{4\pi}{3} - 4\ln\left(2 + \sqrt{3}\right) + \sqrt{3}\right) \text{ cubic units}}.
\end{aligned}
$$

(We can use the integral calculation from (b)(i).)

4. Consider the disk of equation $(y-2)^2 + x^2 \leqslant 1$ centered at $(0, 2)$ of radius 1. A torus (or informally, a donut) is created by revolving the disk about the x-axis.



(a) Find the volume of the torus using the washer method.

Solution. Since the axis of revolution is horizontal, we form washers by revolving vertical strips in the region. To find the inner and outer radii of the washers, we need to find the equations for the top and bottom boundaries of the region as functions of x.

$$(y-2)^2 + x^2 = 1 \Rightarrow y = 2 \pm \sqrt{1-x^2} \Rightarrow y_{\text{bot}} = 2 - \sqrt{1-x^2}, y_{\text{top}} = 2 + \sqrt{1-x^2}, -1 \leqslant x \leqslant 1.$$

Since the axis of revolution is $y = 0$, we have $r_{\text{in}}(x) = y_{\text{bot}} - 0 = 2 - \sqrt{1-x^2}$ and $r_{\text{out}}(x) = y_{\text{top}} - 0 = 2 + \sqrt{1-x^2}$. Therefore

$$\begin{aligned} V &= \int_{-1}^{1} \pi \left(\left(2 + \sqrt{1-x^2}\right)^2 - \left(2 - \sqrt{1-x^2}\right)^2 \right) dx \\ &= \int_{-1}^{1} \pi \left(\left(4 + (1-x^2) + 4\sqrt{1-x^2}\right) - \left(4 + (1-x^2) - 4\sqrt{1-x^2}\right) \right) dx \\ &= 8\pi \int_{-1}^{1} \sqrt{1-x^2} dx. \end{aligned}$$

Observe that $\int_{-1}^{1} \sqrt{1-x^2} dx$ computes the area of a semi-circle of radius 1, so it is equal to $\frac{\pi}{2}$. We deduce that $\boxed{V = 4\pi^2 \text{ cubic units}}$.

(b) Find the volume of the torus using the shell method.

Solution. Since the axis of revolution is horizontal, we form washers by revolving horizontal strips in the region. The shell radius will be the distance between the axis of revolution, $y = 0$ and the slice, so it equal to $r(y) = y - 0 = y$. To find the shell height, we need to find the equations for the left and right boundaries of the region as functions of y.

$$(y-2)^2 + x^2 = 1 \Rightarrow x = \pm\sqrt{1-(y-2)^2} \Rightarrow x_{\text{right}} = \sqrt{1-(y-2)^2}, x_{\text{left}} = -\sqrt{1-(y-2)^2}, 1 \leqslant y \leqslant 3.$$

Thus the shell height is $h(y) = x_{\text{right}} - x_{\text{left}} = 2\sqrt{1-(y-2)^2}$. Therefore

$$V = \int_{1}^{3} 2\pi y \left(2\sqrt{1-(y-2)^2}\right) dy$$

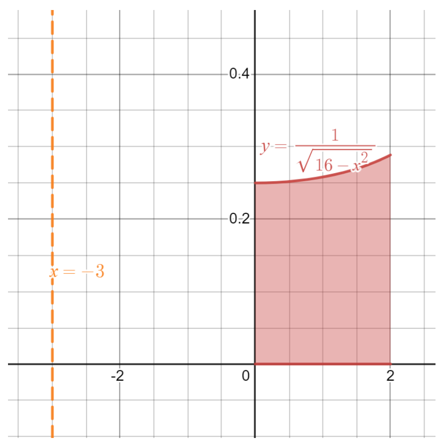
$$= 4\pi \int_1^3 y\sqrt{1-(y-2)^2}dy$$
$$= 4\pi \int_{-1}^1 (u+2)\sqrt{1-u^2}du \quad (\text{with } u = y-2)$$
$$= 4\pi \int_{-1}^1 \left(u\sqrt{1-u^2} + 2\sqrt{1-u^2}\right) du$$

Observe that $\int_{-1}^1 u\sqrt{1-u^2}du = 0$ since the integrand is odd and the interval of integration is centered at 0. In the previous question, we explained that $\int_{-1}^1 \sqrt{1-u^2}du = \frac{\pi}{2}$. So $\boxed{V = 4\pi^2 \text{ cubic units}}$.

