Rutgers University
FA23 Math 152

Final Exam Practice Problems Solutions

1. (a) Find a function $f(x)$ and an interval $[a,b]$ so that the right endpoint Riemann sum of $f(x)$ on the interval $[a,b]$ is

$$\sum_{k=1}^{n} \tan^3\left(\frac{\pi k}{4n}\right)\frac{\pi}{8n}.$$

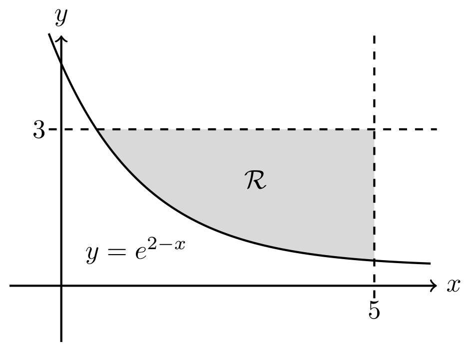
Solution. The function $\boxed{f(x)=\tan^3(2x)}$ on the interval $\boxed{\left[0,\frac{\pi}{8}\right]}$ gives a possible solution.

(b) Evaluate $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n} \tan^3\left(\frac{\pi k}{4n}\right)\frac{\pi}{8n}$.

Solution.

$$\begin{aligned}
\lim_{n\to\infty}\sum_{k=1}^{n} \tan^3\left(\frac{\pi k}{4n}\right)\frac{\pi}{8n} &= \int_0^{\pi/8} \tan^3(2x)dx \\
&= \int_0^{\pi/8} \tan(2x)\tan^2(2x)dx \\
&= \int_0^{\pi/8} \tan(2x)\left(\sec^2(2x)-1\right)dx \\
&= \int_0^{\pi/8} \tan(2x)\sec^2(2x)dx - \int_0^{\pi/8}\tan(2x)dx \\
&= \int_0^{1}\frac{u}{2}du - \left[\frac{\ln|\sec(2x)|}{2}\right]_0^{\pi/8} \qquad (u=\tan(2x)) \\
&= \left[\frac{u^2}{4}\right]_0^1 - \frac{\ln\left(\sqrt{2}\right)}{2} \\
&= \boxed{\frac{1-\ln(2)}{4}}.
\end{aligned}$$

2. Consider the region $\mathcal{R}$ bounded by the curve $y=e^{2-x}$, the line $x=5$ and the line $y=3$. The region $\mathcal{R}$ is sketched below.



Set-up integrals computing the volume of the solid obtained by revolving $\mathcal{R}$ about each axis given below using (i) the disk/washer method, and (ii) the shell method.

(a) x-axis

Solution. (i) $\boxed{V = \int_{2-\ln(3)}^{5} \pi \left(3^2 - \left(e^{2-x}\right)^2\right) dx}$, (ii) $\boxed{V = \int_{e^{-3}}^{3} 2\pi y \left(5 - (2 - \ln(y))\right) dy}$.

(b) y-axis

Solution. (i) $\boxed{V = \int_{e^{-3}}^{3} \pi \left(5^2 - (2 - \ln(y))^2\right) dy}$, (ii) $\boxed{V = \int_{2-\ln(3)}^{5} 2\pi x \left(3 - e^{2-x}\right) dx}$.

(c) $y = -1$

Solution. (i) $\boxed{V = \int_{2-\ln(3)}^{5} \pi \left((3+1)^2 - \left(e^{2-x}+1\right)^2\right) dx}$, (ii) $\boxed{V = \int_{e^{-3}}^{3} 2\pi(y+1) \left(5 - (2 - \ln(y))\right) dy}$.

(d) $x = 7$

Solution. (i) $\boxed{V = \int_{e^{-3}}^{3} \pi \left((7 - (2 - \ln(y)))^2 - (7-5)^2\right) dy}$, (ii) $\boxed{V = \int_{2-\ln(3)}^{5} 2\pi(7-x) \left(3 - e^{2-x}\right) dx}$.

(e) $y = 3$

Solution. (i) $\boxed{V = \int_{2-\ln(3)}^{5} \pi \left(3 - e^{2-x}\right)^2 dx}$, (ii) $\boxed{V = \int_{e^{-3}}^{3} 2\pi(3-y) \left(5 - (2 - \ln(y))\right) dy}$.

(f) $x = -4$

Solution. (i) $\boxed{V = \int_{e^{-3}}^{3} \pi \left((5+4)^2 - (2 - \ln(y) + 4)^2\right) dy}$, (ii) $\boxed{V = \int_{2-\ln(3)}^{5} 2\pi(x+4) \left(3 - e^{2-x}\right) dx}$.

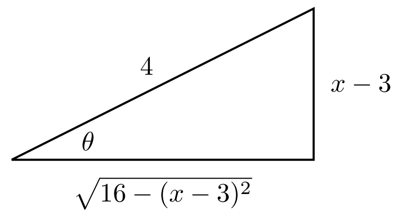
3. Evaluate the following integrals. If an integral diverges, explain why.

