Rutgers University
Math 152

Midterm 2 Practice Problems Solutions

1. Let $\mathcal{R}$ the region between the graph of $y = \cos(3x)\sin(3x)$ and the x-axis on $0 \leqslant x \leqslant \frac{\pi}{6}$. Calculate the volume of the solid obtained by revolving $\mathcal{R}$ about (a) the x-axis, (b) the y-axis.

   Solution. (a) We use the disk method. Revolving the vertical strip at x in the region about the x-axis creates a disk of radius $r(x) = \cos(3x)\sin(3x)$. Therefore, we have

$$\begin{aligned} V &= \int_0^{\pi/6} \pi r(x)^2 dx \\ &= \pi \int_0^{\pi/6} \cos(3x)^2 \sin(3x)^2 dx. \end{aligned}$$

   Since the exponents of cos and sin are both even, we can evaluate this integral using the double angle formulas. This gives

$$\begin{aligned} V &= \pi \int_0^{\pi/6} \frac{1+\cos(6x)}{2} \cdot \frac{1-\cos(6x)}{2} dx \\ &= \frac{\pi}{4} \int_0^{\pi/6} (1 - \cos(6x)^2) dx \\ &= \frac{\pi}{4} \int_0^{\pi/6} \left(1 - \frac{1+\cos(12x)}{2}\right) dx \\ &= \frac{\pi}{8} \int_0^{\pi/6} (1 - \cos(12x))\, dx \\ &= \frac{\pi}{8} \left[x - \frac{\sin(12x)}{12}\right]_0^{\pi/6} \\ &= \boxed{\frac{\pi^2}{48} \text{ cubic units}}. \end{aligned}$$

   (b) We use the method of cylindrical shells. Revolving the vertical strip at x in the region about the y-axis creates a shell of radius $r(x) = x$ and height $h(x) = \cos(3x)\sin(3x)$. Therefore

$$\begin{aligned} V &= \int_0^{\pi/6} 2\pi r(x) h(x) dx \\ &= 2\pi \int_0^{\pi/6} x \cos(3x)\sin(3x) dx. \end{aligned}$$

   We can calculate this integral with an integration by parts, choosing

$$\begin{aligned} &u = x \ \Rightarrow\ du = dx, \\ &dv = \cos(3x)\sin(3x)dx \ \Rightarrow\ v = \frac{\sin(3x)^2}{6}. \end{aligned}$$

This gives

$$\begin{aligned}
V &= 2\pi \left(\left[\frac{x\sin(3x)^2}{6} \right]_0^{\pi/6} - \int_0^{\pi/6} \frac{\sin(3x)^2}{6} dx \right) \\
&= 2\pi \left(\frac{\pi}{36} - \frac{1}{6} \int_0^{\pi/6} \frac{1-\cos(6x)}{2} dx \right) \\
&= 2\pi \left(\frac{\pi}{36} - \frac{1}{6} \left[\frac{x}{2} - \frac{\sin(6x)}{12} \right]_0^{\pi/6} \right) \\
&= 2\pi \left(\frac{\pi}{36} - \frac{\pi}{72} \right) \\
&= \boxed{\frac{\pi^2}{36} \text{ cubic units}}.
\end{aligned}$$

2. Evaluate the following definite or indefinite integrals.

(a) $\int x^5 \ln(x)^2 dx$.

Solution. We use two integration by parts. The first one will use

$$\begin{aligned}
u &= \ln(x)^2 \;\Rightarrow\; du = \frac{2\ln(x)dx}{x}, \\
dv &= x^5 dx, \;\Rightarrow\; v = \frac{x^6}{6}.
\end{aligned}$$

This gives

$$\begin{aligned}
\int x^5 \ln(x)^2 dx &= \frac{x^6 \ln(x)^2}{6} - \int \frac{2\ln(x)x^6}{6x} dx \\
&= \frac{x^6 \ln(x)^2}{6} - \frac{1}{3} \int \ln(x)x^5 dx.
\end{aligned}$$

For the second IBP, we take

$$\begin{aligned}
u &= \ln(x) \;\Rightarrow\; du = \frac{dx}{x}, \\
dv &= x^5 dx, \;\Rightarrow\; v = \frac{x^6}{6}.
\end{aligned}$$

This gives

$$\begin{aligned}
\int x^5 \ln(x)^2 dx &= \frac{x^6 \ln(x)^2}{6} - \frac{1}{3} \left(\frac{\ln(x)x^6}{6} - \int \frac{x^6}{6x} dx \right) \\
&= \frac{x^6 \ln(x)^2}{6} - \frac{\ln(x)x^6}{18} + \frac{1}{18} \int x^5 dx \\
&= \boxed{\frac{x^6 \ln(x)^2}{6} - \frac{\ln(x)x^6}{18} + \frac{x^6}{108} + C}.
\end{aligned}$$

(b) $\displaystyle\int \tan(\theta)^3 \sec(\theta)^5 d\theta.$

Solution. Since the exponent of tan is odd, we can split off a factor $\tan(\theta)\sec(\theta)$, replace the remaining factors of $\tan(\theta)$ using the trigonometric identity $\tan(\theta)^2 = \sec(\theta)^2 - 1$ and then use the substitution $u = \sec(\theta)$, $du = \tan(\theta)\sec(\theta)d\theta$. This gives

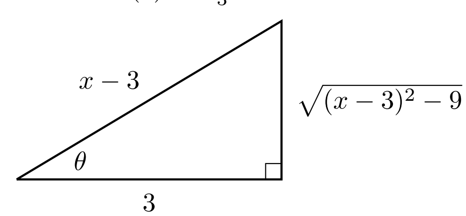
$$\begin{aligned}\int \tan(\theta)^3 \sec(\theta)^5 d\theta &= \int \tan(\theta)^2 \sec(\theta)^4 \tan(\theta)\sec(\theta)d\theta \\ &= \int \left(\sec(\theta)^2 - 1\right)\sec(\theta)^4 \tan(\theta)\sec(\theta)d\theta \\ &= \int \left(u^2 - 1\right) u^4 du \\ &= \int \left(u^6 - u^4\right) du \\ &= \frac{u^7}{7} - \frac{u^5}{5} + C \\ &= \boxed{\frac{\sec(\theta)^7}{7} - \frac{\sec(\theta)^5}{5} + C}.\end{aligned}$$

(c) $\displaystyle\int \frac{dx}{\sqrt{x^2 - 6x}}$, $x > 6$.