5. The two parts of this problem are independent.

(a) Let $\mathcal{R}$ be the region under the graph of $y = \dfrac{1}{\sqrt{16-x^2}}$ for $0 \leqslant x \leqslant 2$. Find the volume of the solid obtained by revolving $\mathcal{R}$ about the line $x = -3$.

Solution.

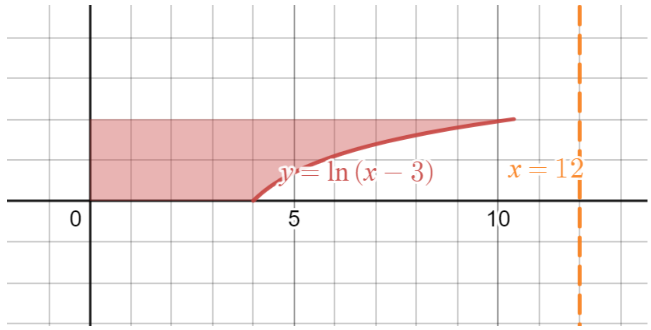


We use the shell method, so we consider vertical strips since the axis of revolution is vertical. The shell formed by the vertical strip at x has radius $r(x) = x-(-3) = x+3$ and height $h(x) = \dfrac{1}{\sqrt{16-x^2}}$. Therefore, the volume is

$$V = \int_0^2 2\pi(x+3)\frac{1}{\sqrt{16-x^2}}dx$$
$$= 2\pi \int_0^2 \left(\frac{x}{\sqrt{16-x^2}} + \frac{3}{\sqrt{16-x^2}}\right) dx$$
$$= 2\pi \left[-\sqrt{16-x^2} + 3\sin^{-1}\left(\frac{x}{4}\right)\right]_0^2$$
$$= 2\pi \left(\left(-\sqrt{16-4} + 3\sin^{-1}\left(\frac{1}{2}\right)\right) - \left(-\sqrt{16} + \sin^{-1}(0)\right)\right)$$
$$= \boxed{2\pi\left(\frac{\pi}{2} - \sqrt{12} + 4\right) \text{ cubic units}}.$$

(b) Let $\mathcal{R}$ be the region bounded by the coordinate axes, the graph of $y = \ln(x-3)$ and the line $y = 2$. Find the volume of the solid obtained by revolving $\mathcal{R}$ about the line $x = 12$.

Solution.



We use the washer method, so we consider horizontal strips in the region since the axis of revolution is vertical. Observe that the region can be described in terms of y as the horizontally simple region between $x = e^y + 3$ and $x = 0$ for $0 \leqslant y \leqslant 2$. The outer radius of the washers is $r_{\text{out}}(y) = 12 - 0 = 12$. The inner radius is $r_{\text{in}}(y) = 12 - (e^y + 3) = 9 - e^y$. Therefore

$$\begin{aligned}
V &= \int_0^2 \pi \left(12^2 - (9 - e^y)^2\right) dy \\
&= \pi \int_0^2 (144 - 81 + 18e^y - e^{2y}) dy \\
&= \pi \int_0^2 (63 + 18e^y - e^{2y}) dy \\
&= \pi \left[63y + 18e^y - \frac{1}{2}e^{2y}\right]_0^2 \\
&= \boxed{\pi\left(18e^2 + \frac{217 - e^4}{2}\right) \text{ cubic units}}.
\end{aligned}$$

6. Calculate the arc length of the given curves.

(a) $x = \frac{6}{7}y^{7/6}$, $0 \leqslant y \leqslant 1$.