(a) $\displaystyle\int \frac{dx}{(7+6x-x^2)^{3/2}}$

Solution. We complete the square and then use a trigonometric substitution. We have

$$7+6x-x^2 = 7-(x^2-6x) = 7-((x-3)^2-9) = 16-(x-3)^2.$$

For the trigonometric substitution, we want $16-(x-3)^2 = 16-16\sin(\theta)^2$, so we will set $x-3 = 4\sin(\theta)$. This gives $dx = 4\cos(\theta)d\theta$ and the following right triangle where $\sin(\theta) = \frac{x-3}{4}$:



Therefore

$$\begin{aligned}
\int \frac{dx}{(7+6x-x^2)^{3/2}} &= \int \frac{dx}{(16-(x-3)^2)^{3/2}} \\
&= \int \frac{4\cos(\theta)d\theta}{(16-16\sin(\theta)^2)^{3/2}} \\
&= \int \frac{4\cos(\theta)d\theta}{(16\cos(\theta)^2)^{3/2}} \\
&= \frac{1}{16}\int \frac{d\theta}{\cos(\theta)^2} \\
&= \frac{1}{16}\int \sec(\theta)^2 d\theta \\
&= \frac{1}{16}\tan(\theta)+C \\
&= \boxed{\frac{x-3}{16\sqrt{16-(x-3)^2}}+C}.
\end{aligned}$$

(b) $\displaystyle\int_0^\infty x^2e^{-3x}dx$

Solution. We start by computing an antiderivative of the integrand using integration by parts twice. For the first IBP, we will use

$$u = x^2 \;\Rightarrow\; du = 2xdx,$$
$$dv = e^{-3x} \;\Rightarrow\; v = -\frac{1}{3}e^{3x}.$$

We get

$$\int x^2e^{-3x} = -\frac{x^2e^{-3x}}{3} + \frac{2}{3}\int xe^{-3x}dx.$$

The second IBP uses

$$u = x \;\Rightarrow\; du = xdx,$$

$$dv = e^{-3x} \ \Rightarrow \ v = -\frac{1}{3}e^{3x}.$$

We get

$$\begin{aligned}
\int x^2 e^{-3x} &= -\frac{x^2 e^{-3x}}{3} - \frac{2xe^{-3x}}{9} + \frac{2}{9}\int e^{-3x}dx \\
&= -\frac{x^2 e^{-3x}}{3} - \frac{2xe^{-3x}}{9} - \frac{2e^{-3x}}{27} + C \\
&= -\frac{9x^2+6x+2}{27e^{3x}} + C.
\end{aligned}$$

We can now evaluate the improper integral.

$$\begin{aligned}
\int_0^\infty x^2 e^{-3x}dx &= \lim_{b\to\infty}\int_0^b x^2 e^{-3x}dx \\
&= \lim_{b\to\infty}\left[-\frac{9x^2+6x+2}{27e^{3x}}\right]_0^b \\
&= \lim_{b\to\infty}\left(-\frac{9b^2+6b+2}{27e^{3b}} + \frac{2}{27}\right) \\
&= \frac{2}{27} - \lim_{b\to\infty}\frac{18b+6}{81e^{3b}} \quad \text{(L'H)} \\
&= \frac{2}{27} - \lim_{b\to\infty}\frac{18}{243e^{3b}} \quad \text{(L'H)} \\
&= \boxed{\frac{2}{27}}.
\end{aligned}$$

(c) $\displaystyle\int \cos^2(5\theta)\sin^2(5\theta)d\theta$

Solution. Trigonometric integrals with powers of sin and cos both even can be dealt with using the double-angle identities. We get

$$\begin{aligned}
\int \cos^2(5\theta)\sin^2(5\theta)d\theta &= \int \frac{1+\cos(10\theta)}{2}\frac{1-\cos(10\theta)}{2}d\theta \\
&= \frac{1}{4}\int\left(1-\cos^2(10\theta)\right)d\theta \\
&= \frac{1}{4}\int\left(1-\frac{1+\cos(20\theta)}{2}\right)d\theta \\
&= \frac{1}{8}\int\left(1-\cos(20\theta)\right)d\theta \\
&= \boxed{\frac{1}{8}\left(\theta-\frac{\sin(20\theta)}{20}\right)+C}.
\end{aligned}$$

(d) $\displaystyle\int_1^{\sqrt{2}} \frac{dx}{\sqrt{4x^2-2}}$

Solution. We use a trigonometric substitution. We want $4x^2 - 2 = 2\sec(\theta)^2 - 2$, so we take $x = \dfrac{\sqrt{2}}{2}\sec(\theta)$, so that $dx = \dfrac{\sqrt{2}}{2}\sec(\theta)\tan(\theta)d\theta$. When $x = 1$, we have $\sec(\theta) = \sqrt{2}$, so $\theta = \frac{\pi}{4}$, When $x = \sqrt{2}$, we have $\sec(\theta) = 2$, so $\theta = \frac{\pi}{3}$. Therefore, the integral becomes

$$\begin{aligned}
\int_1^{\sqrt{2}} \frac{dx}{\sqrt{4x^2-2}} &= \int_{\pi/4}^{\pi/3} \frac{\frac{\sqrt{2}}{2}\sec(\theta)\tan(\theta)}{\sqrt{2\sec(\theta)^2-2}} d\theta \\
&= \int_{\pi/4}^{\pi/3} \frac{\frac{\sqrt{2}}{2}\sec(\theta)\tan(\theta)}{\sqrt{2}\tan(\theta)} d\theta \\
&= \frac{1}{2}\int_{\pi/4}^{\pi/3} \sec(\theta)d\theta \\
&= \frac{1}{2}\left[\ln|\sec(\theta)+\tan(\theta)|\right]_{\pi/4}^{\pi/3} \\
&= \boxed{\frac{1}{2}\left(\ln(2+\sqrt{3}) - \ln(1+\sqrt{2})\right)}.
\end{aligned}$$