Solution. We start by completing the square to see a difference of squares appear in the square root:

$$x^2 - 6x = (x^2 - 6x + 9) - 9 = (x-3)^2 - 9.$$

We will therefore use a trigonometric substitution. We want $(x-3)^2 - 9 = 9\sec(\theta)^2 - 9$, so we substitute $x - 3 = 3\sec(\theta)$ and $dx = 3\sec(\theta)\tan(\theta)d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\sec(\theta) = \frac{x-3}{3}$ as shown below.



We get $\sqrt{(x-3)^3 - 9} = \sqrt{9\sec(\theta)^2 - 9} = \sqrt{9\tan(\theta)^2} = 3\tan(\theta)$, and the integral becomes

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2 - 6x}} &= \int \frac{dx}{\sqrt{(x-3)^2 - 9}} \\ &= \int \frac{3\sec(\theta)\tan(\theta)d\theta}{3\tan(\theta)} \\ &= \int \sec(\theta)d\theta \\ &= \ln|\sec(\theta) + \tan(\theta)| + C \\ &= \boxed{\ln\left|\frac{x-3}{3} + \frac{\sqrt{(x-3)^2 - 9}}{3}\right| + C}.\end{aligned}$$

(d) $\int_{\sqrt{2}}^{2} \sec^{-1}(t)dt$.

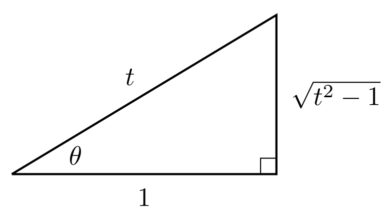
Solution. Method 1: first an IBP and then a trigonometric substitution. For the IBP, we use the parts

$$u = \sec^{-1}(t) \;\Rightarrow\; du = \frac{dt}{|t|\sqrt{t^2-1}},$$
$$dv = dt, \;\Rightarrow\; v = t.$$

This gives

$$\begin{aligned}\int_{\sqrt{2}}^{2} \sec^{-1}(t)dt &= \left[t\sec^{-1}(t)\right]_{\sqrt{2}}^{2} - \int_{\sqrt{2}}^{2} \frac{tdt}{|t|\sqrt{t^2-1}} \\ &= 2\sec^{-1}(2) - \sqrt{2}\sec^{-1}(\sqrt{2}) - \int_{\sqrt{2}}^{2} \frac{tdt}{t\sqrt{t^2-1}} \quad \text{(since } t > 0\text{)} \\ &= \frac{2\pi}{3} - \frac{\sqrt{2}\pi}{4} - \int_{\sqrt{2}}^{2} \frac{dt}{\sqrt{t^2-1}}.\end{aligned}$$

In this last integral, we can use the trigonometric substitution $t = \sec(\theta)$, $dt = \tan(\theta)\sec(\theta)d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\sec(\theta) = t$ as shown below.



The bounds will become

$$t = \sqrt{2} \;\Rightarrow\; \theta = \sec^{-1}(\sqrt{2}) = \frac{\pi}{4}, \qquad\qquad t = 2 \;\Rightarrow\; \theta = \sec^{-1}(2) = \frac{\pi}{3}.$$

We therefore obtain

$$\begin{aligned}\int_{\sqrt{2}}^{2} \sec^{-1}(t)dt &= \frac{2\pi}{3} - \frac{\sqrt{2}\pi}{4} - \int_{\pi/4}^{\pi/3} \frac{\sec(\theta)\tan(\theta)d\theta}{\tan(\theta)} \\ &= \frac{2\pi}{3} - \frac{\sqrt{2}\pi}{4} - \int_{\pi/4}^{\pi/3} \sec(\theta)d\theta \\ &= \frac{2\pi}{3} - \frac{\sqrt{2}\pi}{4} - [\ln|\sec(\theta) + \tan(\theta)|]_{\pi/4}^{\pi/3} \\ &= \boxed{\frac{2\pi}{3} - \frac{\sqrt{2}\pi}{4} - \ln\left(2+\sqrt{3}\right) + \ln(\sqrt{2}+1)}.\end{aligned}$$

Method 2: first a trigonometric substitution and then an IBP. As before, the trigonometric substitution uses $t = \sec(\theta)$ and gives

$$\int_{\sqrt{2}}^{2} \sec^{-1}(t)dt = \int_{\pi/4}^{\pi/3} \sec^{-1}(\sec(\theta))\sec(\theta)\tan(\theta)d\theta = \int_{\pi/4}^{\pi/3} \theta\sec(\theta)\tan(\theta)d\theta.$$

In the new integral, with use an IBP with parts

$$u = \theta \;\Rightarrow\; du = d\theta,$$
$$dv = \sec(\theta)\tan(\theta)d\theta, \;\Rightarrow\; v = \sec(\theta).$$

We obtain

$$\begin{aligned}\int_{\sqrt{2}}^{2} \sec^{-1}(t)dt &= [\theta\sec(\theta)]_{\pi/4}^{\pi/3} - \int_{\pi/4}^{\pi/3} \sec(\theta)d\theta \\ &= \frac{2\pi}{3} - \frac{\sqrt{2}\pi}{4} - [\ln|\sec(\theta)+\tan(\theta)|]_{\pi/4}^{\pi/3} \\ &= \boxed{\frac{2\pi}{3} - \frac{\sqrt{2}\pi}{4} - \ln\left(2+\sqrt{3}\right) + \ln(\sqrt{2}+1)}.\end{aligned}$$

(e) $\int x^2 \tan^{-1}(x)dx$.

