Rutgers University
FA23 Math 152

Midterm Exam 3 Practice Problems Solutions

1. Determine if the sequences below converge or diverge, and find their limit in case of convergence.
Basic:

(a) $\left\{\dfrac{e^{n^2}}{n^3}\right\}$

Solution. The limit of this sequence is an indeterminate form $\frac{\infty}{\infty}$, so we can use L'Hôpital's Rule (twice) to find

$$\begin{aligned}\lim_{n\to\infty}\frac{e^{n^2}}{n^3} &= \lim_{x\to\infty}\frac{e^{x^2}}{x^3}\\ &\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty}\frac{2xe^{x^2}}{3x^2}\\ &= \lim_{x\to\infty}\frac{2e^{x^2}}{3x}\\ &\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty}\frac{4xe^{x^2}}{3}\\ &= \infty.\end{aligned}$$

Therefore, the sequence $\boxed{\left\{\dfrac{e^{n^2}}{n^3}\right\} \text{ diverges}}$.

(b) $\left\{n\ln\left(1-\dfrac{2}{5n}\right)\right\}$

Solution. The limit of this sequence is an indeterminate form $0\times\infty$. We can rewrite the expression as a fraction and then apply L'Hôpital's Rule. This gives

$$\begin{aligned}\lim_{n\to\infty} n\ln\left(1-\frac{2}{5n}\right) &= \lim_{x\to\infty}\frac{\ln\left(1-\frac{2}{5x}\right)}{\frac{1}{x}}\\ &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \frac{\frac{1}{1-2/5x}\cdot\frac{2}{5x^2}}{-\frac{1}{x^2}}\\ &= \lim_{x\to\infty} -\frac{1}{1-2/5x}\cdot\frac{2}{5}\\ &= -\frac{2}{5}\end{aligned}$$

So the sequence $\boxed{\left\{n\ln\left(1-\dfrac{2}{5n}\right)\right\} \text{ converges to the limit } -\dfrac{2}{5}}$.

Advanced:

(c) $\left\{\dfrac{2^n + \arctan(n)}{2^{n+5}}\right\}$

Solution. Since $-\frac{\pi}{2} \leqslant \arctan(n) \leqslant \frac{\pi}{2}$, we have

$$\frac{2^n - \frac{\pi}{2}}{2^{n+5}} \leqslant \frac{2^n + \arctan(n)}{2^{n+5}} \leqslant \frac{2^n + \frac{\pi}{2}}{2^{n+5}}.$$

Also, we have

$$\lim_{n\to\infty} \frac{2^n - \frac{\pi}{2}}{2^{n+5}} \cdot \frac{2^{-n}}{2^{-n}} = \lim_{n\to\infty} \frac{1 - \frac{\pi}{2}2^{-n}}{2^5} = \frac{1-0}{2^5} = \frac{1}{32},$$
$$\lim_{n\to\infty} \frac{2^n + \frac{\pi}{2}}{2^{n+5}} \cdot \frac{2^{-n}}{2^{-n}} = \lim_{n\to\infty} \frac{1 + \frac{\pi}{2}2^{-n}}{2^5} = \frac{1+0}{2^5} = \frac{1}{32}.$$

So by the Squeeze Theorem, the sequence $\boxed{\left\{\dfrac{2^n + \arctan(n)}{2^{n+5}}\right\} \text{ converges to the limit } \dfrac{1}{32}}$.

(d) $\left\{\cos(e^{-n})^{e^{2n}}\right\}$

Solution. This is an indeterminate power 1^∞. We can write the limit using logs as

$$\lim_{n\to\infty} \cos(e^{-n})^{e^{2n}} = \lim_{n\to\infty} e^{e^{2n}\ln(\cos(e^{-n}))}.$$

For the exponent, we have

$$\begin{aligned}
\lim_{n\to\infty} e^{2n}\ln(\cos(e^{-n})) &= \lim_{x\to\infty} \frac{\ln(\cos(e^{-x}))}{e^{-2x}} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty} \frac{-\tan(e^{-x})(-e^{-x})}{-2e^{-2x}} \\
&= \lim_{x\to\infty} -\frac{\tan(e^{-x})}{2e^{-x}} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty} -\frac{\sec^2(e^{-x})(-e^{-x})}{-e^{-x}} \\
&= \lim_{x\to\infty} -\frac{\sec^2(e^{-x})}{2} \\
&= -\frac{\sec^2(0)}{2} \\
&= -\frac{1}{2}.
\end{aligned}$$

Therefore, the original limit turns out to be

$$\lim_{n\to\infty} \cos(e^{-n})^{e^{2n}} = \lim_{n\to\infty} e^{e^{2n}\ln(\cos(e^{-n}))} = e^{-1/2}.$$

So the sequence $\boxed{\left\{\cos(e^{-n})^{e^{2n}}\right\} \text{ converges to the limit } e^{-1/2}}$.

2. Determine if the series below converge absolutely, converge conditionally or diverge. Name any test used and show all work to justify its use. In case of convergence, evaluate the sum when possible.
Basic:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^n + 1}{3^{2n+1}}$

Solution. We can write this series as the sum of two convergent geometric series:

$$\sum_{n=1}^{\infty} \frac{(-5)^n + 1}{3^{2n+1}} = \sum_{n=1}^{\infty} \frac{1}{3}\left(-\frac{5}{9}\right)^n + \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{9}\right)^n.$$

The first series is geometric with common ratio $r = -\frac{5}{9}$ so it converges absolutely since $|r| < 1$. The second series is geometric with common ratio $r = \frac{1}{9}$ so it converges absolutely since $|r| < 1$. Therefore, $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^n + 1}{3^{2n+1}} \text{ converges absolutely}}$. Its sum evaluates to

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-5)^n + 1}{3^{2n+1}} &= \sum_{n=1}^{\infty} \frac{1}{3}\left(-\frac{5}{9}\right)^n + \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{9}\right)^n \\ &= \frac{-\frac{5}{27}}{1 + \frac{5}{9}} + \frac{\frac{1}{27}}{1 - \frac{1}{9}} \\ &= \boxed{-\frac{13}{168}}.\end{aligned}$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$

Solution. We use the Integral Test with the function $f(x) = \dfrac{e^{-\sqrt{x}}}{\sqrt{x}}$.