(e) $\int \sqrt{x}\cos(3\sqrt{x})dx$

Solution. We start with the substitution $w = 3\sqrt{x}$, which gives $dw = \frac{3dx}{2\sqrt{x}}$, so $dx = \frac{2wdw}{3}$. Hence

$$\int \sqrt{x}\cos(3\sqrt{x})dx = \int \frac{w}{3}\cos(w)\frac{2wdw}{3} = \frac{2}{9}\int w^2\cos(w)dw.$$

We now use two successive IBPs. In the first one, we pick $u = w^2$ and $v = \cos(w)$, so $du = 2wdw$ and $v = \sin(w)$, and

$$\int w^2\cos(w)dw = w^2\sin(w) - \int 2w\sin(w)dw.$$

In the second IBP, we pick $u = 2w$ and $dv = \sin(w)$, so that $du = 2dw$ and $v = -\cos(w)$. We obtain

$$\begin{aligned}
\int w^2\cos(w)dw &= w^2\sin(w) - \int 2w\sin(w)dw \\
&= w^2\sin(w) - \left(2w(-\cos(w)) - \int 2(-\cos(w))dw\right) \\
&= w^2\sin(w) + 2w\cos(w) - 2\int \cos(w)dw \\
&= w^2\sin(w) + 2w\cos(w) - 2\sin(w) + C.
\end{aligned}$$

Going back to the x-integral, we obtain

$$\begin{aligned}
\int \sqrt{x}\cos(3\sqrt{x})dx &= \frac{2}{9}\int w^2\cos(w)dw \\
&= \frac{2}{9}\left(w^2\sin(w) + 2w\cos(w) - 2\sin(w)\right) + C \\
&= \frac{2}{9}\left((3\sqrt{x})^2\sin(3\sqrt{x}) + 2(3\sqrt{x})\cos(3\sqrt{x}) - 2\sin(3\sqrt{x})\right) + C \\
&= \boxed{\frac{2}{9}\left(9x\sin(3\sqrt{x}) + 6\sqrt{x}\cos(3\sqrt{x}) - 2\sin(3\sqrt{x})\right) + C}.
\end{aligned}$$

(f) $\displaystyle\int_{\pi/4}^{3\pi/4} \tan(x)\sec^3(x)dx$

Solution. Let us start by finding the antiderivative using the substitution $u = \sec(x)$, $du = \sec(x)\tan(x)dx$. We get

$$\int \tan(x)\sec^3(x)dx = \int u^2 du = \frac{u^3}{3} + C = \frac{\sec^3(x)}{3} + C.$$

For the definite integral, observe that it is a type II improper integral because of the vertical asymptote of the integrand at $x = \frac{\pi}{2}$. We will need to compute this integral by splitting it up as

$$\int_{\pi/4}^{3\pi/4} \tan(x)\sec^3(x)dx = \int_{\pi/4}^{\pi/2} \tan(x)\sec^3(x)dx + \int_{\pi/2}^{3\pi/4} \tan(x)\sec^3(x)dx$$

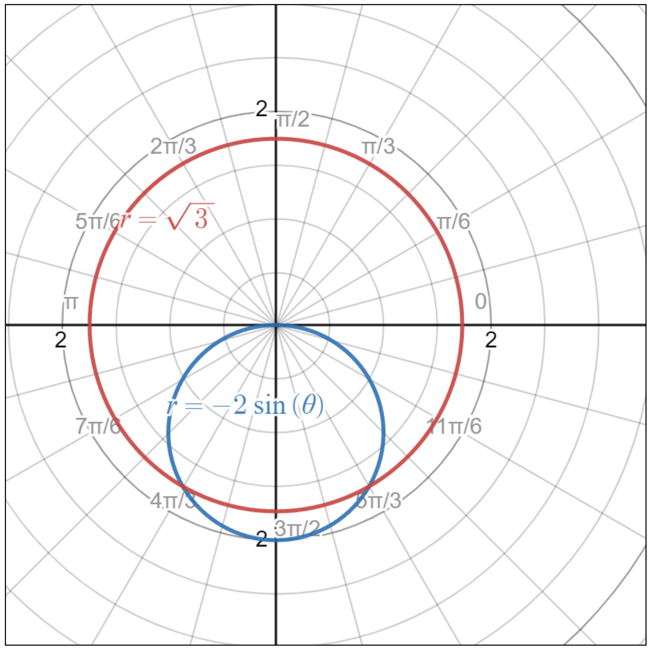
and setting each summand as a limit. For the first summand, we have

$$\begin{aligned}\int_{\pi/4}^{\pi/2} \tan(x)\sec^3(x)dx &= \lim_{b\to\frac{\pi}{2}^-} \int_{\pi/4}^{b} \tan(x)\sec^3(x)dx \\ &= \lim_{b\to\frac{\pi}{2}^-} \left[\frac{\sec^3(x)}{3}\right]_{\pi/4}^{b} \\ &= \lim_{b\to\frac{\pi}{2}^-} \left(\frac{\sec^3(b)}{3} - \frac{(\sqrt{2})^3}{3}\right) \\ &= \infty.\end{aligned}$$

There is no need to go any further. We have found that $\boxed{\int_{\pi/4}^{3\pi/4} \tan(x)\sec^3(x)dx \text{ diverges}}$.

4. Find the area inside the circle $x^2 + y^2 = 3$ and outside the circle $x^2 + y^2 + 2y = 0$. *(Hint: use polar coordinates.)*

   Solution. We start by sketching the circles and converting their equations to polar. The first equation $x^2 + y^2 = 3$ gives $r^2 = 3$, so $r = \sqrt{3}$ is the polar equation. The second equation $x^2 + y^2 + 2y = 0$ becomes $r^2 + 2r\sin(\theta) = 0$, so we get $r = 0$ or $r = -2\sin(\theta)$. Since the origin is already on the graph of $r = -2\sin(\theta)$, we can discard $r = 0$ and keep $r = -2\sin(\theta)$.