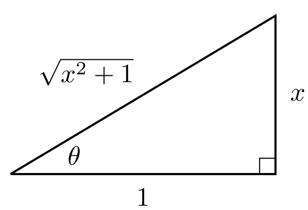
Solution. Method 1: first an IBP and then a trigonometric substitution. For the IBP, we use the parts

$$u = \tan^{-1}(x) \;\Rightarrow\; du = \frac{dx}{x^2+1},$$
$$dv = x^2 dx, \;\Rightarrow\; v = \frac{x^3}{3}.$$

This gives

$$\begin{aligned}\int x^2\tan^{-1}(x)dx &= \frac{x^3\tan^{-1}(x)}{3} - \int \frac{x^3}{3(x^2+1)}dx \\ &= \frac{x^3\tan^{-1}(x)}{3} - \frac{1}{3}\int \frac{x^3}{x^2+1}dx\end{aligned}$$

In this last integral, we can use the trigonometric substitution $x = \tan(\theta)$, so that $dx = \sec(\theta)^2 d\theta$ and $x^2+1 = \tan(\theta)^2+1 = \sec(\theta)^2$. The right triangle for this trigonometric substitution has base angle θ so that $\tan(\theta) = x$ as shown below.



We obtain

$$\begin{aligned}\int \frac{x^3}{x^2+1}dx &= \int \frac{\tan(\theta)^3}{\sec(\theta)^2}\sec(\theta)^2 d\theta \\ &= \int \tan(\theta)^3 d\theta \\ &= \int \tan(\theta)(\sec(\theta)^2-1)d\theta\end{aligned}$$

$$
\begin{aligned}
&= \int \big(\tan(\theta)\sec(\theta)^2 - \tan(\theta)\big)\, d\theta \\
&= \frac{\tan(\theta)^2}{2} - \ln|\sec(\theta)| + C \\
&= \frac{x^2}{2} - \ln\left|\sqrt{x^2+1}\right| + C \\
&= \frac{x^2}{2} - \frac{1}{2}\ln(x^2+1) + C.
\end{aligned}
$$

Putting everything together, we get

$$
\begin{aligned}
\int x^2 \tan^{-1}(x) dx &= \frac{x^3 \tan^{-1}(x)}{3} - \frac{1}{3}\int \frac{x^3}{x^2+1} dx \\
&= \frac{x^3 \tan^{-1}(x)}{3} - \frac{1}{3}\left(\frac{x^2}{2} - \frac{1}{2}\ln(x^2+1)\right) + C \\
&= \boxed{\frac{x^3 \tan^{-1}(x)}{3} - \frac{x^2}{6} + \frac{1}{6}\ln(x^2+1) + C}.
\end{aligned}
$$

Method 2: first a trigonometric substitution and then an IBP. As before, the trigonometric substitution uses $x = \tan(\theta)$, $dx = \sec(\theta)^2 d\theta$ so that

$$
\begin{aligned}
\int x^2 \tan^{-1}(x) dx &= \int \tan(\theta)^2 \tan^{-1}(\tan(\theta)) \sec(\theta)^2 d\theta \\
&= \int \theta \tan(\theta)^2 \sec(\theta)^2 d\theta.
\end{aligned}
$$

We can compute this integral using an IBP with parts

$$
\begin{aligned}
&u = \theta \;\Rightarrow\; du = d\theta, \\
&dv = \tan(\theta)^2 \sec(\theta)^2 d\theta, \;\Rightarrow\; v = \frac{\tan(\theta)^3}{3}.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\int x^2 \tan^{-1}(x) dx &= \int \theta \tan(\theta)^2 \sec(\theta)^2 d\theta \\
&= \frac{\theta \tan(\theta)^3}{3} - \frac{1}{3}\int \tan(\theta)^3 d\theta \\
&= \frac{\theta \tan(\theta)^3}{3} - \frac{1}{3}\int \big(\tan(\theta)\sec(\theta)^2 - \tan(\theta)\big)\, d\theta \\
&= \frac{\theta \tan(\theta)^3}{3} - \frac{1}{3}\left(\frac{\tan(\theta)^2}{2} - \ln|\sec(\theta)|\right) + C \\
&= \boxed{\frac{x^3 \tan^{-1}(x)}{3} - \frac{x^2}{6} + \frac{1}{6}\ln(x^2+1) + C}.
\end{aligned}
$$

(f) $\displaystyle\int \sin(5x)^4 dx$.

Solution. We use double angle formulas twice. We have

$$
\sin(5x)^4 = \left(\frac{1 - \cos(10x)}{2}\right)^2
$$

$$\begin{aligned}
&= \frac{1}{4}\left(1 - 2\cos(10x) + \cos(10x)^2\right) \\
&= \frac{1}{4}\left(1 - 2\cos(10x) + \frac{1+\cos(20x)}{2}\right) \\
&= \frac{1}{8}\left(3 - 4\cos(10x) + \cos(20x)\right).
\end{aligned}$$

So

$$\begin{aligned}
\int \sin(5x)^4 dx &= \frac{1}{8}\int \left(3 - 4\cos(10x) + \cos(20x)\right) dx \\
&= \boxed{\frac{1}{8}\left(3x - \frac{2\sin(10x)}{5} + \frac{\sin(20x)}{20}\right) + C}.
\end{aligned}$$

(g) $\int \sin^{-1}(x)^2 dx$.