Solution. We have

$$\frac{dx}{dy} = y^{1/6},$$

so

$$L = \int_0^1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + y^{1/3}} dy.$$

We can calculate this integral with the substitution $u = \sqrt{1 + y^{1/3}}$. This gives $du = \frac{dy}{6y^{2/3}\sqrt{1+y^{1/3}}}$, or $dy = 6y^{2/3}\sqrt{1 + y^{1/3}} du$. To express this in terms of u, note that $\sqrt{1 + y^{1/3}} = u$, and $y^{2/3} = (u^2 - 1)^2$. So we obtain $dy = 6(u^2 - 1)u du$. The integral becomes

$$L = \int_1^{\sqrt{2}} u \cdot 6(u^2 - 1)^2 u du$$

$$
\begin{aligned}
&= 6\int_1^{\sqrt{2}} u^2(u^4 - 2u^2 + 1)du \\
&= 6\int_1^{\sqrt{2}} (u^6 - 2u^4 1u^2)du \\
&= 6\left[\frac{u^7}{7} = \frac{2u^5}{5} + \frac{u^3}{3}\right]_1^{\sqrt{2}} \\
&= \boxed{\frac{44\sqrt{2} - 16}{35} \text{ units}}.
\end{aligned}
$$

(b) $y = \cos\left(\frac{x}{2}\right) - \ln\left(\csc\left(\frac{x}{2}\right) + \cot\left(\frac{x}{2}\right)\right)$, $\frac{\pi}{2} \leqslant x \leqslant \pi$.

Solution. We have

$$
\frac{dy}{dx} = -\frac{1}{2}\sin\left(\frac{x}{2}\right) + \frac{1}{2}\csc\left(\frac{x}{2}\right).
$$

So

$$
\begin{aligned}
1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \left(-\frac{1}{2}\sin\left(\frac{x}{2}\right) + \frac{1}{2}\csc\left(\frac{x}{2}\right)\right)^2 \\
&= 1 + \frac{1}{4}\sin\left(\frac{x}{2}\right)^2 + \frac{1}{4}\csc\left(\frac{x}{2}\right)^2 - 2\cdot\frac{1}{2}\sin\left(\frac{x}{2}\right)\cdot\frac{1}{2}\csc\left(\frac{x}{2}\right) \\
&= 1 + \frac{1}{4}\sin\left(\frac{x}{2}\right)^2 + \frac{1}{4}\csc\left(\frac{x}{2}\right)^2 - \frac{1}{2} \\
&= \frac{1}{4}\sin\left(\frac{x}{2}\right)^2 + \frac{1}{4}\csc\left(\frac{x}{2}\right)^2 + \frac{1}{2} \\
&= \left(\frac{1}{2}\sin\left(\frac{x}{2}\right)\right)^2 + \left(\frac{1}{2}\csc\left(\frac{x}{2}\right)\right)^2 + 2\cdot\frac{1}{2}\sin\left(\frac{x}{2}\right)\cdot\frac{1}{2}\csc\left(\frac{x}{2}\right) \\
&= \left(\frac{1}{2}\sin\left(\frac{x}{2}\right) + \frac{1}{2}\csc\left(\frac{x}{2}\right)\right)^2.
\end{aligned}
$$

Therefore the arc length is given by

$$
\begin{aligned}
L &= \int_{\pi/2}^{\pi} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int_{\pi/2}^{\pi} \sqrt{\left(\frac{1}{2}\sin\left(\frac{x}{2}\right) + \frac{1}{2}\csc\left(\frac{x}{2}\right)\right)^2}\, dy \\
&= \int_{\pi/2}^{\pi} \left|\frac{1}{2}\sin\left(\frac{x}{2}\right) + \frac{1}{2}\csc\left(\frac{x}{2}\right)\right| dy \\
&= \int_{\pi/2}^{\pi} \left(\frac{1}{2}\sin\left(\frac{x}{2}\right) + \frac{1}{2}\csc\left(\frac{x}{2}\right)\right) dy \\
&= \left[-\cos\left(\frac{x}{2}\right) - \ln\left|\csc\left(\frac{x}{2}\right) + \cot\left(\frac{x}{2}\right)\right|\right]_{\pi/2}^{\pi} \\
&= \left(-\cos\left(\frac{\pi}{2}\right) - \ln\left|\csc\left(\frac{\pi}{2}\right) + \cot\left(\frac{\pi}{2}\right)\right|\right) - \left(-\cos\left(\frac{\pi}{4}\right) - \ln\left|\csc\left(\frac{\pi}{4}\right) + \cot\left(\frac{\pi}{4}\right)\right|\right) \\
&= -0 - \ln(1+0) + \frac{\sqrt{2}}{2} + \ln\left(\sqrt{2}+1\right)
\end{aligned}
$$

$$= \boxed{\frac{\sqrt{2}}{2} + \ln\left(\sqrt{2}+1\right) \text{ units}}.$$

7. Consider the region below the graph $y = \cot(5x) + \csc(5x)$ for $\frac{\pi}{20} \leqslant x \leqslant \frac{\pi}{10}$.