To find the values of θ where the circles intersect, we equate the polar equations and solve for $0 \leqslant \theta \leqslant 2\pi$:

$$-2\sin(\theta) = \sqrt{3} \Rightarrow \sin(\theta) = -\frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{4\pi}{3}, \frac{5\pi}{3}.$$

We can also exploit the symmetry with respect to the y-axis and compute the area of the region in the first and fourth quadrants, and double it. Therefore, the area will be computed by

$$\begin{aligned}
A &= 2\left(\int_{-\pi/3}^{0} \frac{1}{2}\left(\sqrt{3}^2 - (-2\sin(\theta))^2\right) d\theta + \int_0^{\pi/2} \frac{1}{2}\sqrt{3}^2 d\theta\right) \\
&= \int_{-\pi/3}^{0} \left(3 - 4\sin(\theta)^2\right) d\theta + \int_0^{\pi/2} 3 d\theta \\
&= \int_{-\pi/3}^{0} \left(3 - 4\frac{1-\cos(2\theta)}{2}\right) d\theta + \frac{3\pi}{2} \\
&= \int_{-\pi/3}^{0} (1 + 2\cos(2\theta))\, d\theta + \frac{3\pi}{2} \\
&= [\theta + \sin(2\theta)]_{-\pi/3}^{0} + \frac{3\pi}{2} \\
&= -\left(-\frac{\pi}{3} + \sin\left(-\frac{2\pi}{3}\right)\right) + \frac{3\pi}{2} \\
&= \boxed{\frac{11\pi}{6} + \frac{\sqrt{3}}{2}}.
\end{aligned}$$

5. Find the length of the polar curve $r = \theta^4$, $0 \leqslant \theta \leqslant \sqrt{2}$.

Solution. The length is given by

$$L = \int_0^{\sqrt{2}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_0^{\sqrt{2}} \sqrt{\theta^8 + (4\theta^3)^2} d\theta = \int_0^{\sqrt{2}} \theta^3\sqrt{\theta^2 + 16} d\theta.$$

This integral can be calculated with the substitution $u = \theta^2 + 16$, which gives $du = 2\theta d\theta$. The extraneous factor θ^2 in the integrand can be replaced by $u - 16$. The integral becomes

$$\begin{aligned} L &= \int_{16}^{18} \frac{1}{2}(u-16)\sqrt{u}du \\ &= \frac{1}{2}\int_{16}^{18} \left(u^{3/2} - 16u^{1/2}\right) du \\ &= \frac{1}{2}\left[\frac{2}{5}u^{5/2} - \frac{32}{3}u^{3/2}\right]_{16}^{18} \\ &= \boxed{\frac{1}{2}\left(\frac{2}{5}(18^{5/2} - 16^{5/2}) - \frac{32}{3}(18^{3/2} - 16^{3/2})\right)}. \end{aligned}$$

6. Determine if the sequences below converge absolutely, converge conditionally or diverge. If a sequence converges, find its limit.

   (a) $a_n = n^2\left(1 - \sec\left(\frac{5}{n}\right)\right)$

   Solution. This is an indeterminate form $\infty \cdot 0$. We can calculate the limit by writing the expression as a $\frac{0}{0}$ quotient and using L'Hopital's Rule. This gives

$$\begin{aligned} \lim_{n\to\infty} n^2\left(1 - \sec\left(\frac{5}{n}\right)\right) &= \lim_{x\to\infty} \frac{1 - \sec\left(\frac{5}{x}\right)}{\frac{1}{x^2}} \\ &= \lim_{x\to\infty} \frac{\frac{5}{x^2}\sec\left(\frac{5}{x}\right)\tan\left(\frac{5}{x}\right)}{-\frac{2}{x^3}} \\ &= \lim_{x\to\infty} -\frac{5}{2}\sec\left(\frac{5}{x}\right)\frac{\tan\left(\frac{5}{x}\right)}{\frac{1}{x}} \quad \text{(L'H)} \\ &= \left(\lim_{x\to\infty} -\frac{5}{2}\sec\left(\frac{5}{x}\right)\right)\left(\lim_{x\to\infty}\frac{\tan\left(\frac{5}{x}\right)}{\frac{1}{x}}\right) \\ &= -\frac{5}{2}\lim_{x\to\infty}\frac{-\frac{5}{x^2}\sec^2\left(\frac{5}{x}\right)}{-\frac{1}{x^2}} \quad \text{(L'H)} \\ &= -\frac{5}{2}\lim_{x\to\infty} 5\sec^2\left(\frac{5}{x}\right) \\ &= \boxed{-\frac{25}{2}}. \end{aligned}$$

   (b) $a_n = \dfrac{\ln(2^n + 1)}{\ln(n)}$

   Solution. This is an indeterminate form $\frac{\infty}{\infty}$, so we can use L'Hopital's Rule. We have

$$\begin{aligned} \lim_{x\to\infty}\frac{\ln(2^x+1)}{\ln(x)} &= \lim_{x\to\infty}\frac{\frac{\ln(2)2^x}{2^x+1}}{\frac{1}{x}} \quad \text{(L'H)} \\ &= \lim_{x\to\infty}\frac{\ln(2)x}{1 + \frac{1}{2^x}} \end{aligned}$$

$$= \infty.$$

Therefore, the sequence diverges.