Solution. We use an IBP with parts

$$\begin{aligned}
&u = \sin^{-1}(x)^2 \;\Rightarrow\; du = \frac{2\sin^{-1}(x)}{\sqrt{1-x^2}}dx, \\
&dv = dx, \;\Rightarrow\; v = x.
\end{aligned}$$

This gives

$$\begin{aligned}
\int \sin^{-1}(x)^2 dx &= x\sin^{-1}(x)^2 - \int \frac{2x\sin^{-1}(x)}{\sqrt{1-x^2}}dx \\
&= x\sin^{-1}(x)^2 - 2\int \frac{x\sin^{-1}(x)}{\sqrt{1-x^2}}dx.
\end{aligned}$$

In this new integral, we use another IBP with parts

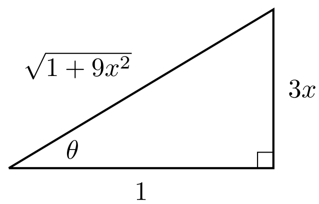
$$\begin{aligned}
&u = \sin^{-1}(x) \;\Rightarrow\; du = \frac{dx}{\sqrt{1-x^2}}, \\
&dv = \frac{x}{\sqrt{1-x^2}}, \;\Rightarrow\; v = -\sqrt{1-x^2}.
\end{aligned}$$

We obtain

$$\begin{aligned}
\int \sin^{-1}(x)^2 dx &= x\sin^{-1}(x)^2 - 2\int \frac{x\sin^{-1}(x)}{\sqrt{1-x^2}}dx \\
&= x\sin^{-1}(x)^2 - 2\left(-\sin^{-1}(x)\sqrt{1-x^2} - \int (-\sqrt{1-x^2})\frac{dx}{\sqrt{1-x^2}}\right) \\
&= x\sin^{-1}(x)^2 + 2\sin^{-1}(x)\sqrt{1-x^2} - 2\int dx \\
&= \boxed{x\sin^{-1}(x)^2 + 2\sin^{-1}(x)\sqrt{1-x^2} - 2x + C}.
\end{aligned}$$

(h) $\displaystyle\int \frac{x^2}{(1+9x^2)^{5/2}}dx.$

Solution. We use a trigonometric substitution. We will want $1+9x^2 = 1+\tan(\theta)^2$, so we substitute $x = \frac{1}{3}\tan(\theta)$, so that $dx = \frac{1}{3}\sec(\theta)^2 d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\tan(\theta) = 3x$ as shown below.



We get $(1+9x^2)^{5/2} = (1+\tan(\theta)^2)^{5/2} = (\sec(\theta)^2)^{5/2} = \sec(\theta)^5$ and the integral becomes

$$\begin{aligned}
\int \frac{x^2}{(1+9x^2)^{5/2}}dx &= \int \frac{\left(\frac{1}{3}\tan(\theta)\right)^2}{\sec(\theta)^5}\frac{1}{3}\sec(\theta)^2 d\theta \\
&= \frac{1}{27}\int \frac{\tan(\theta)^2}{\sec(\theta)^3}d\theta \\
&= \frac{1}{27}\int \frac{\frac{\cos(\theta)^2}{\sin(\theta)^2}}{\frac{1}{\cos(\theta)^3}}d\theta \\
&= \frac{1}{27}\int \sin(\theta)^2\cos(\theta)d\theta \\
&= \frac{1}{81}\sin(\theta)^3 + C \\
&= \frac{1}{81}\left(\frac{3x}{\sqrt{1+9x^2}}\right)^3 + C \\
&= \boxed{\frac{x^3}{3(1+9x^2)^{3/2}} + C}.
\end{aligned}$$

(i) $\displaystyle\int_0^8 \sqrt{64+x^2}dx.$

Solution. We substitute $x = 8\tan(\theta)$, so that $dx = 8\sec(\theta)^2 d\theta$ and $\sqrt{64+x^2} = \sqrt{64+64\tan(\theta)^2} = \sqrt{64\sec(\theta)^2} = 8\sec(\theta)$. The bounds change as follows:

$$\begin{aligned}
x = 0 \;\Rightarrow\; \theta &= \tan^{-1}\left(\frac{0}{8}\right) = 0, \\
x = 8 \;\Rightarrow\; \theta &= \tan^{-1}\left(\frac{8}{8}\right) = \frac{\pi}{4}.
\end{aligned}$$

The integral becomes

$$\begin{aligned}
\int_0^8 \sqrt{64+x^2}dx &= \int_0^{\pi/4} 8\sec(\theta)\cdot 8\sec(\theta)^2 d\theta \\
&= 64\int_0^{\pi/4}\sec(\theta)^3 d\theta.
\end{aligned}$$

Using integration by parts, we have previously found that

$$\int \sec(\theta)^3 d\theta = \frac{1}{2}\left(\sec(\theta)\tan(\theta) - \ln|\sec(\theta)+\tan(\theta)|\right) + C.$$

So

$$\begin{aligned}\int_0^8 \sqrt{64+x^2}dx &= 64\left[\frac{1}{2}\left(\sec(\theta)\tan(\theta) - \ln|\sec(\theta)+\tan(\theta)|\right)\right]_0^{\pi/4} \\ &= \boxed{32\left(\sqrt{2} - \ln(\sqrt{2}+1)\right)}.\end{aligned}$$

3. Calculate the surface area obtained by revolving the curve $y = \frac{1}{2}x^2$, $0 \leqslant x \leqslant 1$, about the x-axis.

Solution. The surface area for revolution about the x-axis is given by

$$\begin{aligned}A &= \int_0^1 2\pi y(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx \\ &= \int_0^1 2\pi \cdot \frac{1}{2}x^2\sqrt{1+x^2}dx \\ &= \pi\int_0^1 x^2\sqrt{1+x^2}.\end{aligned}$$

Using the trigonometric substitution $x = \tan(\theta)$, $dx = \sec(\theta)^2 d\theta$, we get

$$\begin{aligned}A &= \pi\int_0^{\pi/4} \tan(\theta)^2\sec(\theta)^3 \\ &= \pi\int_0^{\pi/4} (\sec(\theta)^2 - 1)\sec(\theta)^3 d\theta \\ &= \pi\int_0^{\pi/4} (\sec(\theta)^5 - \sec(\theta)^3)d\theta.\end{aligned}$$