   (a) Find the area of the region.

   Solution. The area is given by

$$\begin{aligned}
A &= \int_{\pi/20}^{\pi/10} (\cot(5x) + \csc(5x))dx \\
&= \frac{1}{5}\int_{\pi/4}^{\pi/2} (\cot(u) + \csc(u))\, du \quad (u = 5x) \\
&= \frac{1}{5}\left[\ln|\sin(u)| - \ln|\csc(u) + \cot(u)|\right]_{\pi/4}^{\pi/2} \\
&= \frac{1}{5}\left(\ln(1) - \ln(1+0) - \ln\left(\frac{1}{\sqrt{2}}\right) + \ln\left(\sqrt{2}+1\right)\right) \\
&= \boxed{\frac{1}{5}\ln(2+\sqrt{2}) \text{ square units}}.
\end{aligned}$$

   (b) Find the volume of the solid obtained by revolving the region about the x-axis.

   Solution. We use the disk method. Revolving the vertical strip at x about the x-axis gives a disk of radius $r(x) = \cot(5x) + \csc(5x)$. Therefore, the volume is

$$\begin{aligned}
V &= \int_{\pi/20}^{\pi/10} \pi\left(\cot(5x) + \csc(5x)\right)^2 dx \\
&= \frac{\pi}{5}\int_{\pi/4}^{\pi/2} \left(\cot(u) + \csc(u)\right)^2 du \quad (u = 5x) \\
&= \frac{\pi}{5}\int_{\pi/4}^{\pi/2} \left(\cot(u)^2 + \csc(u)^2 + 2\cot(u)\csc(u)\right) du \\
&= \frac{\pi}{5}\int_{\pi/4}^{\pi/2} \left((\csc(u)^2 - 1) + \csc(u)^2 + 2\cot(u)\csc(u)\right) du \\
&= \frac{\pi}{5}\int_{\pi/4}^{\pi/2} \left(2\csc(u)^2 + 2\cot(u)\csc(u) - 1\right) du \\
&= \frac{\pi}{5}\left[-2\cot(u) - 2\csc(u) - u\right]_{\pi/4}^{\pi/2} \\
&= \frac{\pi}{5}\left(-2\cdot 0 - 2 - \frac{\pi}{2} + 2 + 2\sqrt{2} + \frac{\pi}{4}\right) \\
&= \boxed{\frac{\pi}{5}\left(2\sqrt{2} - \frac{\pi}{4}\right) \text{ cubic units}}.
\end{aligned}$$

8. Calculate the surface area obtained by revolving the given curve about the given axis.

(a) The curve $x = \dfrac{4}{3}y^{3/4}$, $0 \leqslant y \leqslant 1$, revolved about the y-axis.

Solution. The surface area for revolution about the y-axis is given by

$$\begin{aligned} A &= \int_0^1 2\pi x(y)\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \\ &= \int_0^1 2\pi\frac{4}{3}y^{3/4}\sqrt{1+\left(y^{-1/4}\right)^2}dy \\ &= \frac{8\pi}{3}\int_0^1 y^{3/4}\sqrt{1+\frac{1}{y^{1/2}}}dy \\ &= \frac{8\pi}{3}\int_0^1 y^{3/4}\frac{\sqrt{y^{1/2}+1}}{y^{1/4}}dy \\ &= \frac{8\pi}{3}\int_0^1 y^{1/2}\sqrt{y^{1/2}+1}dy. \end{aligned}$$

We can now compute this integral with the substitution $u = y^{1/2}+1$. This gives $du = \frac{dy}{2y^{1/2}}$, or $dy = 2udu$. The extraneous factor $y^{1/2}$ in the numerator is equal to $u-1$. This gives

$$\begin{aligned} A &= \frac{8\pi}{3}\int_1^2 (u-1)\sqrt{u}2udu \\ &= \frac{16\pi}{3}\int_1^2 \left(u^{5/2}-u^{3/2}\right)du \\ &= \frac{16\pi}{3}\left[\frac{2}{7}u^{7/2}-\frac{2}{5}u^{5/2}\right]_2^{\sqrt{2}+1} \\ &= \boxed{\frac{8\pi(11\cdot 2^{7/2}-32)}{315}\text{ units}}. \end{aligned}$$

(b) The curve $y = 1+\sqrt{2-x^2}$, $0 \leqslant x \leqslant 1$, revolved about the x-axis.