7. Determine if the series below converge or diverge. If a series converges, find its sum when possible.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(n) - 5^n}{3^{2n}}$

Solution. Observe that

$$\frac{\cos(n) - 5^n}{3^{2n}} = \frac{\cos(n)}{9^n} - \left(\frac{5}{9}\right)^n.$$

Since $0 \leqslant \left|\frac{\cos(n)}{9^n}\right| \leqslant \frac{1}{9^n}$ and $\displaystyle\sum_{n=0}^{\infty} \frac{1}{9^n}$ converges as a geometric series with $|r| = \frac{1}{9} < 1$, we deduce that $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(n)}{9^n}$ converges absolutely by the DCT. Also, $\displaystyle\sum_{n=0}^{\infty} \left(\frac{5}{9}\right)^n$ converges absolutely as a geometric series with $|r| = \frac{5}{9} < 1$. Therefore, we have

$$\sum_{n=0}^{\infty} \frac{\cos(n) - 5^n}{3^{2n}} = \sum_{n=0}^{\infty} \frac{\cos(n)}{9^n} - \sum_{n=0}^{\infty} \left(\frac{5}{9}\right)^n$$

and $\boxed{\displaystyle\sum_{n=0}^{\infty} \frac{\cos(n) - 5^n}{3^{2n}} \text{ converges absolutely}}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{3 \cdot 2^{2n}}{5^{n+1}}$

Solution. The series can be written as

$$\sum_{n=1}^{\infty} \frac{3 \cdot 2^{2n}}{5^{n+1}} = \sum_{n=1}^{\infty} \frac{3}{5}\left(\frac{4}{5}\right)^n$$

so it is geometric. Since the common ratio $r = \frac{4}{5}$ satisfies $|r| < 1$, the series converges absolutely and its sum is

$$\boxed{\sum_{n=1}^{\infty} \frac{3 \cdot 2^{2n}}{5^{n+1}} = \frac{\frac{12}{25}}{1 - \frac{4}{5}} = \frac{12}{5}}.$$

(c) $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{n\sqrt{\ln(n)^2 + 1}}$

Solution. We start by using the AST with $a_n = \frac{1}{n\sqrt{\ln(n)^2+1}}$, which is positive, decreasing and converges to 0 as the reciprocal of an increasing positive sequence going to infinity. Therefore, the series converges.

We now need to determine if the convergence is absolute or conditional by inspecting the series

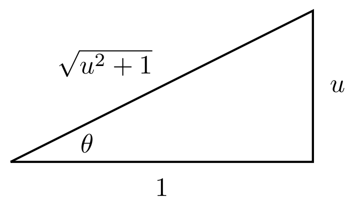
$$\sum_{n=3}^{\infty} \left| \frac{(-1)^n}{n\sqrt{\ln(n)^2+1}} \right| = \sum_{n=3}^{\infty} \frac{1}{n\sqrt{\ln(n)^2+1}}.$$

For this series, we use the Integral Test. The function $f(x) = \dfrac{1}{x\sqrt{\ln(x)^2+1}}$ is continuous and positive on $[3, \infty)$. It is also decreasing as the reciprocal of a positive increasing function. Therefore, the Integral Test applies and we can test for the convergence of the series by calculating the improper integral $\displaystyle\int_3^{\infty} \frac{dx}{x\sqrt{\ln(x)^2+1}}$.

Before we calculate the improper integral, let us start by finding an antiderivative. We first use the substitution $u = \ln(x)$, $du = \frac{dx}{x}$ to get

$$\int \frac{dx}{x\sqrt{\ln(x)^2+1}} = \int \frac{du}{\sqrt{u^2+1}}.$$

This integral can be calculated with the substitution $u = \tan(\theta)$, which gives $du = \sec^2(\theta)d\theta$ and $\sqrt{u^2+1} = \sqrt{\tan^2(\theta)+1} = \sec(\theta)$. The right triangle for this substitution has base angle θ so that $\tan(\theta) = u$, as shown below.



The integral becomes

$$\begin{aligned}
\int \frac{dx}{x\sqrt{\ln(x)^2+1}} &= \int \frac{du}{\sqrt{u^2+1}} \\
&= \int \frac{\sec^2(\theta)d\theta}{\sec(\theta)} \\
&= \int \sec(\theta)d\theta \\
&= \ln|\sec(\theta)+\tan(\theta)| + C \\
&= \ln\left|\sqrt{u^2+1}+u\right| + C \\
&= \ln\left|\sqrt{\ln(x)^2+1}+\ln(x)\right| + C.
\end{aligned}$$

Now for the improper integral, we get

$$\begin{aligned}
\int_3^{\infty} \frac{dx}{x\sqrt{\ln(x)^2+1}} &= \lim_{b\to\infty} \int_3^{b} \frac{dx}{x\sqrt{\ln(x)^2+1}} \\
&= \lim_{b\to\infty} \left[\ln\left|\sqrt{\ln(x)^2+1}+\ln(x)\right|\right]_3^{\infty} \\
&= \lim_{b\to\infty} \left(\ln\left|\sqrt{\ln(b)^2+1}+\ln(b)\right| - \ln\left|\sqrt{\ln(3)^2+1}+\ln(3)\right|\right) \\
&= \infty.
\end{aligned}$$

So $\displaystyle\int_3^\infty \frac{dx}{x\sqrt{\ln(x)^2+1}}$ diverges, and thus $\displaystyle\sum_{n=3}^\infty \frac{(-1)^n}{n\sqrt{\ln(n)^2+1}}$ does not converge absolutely.

In conclusion, $\boxed{\displaystyle\sum_{n=3}^\infty \frac{(-1)^n}{n\sqrt{\ln(n)^2+1}} \text{ converges conditionally}}$.