We have seen previously that

$$\int \sec(\theta)^3 d\theta = \frac{1}{2}\left(\sec(\theta)\tan(\theta) - \ln|\sec(\theta)+\tan(\theta)|\right) + C,$$

and we can express $\int \sec(\theta)^5 d\theta$ in terms of $\int \sec(\theta)^3 d\theta$ using an IBP with $u = \sec(\theta)^3$, $du = 3\sec(\theta)^3\tan(\theta)$ and $dv = \sec(\theta)^2 d\theta$, $v = \tan(\theta)$. We get

$$\begin{aligned}\int \sec(\theta)^5 d\theta &= \int \sec(\theta)^3\sec(\theta)^2 d\theta \\ &= \sec(\theta)^3\tan(\theta) - 3\int \sec(\theta)^3\tan(\theta)^2 d\theta \\ &= \sec(\theta)^3\tan(\theta) - 3\int \sec(\theta)^3(\sec(\theta)^2 - 1)d\theta \\ &= \sec(\theta)^3\tan(\theta) - 3\int \sec(\theta)^5 + 3\int \sec(\theta)^3 d\theta\end{aligned}$$

$$= \sec(\theta)^3\tan(\theta) - 3\int \sec(\theta)^5 + \frac{3}{2}\left(\sec(\theta)\tan(\theta) - \ln|\sec(\theta)+\tan(\theta)|\right).$$

Collecting the terms involving $\int \sec(\theta)^5 d\theta$ to the left-hand side gives

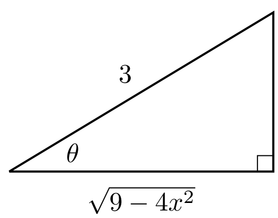
$$4\int \sec(\theta)^5 d\theta = \sec(\theta)^3\tan(\theta) + \frac{3}{2}\left(\sec(\theta)\tan(\theta) - \ln|\sec(\theta)+\tan(\theta)|\right)$$
$$\Rightarrow \int \sec(\theta)^5 d\theta = \frac{1}{4}\sec(\theta)^3\tan(\theta) + \frac{3}{8}\left(\sec(\theta)\tan(\theta) - \ln|\sec(\theta)+\tan(\theta)|\right) + C.$$

Going back to the surface area, we get

$$\begin{aligned} A &= \pi\int_0^{\pi/4} (\sec(\theta)^5 - \sec(\theta)^3)d\theta \\ &= \left[\frac{1}{4}\sec(\theta)^3\tan(\theta) - \frac{1}{8}\left(\sec(\theta)\tan(\theta) - \ln|\sec(\theta)+\tan(\theta)|\right)\right]_0^{\pi/4} \\ &= \boxed{\frac{\sqrt{2}}{2} - \frac{1}{8}\left(\sqrt{2} + \ln|\sqrt{2}+1|\right) \text{ square units}}. \end{aligned}$$

4. Evaluate $\int \frac{\sqrt{9-4x^2}}{x^2}dx$ two ways.

Solution. Method 1. We use the trigonometric substitution $x = \frac{3}{2}\sin(\theta)$, $dx = \frac{3}{2}\cos(\theta)d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\sin(\theta) = \frac{2x}{3}$ as shown below.



We get $\sqrt{9-4x^2} = \sqrt{9-9\sin(\theta)^2} = \sqrt{9\cos(\theta)^2} = 3\cos(\theta)$, and the integral becomes

$$\begin{aligned} \int \frac{\sqrt{9-4x^2}}{x^2}dx &= \int \frac{3\cos(\theta)}{\left(\frac{3}{2}\sin(\theta)\right)^2}\frac{3}{2}\cos(\theta)d\theta \\ &= 2\int \cot(\theta)^2 d\theta \\ &= 2\int (\csc(\theta)^2 - 1)d\theta \\ &= 2\left(-\cot(\theta) - \theta\right) + C. \end{aligned}$$

We can now use the right triangle above to express $\cot(\theta)$ in terms of x, while θ can be replaced by $\sin^{-1}\left(\frac{2x}{3}\right)$. This gives

$$\int \frac{\sqrt{9-4x^2}}{x^2}dx = 2\left(-\frac{\sqrt{9-4x^2}}{2x} - \sin^{-1}\left(\frac{2x}{3}\right)\right) + C$$

$$= \boxed{-\frac{\sqrt{9-4x^2}}{x} - 2\sin^{-1}\left(\frac{2x}{3}\right) + C}.$$

Method 2. We use an IBP with parts

$$u = \sqrt{9-4x^2} \;\Rightarrow\; du = \frac{-4x}{\sqrt{9-4x^2}},$$
$$dv = \frac{dx}{x^2} \;\Rightarrow\; v = -\frac{1}{x}.$$

This gives

$$\begin{aligned}
\int \frac{\sqrt{9-4x^2}}{x^2}dx &= -\frac{\sqrt{9-4x^2}}{x} - \int \frac{-4x}{\sqrt{9-4x^2}}\left(-\frac{1}{x}\right)dx \\
&= -\frac{\sqrt{9-4x^2}}{x} - 4\int \frac{dx}{\sqrt{9-4x^2}} \\
&= -\frac{\sqrt{9-4x^2}}{x} - \frac{4}{3}\int \frac{dx}{\sqrt{1-\left(\frac{2x}{3}\right)^2}} \\
&= -\frac{\sqrt{9-4x^2}}{x} - \frac{4}{3}\cdot\frac{3}{2}\sin^{-1}\left(\frac{2x}{3}\right) + C \\
&= \boxed{-\frac{\sqrt{9-4x^2}}{x} - 2\sin^{-1}\left(\frac{2x}{3}\right) + C}.
\end{aligned}$$

5. Use a comparison test to determine if the following improper integrals converge or diverge.

(a) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}e^{3x}}$.

Solution. We use the DCT. Since $1 \leqslant e^{3x} \leqslant e^3$ when $0 \leqslant x \leqslant 1$, we have

$$0 \leqslant \frac{1}{\sqrt{x}e^{3x}} \leqslant \frac{1}{\sqrt{x}e^3}.$$

Furthermore, $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}e^3} = \frac{1}{e^3}\int_0^1 \frac{dx}{\sqrt{x}}$ converges as a type *II* p-integral with $p = \frac{1}{2} < 1$. Therefore, $\boxed{\displaystyle\int_0^1 \frac{dx}{\sqrt{x}e^{3x}} \text{ converges}}$.