Solution. The surface area for revolution about the x-axis is given by

$$\begin{aligned} A &= \int_0^1 2\pi y(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dy \\ &= \int_0^1 2\pi\left(1+\sqrt{2-x^2}\right)\sqrt{1+\left(-\frac{2x}{2\sqrt{2-x^2}}\right)^2}\,dy \\ &= 2\pi\int_0^1 \left(1+\sqrt{2-x^2}\right)\sqrt{1+\frac{x^2}{2-x^2}}dy \\ &= 2\pi\int_0^1 \left(1+\sqrt{2-x^2}\right)\sqrt{\frac{2-x^2+x^2}{2-x^2}}dy \\ &= 2\pi\int_0^1 \left(1+\sqrt{2-x^2}\right)\frac{\sqrt{2}}{\sqrt{2-x^2}}dy \end{aligned}$$

$$
\begin{aligned}
&= 2\sqrt{2}\pi \int_0^1 \left(\frac{1}{\sqrt{2-x^2}} + 1 \right) dy \\
&= 2\sqrt{2}\pi \left[\sin^{-1}\left(\frac{x}{\sqrt{2}} \right) + x \right]_0^1 \\
&= 2\sqrt{2}\pi \left(\sin^{-1}\left(\frac{1}{\sqrt{2}} \right) + 1 \right) \\
&= \boxed{2\sqrt{2}\pi \left(\frac{\pi}{4} + 1 \right) \text{ square units}}.
\end{aligned}
$$

9. Find the area of the surface obtained by revolving the curve $y = ax^2$, $0 \leqslant x \leqslant 1$, about the y-axis (where a is a positive constant).

Solution. Method 1: we use integration with respect to y. Observe that the curve can be written in terms of y as $x = \sqrt{\frac{y}{a}}$, $0 \leqslant y \leqslant a$. Therefore,

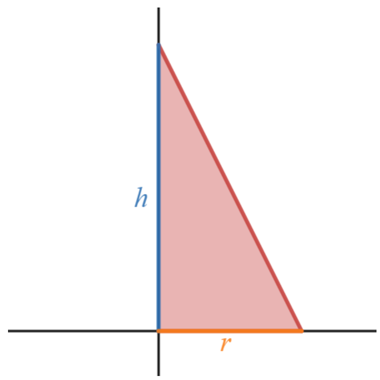
$$
\begin{aligned}
A &= \int_0^a 2\pi x(y) \sqrt{1 + \left(\frac{dx}{dy} \right)^2}\, dy \\
&= 2\pi \int_0^a \sqrt{\frac{y}{a}} \sqrt{1 + \left(\frac{1}{2\sqrt{ay}} \right)^2}\, dy \\
&= 2\pi \int_0^a \sqrt{\frac{y}{a} + \frac{1}{4a^2}}\, dy \\
&= 2\pi \left[\frac{2a}{3} \left(\frac{y}{a} + \frac{1}{4a^2} \right)^{3/2} \right]_0^a \\
&= \frac{4a\pi}{3} \left(\left(1 + \frac{1}{4a^2} \right)^{3/2} - \left(\frac{1}{4a^2} \right)^{3/2} \right) \\
&= \frac{4a\pi}{3} \cdot \frac{\left(4a^2+1\right)^{3/2} - 1}{(4a^2)^{3/2}} \\
&= \boxed{\frac{\pi}{6a^2} \left(\left(4a^2+1\right)^{3/2} - 1 \right) \text{ square units}}.
\end{aligned}
$$