(d) $\displaystyle\sum_{n=1}^\infty \frac{\sqrt{9n^2+2}}{2n^4}$

Solution. We use the LCT with the reference series $\displaystyle\sum_{n=1}^\infty \frac{\sqrt{n^2}}{n^4} = \sum_{n=1}^\infty \frac{1}{n^3}$, which converges as a p-series with $p = 3 > 1$. The limit for the LCT is

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{a_n}{b_n} \\ &= \lim_{n\to\infty} \frac{\frac{\sqrt{9n^2+2}}{2n^4}}{\frac{\sqrt{n^2}}{n^4}} \\ &= \lim_{n\to\infty} \frac{\sqrt{9+\frac{2}{n^2}}}{2} \\ &= \frac{3}{2}. \end{aligned}$$

Since $0 < L < \infty$, both series have the same behavior, so $\boxed{\displaystyle\sum_{n=1}^\infty \frac{\sqrt{9n^2+2}}{2n^4} \text{ converges absolutely}}$.

8. Find the radius and interval of convergence of the following power series.

(a) $\displaystyle\sum_{n=1}^\infty \frac{2^n(x+3)^n}{n}$

Solution. We start by using the Root Test. We have

$$\rho = \lim_{n\to\infty} \left| \frac{2^n(x+3)^n}{n} \right|^{1/n} = \lim_{n\to\infty} \frac{2|x+3|}{n^{1/n}} = 2|x+3|.$$

The series converges absolutely when $2|x+3| < 1$, which gives $-\frac{7}{2} < x < -\frac{5}{2}$. We now need to test both endpoints.

- At $x = -\frac{7}{2}$, we have
$$\sum_{n=1}^\infty \frac{2^n(-\frac{7}{2}+3)^n}{n} = \sum_{n=1}^\infty \frac{(-1)^n}{n}.$$
This series converges by the AST since $a_n = \frac{1}{n}$ is positive, decreasing and converges to 0.
- At $x = -\frac{5}{2}$, we have
$$\sum_{n=1}^\infty \frac{2^n(-\frac{5}{2}+3)^n}{n} = \sum_{n=1}^\infty \frac{1}{n}.$$
This series diverges as a p-series with $p = 1$.

In conclusion, we have $\boxed{R = \frac{1}{2}}$ and $\boxed{\text{IOC} = \left[-\frac{7}{2}, -\frac{5}{2}\right)}$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 x^n}{5^{n^2}}$

Solution. We start by using the Root Test. We have

$$\rho = \lim_{n\to\infty} \sqrt[n]{\left|\frac{n^2 x^n}{5^{n^2}}\right|} = \lim_{n\to\infty} \frac{n^{2/n}|x|}{5^n} = 0.$$

Therefore, the series converges absolutely for any value of x. So $\boxed{R = \infty}$ and $\boxed{\text{IOC} = (-\infty, \infty)}$.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x+5)^{3n}}{\sqrt{64^n n + 1}}$

Solution. We use the Root Test. We have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^{n+1}(x+5)^{3n}}{\sqrt{64^n n + 1}}\right|} \\
&= \lim_{n\to\infty} \frac{|x+5|^3}{(64^n n + 1)^{1/2n}} \\
&= \lim_{n\to\infty} \frac{|x+5|^3}{8n^{1/2n}\left(1 + \frac{1}{64^n n}\right)^{2/n}} \\
&= \frac{|x+5|^3}{8}
\end{aligned}$$

since

$$\lim_{n\to\infty} n^{1/2n} = e^{\lim_{n\to\infty} \frac{\ln(n)}{2n}} = e^{\lim_{n\to\infty} \frac{1/n}{2}} = e^0 = 1.$$

So the series converges absolutely when $\frac{|x+5|^3}{8} < 1$, that is $-2 < x+5 < 2$, so $-7 < x < -3$. When $x > -3$ or $x < -7$, the series diverges. We now need to test the endpoints $x = -3, -7$.

When $x = -3$, we have

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(-3+5)^{3n}}{\sqrt{64^n n + 1}} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 8^n}{\sqrt{64^n n + 1}} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n + 64^{-n}}}.$$

Let us check the assumptions of the AST for this series. The sequence $a_n = \frac{1}{\sqrt{n+64^{-n}}}$ is positive. We can see that a_n is decreasing by observing that the function $f(x) = x + 64^{-x}$ has a positive derivative on $[1, \infty)$:

$$f'(x) = 1 - \ln(64)64^{-x} = \frac{64^x - \ln(64)}{64^x} > 0 \quad \text{when} \quad x \geqslant 1.$$

Also, $\lim_{n\to\infty} \frac{1}{\sqrt{n+64^{-n}}} = 0$. Therefore, the AST applies and the series converges.

When $x = -7$, we have

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(-7+5)^{3n}}{\sqrt{64^n n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(-1)^{3n}8^n}{\sqrt{64^n n+1}} = -\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+64^{-n}}}.$$

We can use the LCT for this series with $b_n = \frac{1}{\sqrt{n}}$. We have

$$\begin{aligned}\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+64^{-n}}}}{\frac{1}{\sqrt{n}}} &= \lim_{n\to\infty} \frac{1}{\sqrt{1+64^{-n}n^{-1/2}}} \\ &= \frac{1}{\sqrt{1+0}} \\ &= 1.\end{aligned}$$

Furthermore, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges as a p-series with $p = \frac{1}{2} \leqslant 1$. Therefore, the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+64^{-n}}}$ diverges.