(b) $\displaystyle\int_3^\infty \frac{2\cos(x) + 5x^3}{13x^5 + 6x + 7}dx$.

Solution. We use the LCT to compare with $\displaystyle\int_3^\infty \frac{x^3}{x^5}dx = \int_3^\infty \frac{dx}{x^2}$, which converges as a type I p-integral with $p = 2 > 1$. We have

$$L = \lim_{x\to\infty} \frac{f(x)}{g(x)}$$

$$= \lim_{x\to\infty} \frac{\frac{2\cos(x)+5x^3}{13x^5+6x+7}}{\frac{x^3}{x^5}}$$

$$= \lim_{x\to\infty} \frac{\frac{2\cos(x)}{x^3}+5}{13+\frac{6}{x^4}+\frac{7}{x^5}}.$$

To compute the limit of the numerator, observe that $-1 \leqslant \cos(x) \leqslant 1$, so

$$-\frac{2}{x^3} \leqslant \frac{2\cos(x)}{x^3} \leqslant \frac{2}{x^3}.$$

Since $\lim\limits_{x\to\infty} -\frac{2}{x^3} = \lim\limits_{x\to\infty} \frac{2}{x^3}$, the Sandwich Theorem guarantees that $\lim\limits_{x\to\infty} \frac{2\cos(x)}{x^3} = 0$. Using this in our main limit computation, we have

$$L = \lim_{x\to\infty} \frac{\frac{2\cos(x)}{x^3}+5}{13+\frac{6}{x^4}+\frac{7}{x^5}} = \frac{5}{13}.$$

Since $L > 0$ and $\displaystyle\int_3^\infty \frac{x^3}{x^5}dx$ converges, we conclude that $\boxed{\displaystyle\int_3^\infty \frac{2\cos(x)+5x^3}{13x^5+6x+7}dx \text{ converges}}$.

(c) $\displaystyle\int_1^\infty \frac{\sqrt{x^3-1}}{x^2}dx$.

Solution. We use the LCT to compare with $\displaystyle\int_1^\infty \frac{\sqrt{x^3}}{x^2}dx = \int_1^\infty \frac{dx}{x^{1/2}}$, which diverges as a type I p-integral with $p = \frac{1}{2} \leqslant 1$. We have

$$\begin{aligned} L &= \lim_{x\to\infty} \frac{f(x)}{g(x)} \\ &= \lim_{x\to\infty} \frac{\frac{\sqrt{x^3-1}}{x^2}}{\frac{\sqrt{x^3}}{x^2}} \\ &= \lim_{x\to\infty} \sqrt{1-\frac{1}{x^3}} \\ &= 1. \end{aligned}$$

Since $L > 0$ and $\displaystyle\int_1^\infty \frac{\sqrt{x^3}}{x^2}dx$ diverges, we conclude that $\boxed{\displaystyle\int_1^\infty \frac{\sqrt{x^3-1}}{x^2}dx \text{ diverges}}$.

6. Evaluate the following improper integrals.

(a) $\displaystyle\int_0^\infty \frac{dx}{(x^2+16)^2}$.

Solution. We can set-up the improper integral as a limit as follows:

$$\int_0^\infty \frac{dx}{(x^2+16)^2} = \lim_{b\to\infty} \int_0^b \frac{dx}{(x^2+16)^2}.$$

We can compute the proper integral using the trigonometric substitution $x = 4\tan(\theta)$, which gives $dx = 4\sec(\theta)^2 d\theta$. The bounds of the integral become

$$x = 0 \;\Rightarrow\; \theta = \tan^{-1}\left(\frac{0}{4}\right) = 0,$$
$$x = b \;\Rightarrow\; \theta = \tan^{-1}\left(\frac{b}{4}\right).$$

So

$$\begin{aligned}
\int_0^\infty \frac{dx}{(x^2+16)^2} &= \lim_{b\to\infty}\int_0^{\tan^{-1}(b/4)} \frac{4\sec(\theta)^2 d\theta}{(16\tan(\theta)^2+16)^2} \\
&= \lim_{b\to\infty}\int_0^{\tan^{-1}(b/4)} \frac{4\sec(\theta)^2 d\theta}{(16\sec(\theta)^2)^2} \\
&= \lim_{b\to\infty}\frac{1}{64}\int_0^{\tan^{-1}(b/4)} \cos(\theta)^2 d\theta \\
&= \frac{1}{64}\int_0^{\pi/2} \cos(\theta)^2 d\theta
\end{aligned}$$

since $\tan^{-1}\left(\frac{b}{4}\right) \to \frac{\pi}{2}$ as $b \to \infty$. We can compute this last integral with a double-angle formula to obtain

$$\begin{aligned}
\int_0^\infty \frac{dx}{(x^2+16)^2} &= \frac{1}{64}\int_0^{\pi/2} \frac{1+\cos(2\theta)}{2} d\theta \\
&= \frac{1}{64}\left[\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right]_0^{\pi/2} \\
&= \boxed{\frac{\pi}{256}}.
\end{aligned}$$

(b) $\displaystyle\int_0^1 \frac{\ln(x)}{\sqrt[3]{x}} dx$.

Solution. Note that this ia type II improper integral due to the vertical asymptote of the integrand at $x = 0$. We can start by finding an antiderivative using an IBP with parts

$$u = \ln(x) \;\Rightarrow\; du = \frac{dx}{x},$$
$$dv = \frac{dx}{\sqrt[3]{x}} \;\Rightarrow\; v = \frac{3x^{2/3}}{2}.$$