Method 2: we use integration with respect to x. We have

$$
\begin{aligned}
A &= \int_0^1 2\pi x \sqrt{1 + \left(\frac{dy}{dx} \right)^2}\, dx \\
&= 2\pi \int_0^1 x \sqrt{1 + (2ax)^2}\, dx \\
&= 2\pi \int_0^1 x \sqrt{1 + 4a^2x^2}\, dx \\
&= 2\pi \left[\frac{2}{3} \cdot \frac{1}{8a^2} \left(1 + 4a^2x^2\right)^{3/2} \right]_0^1
\end{aligned}
$$

$$= \boxed{\frac{\pi}{6a^2}\left(\left(1+4a^2\right)^{3/2}-1\right) \text{ square units}}.$$

10. A circular cone of height h and radius r is created by revolving a right triangle with base r, height h and right angle at the origin about the y-axis, see figure below.



(a) The triangle is bounded by the coordinate axes and a line. Find an equation of the line bounding the triangle in terms of the constants r and h.

Solution. The line bounding the triangle passes through the points $(0, h)$ and $(r, 0)$. Therefore, it has equation

$$\boxed{y = -\frac{h}{r}x + h = h\left(-\frac{x}{r}+1\right)}.$$

(b) Calculate the volume of the cone using the disk/washer method.

Solution. Since the axis of revolution is the left boundary of the region, the washers are just disks. To find the radius, we express the right boundary (which is the line) as a function of y:

$$y = -\frac{h}{r}x + h \quad \Rightarrow \quad x = -\frac{r}{h}(y+h) = r\left(-\frac{y}{h}+1\right).$$

Therefore the volume is

$$\begin{aligned} V &= \int_0^h \pi\left(r\left(-\frac{y}{h}+1\right)\right)^2 dy \\ &= \pi r^2 \int_0^h \left(-\frac{y}{h}+1\right)^2 dy \\ &= \pi r^2 \left[-\frac{h}{3}\left(-\frac{y}{h}+1\right)^3\right]_0^h \\ &= \boxed{\frac{\pi r^2 h}{3}}. \end{aligned}$$

(c) Calculate the volume of the cone using the shell method.

Solution. The shells have radius $r(x) = x$ and height $h(x) = h\left(-\frac{x}{r} + 1\right)$. Therefore

$$\begin{aligned} V &= \int_0^r 2\pi x h \left(-\frac{x}{r} + 1\right) dx \\ &= 2\pi h \int_0^r \left(-\frac{x^2}{r} + x\right) dx \\ &= 2\pi h \left[-\frac{x^3}{3r} + \frac{x^2}{2}\right]_0^r \\ &= \boxed{\frac{\pi r^2 h}{3}}. \end{aligned}$$

(d) Calculate the surface area of the side of the cone.

Solution. The side of the cone is obtained by revolving the line segment $y = -\frac{h}{r}x + h$, $0 \leqslant x \leqslant r$, about the y-axis. So the area, computed with an x-integral is

$$\begin{aligned} A &= \int_0^r 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_0^r 2\pi x \sqrt{1 + \left(-\frac{h}{r}\right)^2}\, dx \\ &= 2\pi \sqrt{1 + \frac{h^2}{r^2}} \int_0^r x dx \\ &= 2\pi \sqrt{\frac{r^2 + h^2}{r^2}} \frac{r^2}{2} \\ &= \boxed{\pi r \sqrt{r^2 + h^2}}. \end{aligned}$$

Remark: we can also use a y-integral to compute the surface area. The computation goes as follows:

$$\begin{aligned} A &= \int_0^h 2\pi x(y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \\ &= \int_0^h 2\pi r \left(-\frac{y}{h} + 1\right) \sqrt{1 + \left(-\frac{r}{h}\right)^2}\, dy \\ &= 2\pi r \sqrt{1 + \frac{r^2}{h^2}} \int_0^h \left(-\frac{y}{h} + 1\right) dy \\ &= 2\pi r \sqrt{\frac{h^2 + r^2}{h^2}} \left[-\frac{y^2}{2h} + y\right]_0^h \\ &= 2\pi r \frac{\sqrt{h^2 + r^2}}{h} \frac{h}{2} \\ &= \boxed{\pi r \sqrt{h^2 + r^2}}. \end{aligned}$$