Conclusion: the radius of convergence is $\boxed{R=2}$ and the interval of convergence is $\boxed{(-7,-3]}$.

9. Consider the curve with parametric equations $x = 2t, y = 3\ln(t) + 2, \dfrac{3}{2} \leqslant t \leqslant 3$.

(a) Calculate the length of the curve.

Solution.

$$\begin{aligned}L &= \int_{3/2}^{3} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_{3/2}^{3} \sqrt{(2)^2 + \left(\frac{3}{t}\right)^2}\, dt \\ &= \int_{3/2}^{3} \sqrt{4 + \frac{9}{t^2}}\, dt \\ &= \int_{3/2}^{3} \frac{\sqrt{4t^2+9}}{t}\, dt.\end{aligned}$$

We will compute this integral with a trigonometric substitution. We want $4t^2 + 9 = 9\tan^2(\theta) + 9 = 9\sec^2(\theta)$, so we choose $t = \frac{3\tan(\theta)}{2}$. Then we have $dt = \frac{3\sec^2(\theta)d\theta}{2}$. When $t = 3/2$, we have $\tan(\theta) = 1$ so $\theta = \pi/4$. When $t = 3$, we have $\tan(\theta) = 2$ so $\theta = \tan^{-1}(2)$. It follows that

$$\begin{aligned}L &= \int_{\pi/4}^{\tan^{-1}(2)} \frac{\sqrt{9\sec^2(\theta)}}{\frac{3\tan(\theta)}{2}} \frac{3}{2} \sec^2(\theta) d\theta \\ &= 3\int_{\pi/4}^{\tan^{-1}(2)} \frac{\sec(\theta)^3}{\tan(\theta)} d\theta \\ &= 3\int_{\pi/4}^{\tan^{-1}(2)} \frac{\sec(\theta)}{\tan(\theta)} \left(\tan(\theta)^2 + 1\right) d\theta\end{aligned}$$

$$\begin{aligned}
&= 3\int_{\pi/4}^{\tan^{-1}(2)} \left(\sec(\theta)\tan(\theta) + \csc(\theta)\right) d\theta \\
&= 3\left[\sec(\theta) + \ln|\csc(\theta) - \cot(\theta)|\right]_{\pi/4}^{\tan^{-1}(2)} \\
&= \boxed{3\left(\sqrt{5} - \sqrt{2} + \ln\left(\frac{\sqrt{5}-1}{2\left(\sqrt{2}-1\right)}\right)\right)}.
\end{aligned}$$

(b) Calculate the area of the surface of revolution obtained by revolving the curve about the y-axis.

Solution. We have

$$\begin{aligned}
A &= \int_{3/2}^{3} 2\pi x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= 2\pi\int_{3/2}^{3} 2t\frac{\sqrt{4t^2+9}}{t} dt \\
&= 4\pi\int_{3/2}^{3} \sqrt{4t^2+9} dt.
\end{aligned}$$

We use the same trigonometric substitution $t = \frac{3\tan(\theta)}{2}$, which gives

$$\begin{aligned}
A &= 4\pi\int_{\pi/4}^{\tan^{-1}(2)} \sqrt{9\sec^2(\theta)}\frac{3}{2}\sec^2(\theta)d\theta \\
&= 18\pi\int_{\pi/4}^{\tan^{-1}(2)} \sec(\theta)^3 d\theta.
\end{aligned}$$

Remember that we can compute an antiderivative of $\sec(\theta)^3$ using an integration by parts with $u = \sec(\theta)$ and $dv = \sec(\theta)^2 d\theta$, a trigonometric identity and collecting terms, as follows:

$$\begin{aligned}
\int \sec(\theta)^3 d\theta &= \sec(\theta)\tan(\theta) - \int \sec(\theta)\tan(\theta)^2 d\theta \\
&= \sec(\theta)\tan(\theta) - \int \sec(\theta)\left(\sec(\theta)^2 - 1\right) d\theta \\
&= \sec(\theta)\tan(\theta) - \int \sec(\theta)^3 d\theta + \int \sec(\theta) d\theta \\
&= \sec(\theta)\tan(\theta) - \int \sec(\theta)^3 d\theta + \ln|\sec(\theta) + \tan(\theta)| \\
\Rightarrow\ 2\int \sec(\theta)^3 d\theta &= \sec(\theta)\tan(\theta) + \ln|\sec(\theta) + \tan(\theta)| \\
\Rightarrow\ \int \sec(\theta)^3 d\theta &= \frac{1}{2}\left(\sec(\theta)\tan(\theta) + \ln|\sec(\theta) + \tan(\theta)|\right) + C.
\end{aligned}$$

Using this for the surface area gives

$$A = 18\pi\left[\frac{1}{2}\left(\sec(\theta)\tan(\theta) + \ln|\sec(\theta) + \tan(\theta)|\right)\right]_{\pi/4}^{\tan^{-1}(2)}$$

$$= \boxed{9\pi\left(2\sqrt{5} - \sqrt{2} + \ln\left(\frac{2+\sqrt{5}}{1+\sqrt{2}}\right)\right)}.$$