We get

$$\begin{aligned}
\int \frac{\ln(x)}{\sqrt[3]{x}} &= \frac{3x^{2/3}\ln(x)}{2} - \int \frac{3x^{2/3}}{2}\cdot\frac{dx}{x} \\
&= \frac{3x^{2/3}\ln(x)}{2} - \frac{3}{2}\int \frac{dx}{x^{1/3}} \\
&= \frac{3x^{2/3}\ln(x)}{2} - \frac{3}{2}\cdot\frac{3x^{2/3}}{2} + C
\end{aligned}$$

$$= \frac{3x^{2/3}}{4}\left(2\ln(x) - 3\right) + C.$$

We can now use this to compute the improper integral, as follows:

$$\begin{aligned}
\int_0^1 \frac{\ln(x)}{\sqrt[3]{x}}dx &= \lim_{a\to 0}\int_a^1 \frac{\ln(x)}{\sqrt[3]{x}}dx \\
&= \lim_{a\to 0}\left[\frac{3x^{2/3}}{4}\left(2\ln(x) - 3\right)\right]_a^1 \\
&= \lim_{a\to 0}\left(-\frac{9}{4} - \frac{3a^{2/3}}{4}\left(2\ln(a) - 3\right)\right) \\
&= -\frac{9}{4} - \frac{3}{4}\lim_{a\to 0}\frac{2\ln(a) - 3}{a^{-2/3}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} -\frac{9}{4} - \frac{3}{4}\lim_{a\to 0}\frac{\frac{2}{a}}{-\frac{2}{3}a^{-5/3}} \\
&= -\frac{9}{4} - \frac{3}{4}\lim_{a\to 0} 3a^{2/3} \\
&= \boxed{-\frac{9}{4}}.
\end{aligned}$$

(c) $\displaystyle\int_0^\infty e^{-\sqrt{z}}dz.$

Solution. We can start by computing an antiderivative using a substitution followed by an IBP. For the substitution, we will let $t = \sqrt{z}$, so that $dt = \frac{dz}{2\sqrt{z}}$, or $dz = 2tdt$. This gives

$$\int e^{-\sqrt{z}}dz = 2\int te^{-t}dt.$$

This last integral can be computed using an IBP with $u = t$, $du = dt$ and $dv = e^{-t}dt$, $v = -e^{-t}$. We obtain

$$\begin{aligned}
\int e^{-\sqrt{z}}dz &= 2\left(-te^{-t} - \int(-e^{-t})dt\right) \\
&= 2\left(-te^{-t} - e^{-t}\right) + C \\
&= -2e^{-t}(t+1) + C \\
&= -2e^{-\sqrt{z}}(\sqrt{z}+1) + C.
\end{aligned}$$

We now use this for the improper integral. We obtain

$$\begin{aligned}
\int_0^\infty e^{-\sqrt{z}}dz &= \lim_{b\to\infty}\int_0^b e^{-\sqrt{z}}dz \\
&= \lim_{b\to\infty}\left[-2e^{-\sqrt{z}}(\sqrt{z}+1)\right]_0^b \\
&= \lim_{b\to\infty}\left(2 - 2e^{-\sqrt{b}}(\sqrt{b}+1)\right) \\
&= 2 - 2\lim_{b\to\infty}\frac{\sqrt{b}+1}{e^{\sqrt{b}}}
\end{aligned}$$

$$\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} 2 - 2 \lim_{b\to\infty} \frac{\frac{1}{2\sqrt{b}}}{\frac{1}{2\sqrt{b}} e^{\sqrt{b}}}$$
$$= 2 - 2 \lim_{b\to\infty} \frac{1}{e^{\sqrt{b}}}$$
$$= \boxed{2}.$$

7. Find the sums of the following infinite series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^n}{3^{2n+1}}$.

Solution. Observe that

$$\sum_{n=1}^{\infty} \frac{(-5)^n}{3^{2n+1}} = \sum_{n=1}^{\infty} \frac{(-5)^n}{3^{2n} \cdot 3} = \sum_{n=1}^{\infty} \frac{1}{3}\left(-\frac{5}{9}\right)^n,$$

so we have a geometric series with first term $-\frac{5}{27}$ and common ratio $r = -\frac{5}{9}$. Since $|r| < 1$, the series converges and the sum is given by

$$\sum_{n=1}^{\infty} \frac{(-5)^n}{3^{2n+1}} = \frac{\text{first term}}{1-r}$$
$$= \frac{-\frac{5}{27}}{1-\left(-\frac{5}{9}\right)}$$
$$= \boxed{-\frac{5}{42}}.$$

(b) $\displaystyle\sum_{n=3}^{\infty} \left(\tan\left(\frac{\pi}{n}\right) - \tan\left(\frac{\pi}{n+2}\right)\right)$.

Solution. Writing out the first few terms of the partial sum $S_N = \displaystyle\sum_{n=3}^{N} \left(\tan\left(\frac{\pi}{n}\right) - \tan\left(\frac{\pi}{n+2}\right)\right)$, we see that we have a telescoping series. More precisely, observe that

$$S_N = \left(\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{5}\right)\right) + \left(\tan\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{6}\right)\right) + \left(\tan\left(\frac{\pi}{5}\right) - \tan\left(\frac{\pi}{7}\right)\right) + \cdots$$
$$\cdots + \left(\tan\left(\frac{\pi}{N-1}\right) - \tan\left(\frac{\pi}{N+1}\right)\right) + \left(\tan\left(\frac{\pi}{N}\right) - \tan\left(\frac{\pi}{N+2}\right)\right)$$
$$= \tan\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{N+1}\right) - \tan\left(\frac{\pi}{N+2}\right).$$

Therefore

$$\sum_{n=3}^{\infty} \left(\tan\left(\frac{\pi}{n}\right) - \tan\left(\frac{\pi}{n+2}\right)\right) = \lim_{N\to\infty} S_N$$
$$= \lim_{N\to\infty} \left(\tan\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{N+1}\right) - \tan\left(\frac{\pi}{N+2}\right)\right)$$
$$= \tan\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{4}\right) - \tan(0) - \tan(0)$$
$$= \boxed{\sqrt{3}+1}.$$

8. Use geometric series to write the number $3.41232323\cdots$ as a quotient of two integers.

Solution. We have

$$\begin{aligned}
3.41232323\cdots &= 3.41 + 0.0023 + 0.000023 + 0.00000023\cdots \\
&= \frac{341}{100} + \frac{23}{10^4} + \frac{23}{10^6} + \frac{23}{10^8}\cdots \\
&= \frac{341}{100} + \frac{23}{10000}\left(1 + \frac{1}{100} + \frac{1}{100^2} + \cdots\right) \\
&= \frac{341}{100} + \frac{23}{10000} \cdot \frac{1}{1-\frac{1}{100}} \\
&= \frac{341}{100} + \frac{23}{10000} \cdot \frac{100}{99} \\
&= \frac{341}{100} + \frac{23}{100 \cdot 99} \\
&= \frac{33782}{9900} \\
&= \boxed{\frac{16891}{4950}}.
\end{aligned}$$

9. Consider the series $\sum_{n=0}^{\infty} \frac{3 \cdot 4^n}{(A-5)^{2n}}$, where A is an unspecified constant. Use this series in parts (a)-(b).

(a) For which values of the constant A does the series converge?

Solution. Observe that

$$\sum_{n=0}^{\infty} \frac{3 \cdot 4^n}{(A-5)^{2n}} = \sum_{n=0}^{\infty} 3\left(\frac{4}{(A-5)^2}\right)^n,$$

so this is a geometric series with common ratio $r = \frac{4}{(A-5)^2}$. The series converges when $|r| < 1$, that is

$$\begin{aligned}
&\frac{4}{(A-5)^2} < 1 \\
&\Rightarrow\ 4 < (A-5)^2 \\
&\Rightarrow\ 2 < |A-5| \\
&\Rightarrow A-5 > 2 \text{ or } A-5 < -2 \\
&\Rightarrow \boxed{A > 7 \text{ or } A < 3}.
\end{aligned}$$

(b) For the values of A that you found in part (a), find the sum of the series.

Solution. When $|r| < 1$, we have

$$\sum_{n=0}^{\infty} \frac{3 \cdot 4^n}{(A-5)^{2n}} = \frac{\text{first term}}{1-r}$$

$$= \frac{3}{1 - \frac{4}{(A-5)^2}}$$

$$= \boxed{\frac{3(A-5)^2}{(A-5)^2 - 4}}.$$

10. Determine if the following series converge or diverge.

(a) $\sum_{n=0}^{\infty} n^2 e^{-n}$.

Solution. We use the Integral Test. The function $f(x) = x^2 e^{-x}$ is positive and continuous on $[0, \infty)$. Furthermore we have

$$f'(x) 2xe^{-x} - x^2 e^{-x} = xe^{-x}(2 - x),$$

from which we see that $f'(x) < 0$, and so f is decreasing, on $(2, \infty)$. Therefore we can apply the Integral Test and test whether the series converges by computing the corresponding improper integral. For the integral, we will use two successive IBPs, taking $dv = e^{-x}dx$ each time. We get

$$\begin{aligned}
\int x^2 e^{-x} dx &= -x^2 e^{-x} + 2 \int xe^{-x} dx \\
&= -x^2 e^{-x} - 2xe^{-x} + \int e^{-x} dx \\
&= -x^2 e^{-x} - 2xe^{-x} - e^{-x} + C \\
&= -\frac{x^2 + 2x + 2}{e^x} + C.
\end{aligned}$$

So

$$\begin{aligned}
\int_0^{\infty} x^2 e^{-x} dx &= \lim_{b \to \infty} \int_0^b x^2 e^{-x} dx \\
&= \lim_{b \to \infty} \left[-\frac{x^2 + 2x + 2}{e^x} \right]_0^b \\
&= \lim_{b \to \infty} \left(2 - \frac{b^2 + 2b + 2}{e^b} \right) \\
&= 2 - \lim_{b \to \infty} \frac{b^2 + 2b + 2}{e^b} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} 2 - \lim_{b \to \infty} \frac{2b + 2}{e^b} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} 2 - \lim_{b \to \infty} \frac{2}{e^b} \\
&= 2.
\end{aligned}$$

So $\int_0^{\infty} x^2 e^{-x} dx$ converges. It follows that $\boxed{\sum_{n=0}^{\infty} n^2 e^{-n} \text{ converges}}$.

(b) $\displaystyle\sum_{n=3}^{\infty}\left(\frac{n+3}{n}\right)^n$.

Solution. The limit of the general term is an indeterminate form 1^∞. If this indetermination resolves into something not equal to zero, the Term Divergence Test will immediately tell us that the series diverges. So let us try to compute the limit of the general term.

We can start by writing the power in exponential form

$$\lim_{n\to\infty}\left(\frac{n+3}{n}\right)^n = \lim_{n\to\infty} e^{n\ln\left(1+\frac{3}{n}\right)}.$$

We now compute the limit of the exponent using L'Hôpital's Rule:

$$\begin{aligned}\lim_{n\to\infty} n\ln\left(1+\frac{3}{n}\right) &= \lim_{x\to\infty}\frac{\ln(1+3/x)}{\frac{1}{x}}\\ &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \frac{-\frac{3}{x^2}\cdot\frac{1}{1+3/x}}{-\frac{1}{x^2}}\\ &= \lim_{x\to\infty}\frac{3}{1+3/x}\\ &= 3.\end{aligned}$$

So

$$\lim_{n\to\infty} e^{n\ln\left(1+\frac{3}{n}\right)} = e^3 \neq 0.$$

By the Term Divergence Test, it follows that $\boxed{\displaystyle\sum_{n=3}^{\infty}\left(\frac{n+3}{n}\right)^n \text{ diverges}}$.

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{2^{3n-2}}{5^{n+1}}$.

Solution. Observe that

$$\sum_{n=0}^{\infty}\frac{2^{3n-2}}{5^{n+1}} = \sum_{n=0}^{\infty}\frac{2^{3n}\cdot 2^{-2}}{5^n\cdot 5} = \sum_{n=0}^{\infty}\frac{1}{20}\left(\frac{8}{5}\right)^n.$$

So this series is a geometric series with common ratio $r=\frac{8}{5}$. Since $|r|\geqslant 1$, we conclude that $\boxed{\displaystyle\sum_{n=0}^{\infty}\frac{2^{3n-2}}{5^{n+1}} \text{ diverges}}$.