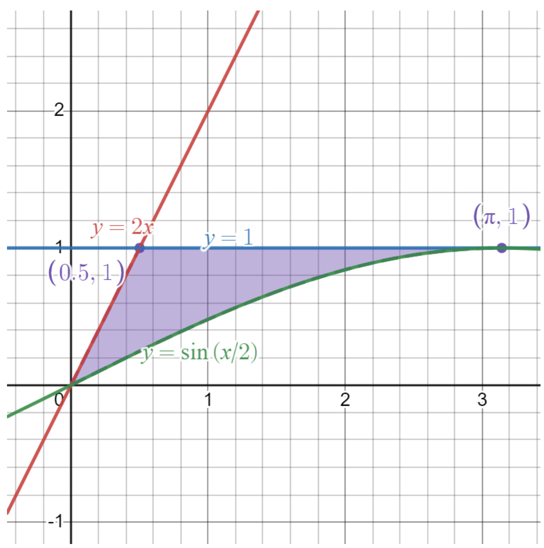
(c) Set-up (but do not evaluate) an integral that computes the area of the surface of revolution obtained by revolving the curve about the x-axis. *Solution.*

$$A = \int_{3/2}^{3} 2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$
$$= \boxed{\int_{3/2}^{3} 2\pi(3\ln(t)+2)\frac{\sqrt{4t^2+9}}{t}\,dt}$$

10. Consider the region $\mathcal{R}$ in the xy-plane bounded by the lines $y = 2x$, $y = 1$ and the graph $y = \sin\left(\frac{x}{2}\right)$.

(a) Sketch the region $\mathcal{R}$.

Solution.



(b) Calculate the area of the region using (i) an x-integral, and (i) a y-integral.

Solution. (i) The area is given by

$$A = \int_0^{1/2} \left(2x - \sin\left(\frac{x}{2}\right)\right) dx + \int_{1/2}^{\pi} \left(1 - \sin\left(\frac{x}{2}\right)\right) dx$$
$$= \left[x^2 + 2\cos\left(\frac{x}{2}\right)\right]_0^{1/2} + \left[x + 2\cos\left(\frac{x}{2}\right)\right]_{1/2}^{\pi}$$
$$= \frac{1}{4} + 2\cos\left(\frac{1}{4}\right) - 2 + \pi - \frac{1}{2} - 2\cos\left(\frac{1}{4}\right)$$

$$= \boxed{\pi - \frac{9}{4}}$$

(ii) To use a y-integral, observe that the region can be described as $\frac{y}{2} \leqslant x \leqslant 2\sin^{-1}(y)$ for $0 \leqslant y \leqslant 1$. Therefore we obtain

$$\begin{aligned} A &= \int_0^1 \left(2\sin^{-1}(y) - \frac{y}{2}\right) dy \\ &= 2\int_0^1 \sin^{-1}(y)dy - \left[\frac{y^2}{4}\right]_0^1 \\ &= 2\int_0^1 \sin^{-1}(y)dy - \frac{1}{4}. \end{aligned}$$

The remaining integral can be computed using IBP with $u = \sin^{-1}(y)$ and $dv = dy$, so $du = \frac{dy}{\sqrt{1-y^2}}$ and $v = y$. This gives

$$\begin{aligned} \int_0^1 \sin^{-1}(y)dy &= \left[y\sin^{-1}(y)\right]_0^1 - \int_0^1 \frac{y}{\sqrt{1-y^2}}dy \\ &= \frac{\pi}{2} - \int_1^0 -\,dw \qquad \left(w = \sqrt{1-y^2}, dw = -\frac{y}{\sqrt{1-y^2}}dt\right) \\ &= \frac{\pi}{2} - 1. \end{aligned}$$

Therefore

$$A = 2\left(\frac{\pi}{2} - 1\right) - \frac{1}{4} = \boxed{\pi - \frac{9}{4}}.$$

(c) A solid is obtained by revolving the region $\mathcal{R}$ about the line $y = 2$. Set-up integrals that calculate the volume of the solid using (i) the disk/washer method, and (ii) the shell method. Then evaluate one of the integrals to find the volume of the solid.

Solution. (i) Washers:

$$\boxed{V = \int_0^{1/2} \pi\left(\left(2 - \sin\left(\frac{x}{2}\right)\right)^2 - (2-2x)^2\right) dx + \int_{1/2}^{\pi} \pi\left(\left(2 - \sin\left(\frac{x}{2}\right)\right)^2 - (2-1)^2\right) dx}$$

(ii) Shells:

$$\boxed{V = \int_0^1 2\pi(2-y)\left(2\sin^{-1}(y) - \frac{y}{2}\right) dy}.$$

Let us evaluate the integral from the washer method:

$$\begin{aligned} V &= \int_0^{\pi} \pi\left(2 - \sin\left(\frac{x}{2}\right)\right)^2 dx - \int_0^{1/2} \pi(2-2x)^2 dx - \int_{1/2}^{\pi} \pi dx \\ &= \pi\left(\int_0^{\pi}\left(4 - 4\sin\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right)\right) dx - 4\int_0^{1/2}(1-x)^2 dx - \left(\pi - \frac{1}{2}\right)\right) \end{aligned}$$

$$
\begin{aligned}
&= \pi \left(\int_0^{\pi} \left(4 - 4\sin\left(\frac{x}{2}\right) + \frac{1-\cos(x)}{2} \right) dx - 4\left[-\frac{(1-x)^3}{3} \right]_0^{1/2} - \left(\pi - \frac{1}{2} \right) \right) \\
&= \pi \left(\left[4x + 8\cos\left(\frac{x}{2}\right) + \frac{x - \sin(x)}{2} \right]_0^{\pi} - \frac{4}{3}\left(-\frac{1}{8} + 1 \right) - \pi + \frac{1}{2} \right) \\
&= \pi \left(4\pi + \frac{\pi}{2} - 8 - \frac{7}{6} - \pi + \frac{1}{2} \right) \\
&= \boxed{\pi \left(\frac{7\pi}{2} - \frac{26}{3} \right)}.
\end{aligned}
$$