- $f(x) = \dfrac{e^{-\sqrt{x}}}{\sqrt{x}} > 0$ on $[1, \infty)$.
- $f(x) = \dfrac{e^{-\sqrt{x}}}{\sqrt{x}}$ is continuous on $[1, \infty)$.
- Writing the function as $f(x) = \dfrac{1}{e^{\sqrt{x}}\sqrt{x}}$, we also see that the function is decreasing as the reciprocal of a positive increasing function.

So the Integral Test applies and we can determine whether the series converges or diverges by computing the corresponding improper integral. We have

$$\begin{aligned}\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx &= \lim_{b\to\infty} \int_1^{b} \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx \\ &= \lim_{b\to\infty} \int_1^{\sqrt{b}} 2e^{-u} du \qquad \left(u = \sqrt{x}, du = \frac{dx}{2\sqrt{x}}\right) \\ &= \lim_{b\to\infty} \left[-2e^{-u}\right]_1^{\sqrt{b}} \\ &= \lim_{b\to\infty} \left(2e^{-1} - 2e^{-\sqrt{b}}\right) \\ &= 2e^{-1}.\end{aligned}$$

Therefore, the improper integral $\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$ converges. It follows that $\boxed{\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}} \text{ converges (absolutely)}}$.

(c) $\sum_{n=1}^{\infty} \left(5 + \frac{3}{n}\right)^2$

Solution. Since

$$\lim_{n\to\infty} \left(5 + \frac{3}{n}\right)^2 = (5+0)^2 = 25 \neq 0,$$

the Term Divergence Test guarantees that $\boxed{\sum_{n=1}^{\infty} \left(5 + \frac{3}{n}\right)^2 \text{ diverges}}$.

(d) $\sum_{n=0}^{\infty} \frac{3^n + 2}{7\sqrt{25^n + 1}}$

Solution. We will use the LCT with the reference series $\sum_{n=0}^{\infty} \frac{3^n}{\sqrt{25^n}} = \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n$, which converges as a geometric series with common ratio $r = \frac{3}{5}$, $|r| < 1$. We have

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{a_n}{b_n} \\ &= \lim_{n\to\infty} \frac{\frac{3^n+2}{7\sqrt{25^n+1}}}{\frac{3^n}{\sqrt{25^n}}} \\ &= \lim_{n\to\infty} \frac{1 + \frac{2}{3^n}}{7\sqrt{1 + \frac{1}{25^n}}} \\ &= \frac{1}{7}. \end{aligned}$$

Since $0 < L < \infty$, both series have the same behavior. So $\boxed{\sum_{n=0}^{\infty} \frac{3^n + 2}{7\sqrt{25^n + 1}} \text{ converges (absolutely)}}$.

(e) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\pi}}$

Solution. We test for absolute convergence. Observe that

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^{\pi}}\right| = \sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$$

which converges as a p-series with $p = \pi > 1$. Therefore, $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\pi}} \text{ converges absolutely}}$.

Remark: the AST applies here, but only gives convergence (does not say if the converge is absolute or conditional).

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+5\sqrt{n}}$

Solution. This series is alternating. We apply the AST with the sequence $a_n = \dfrac{1}{2n+5\sqrt{n}}$. We have

- $\dfrac{1}{2n+5\sqrt{n}} \geqslant 0$ for all $n > 0$.
- $\left\{\dfrac{1}{2n+5\sqrt{n}}\right\}$ is decreasing as the reciprocal of a positive increasing sequence.
- $\displaystyle\lim_{n\to\infty} \frac{1}{2n+5\sqrt{n}} = 0$.

So the AST applies and we deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+5\sqrt{n}}$ converges.

To determine if the convergence is absolute or conditional, observe that

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{2n+5\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{2n+5\sqrt{n}}.$$

For this series, we use the LCT with the reference series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges as a p-series with $p = 1$. We have

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{a_n}{b_n} \\ &= \lim_{n\to\infty} \frac{\frac{1}{2n+5\sqrt{n}}}{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{1}{2+\frac{5}{\sqrt{n}}} \\ &= \frac{1}{2}. \end{aligned}$$

Since $0 < L < \infty$, both series have the same behavior. So $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{2n+5\sqrt{n}}\right|$ diverges.

In conclusion, $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+5\sqrt{n}} \text{ converges conditionally}}$.

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+3)!}{(-5)^n}$

Solution. The presence of factorials strongly suggests using the Ratio Test. We have

$$\begin{aligned} \rho &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty} \left|\frac{(n+4)!}{(-5)^{n+1}} \cdot \frac{(-5)^n}{(n+3)!}\right| \\ &= \lim_{n\to\infty} \frac{n+4}{5} \\ &= \infty. \end{aligned}$$

Since $\rho > 1$, we conclude that $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{(n+3)!}{(-5)^n} \text{ diverges}}$.

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^{n^2}}$

Solution. Using the Root Test, we have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} |a_n|^{1/n} \\
&= \lim_{n\to\infty} \left(\frac{n^n}{3^{n^2}}\right)^{1/n} \\
&= \lim_{n\to\infty} \frac{n}{3^n} \\
&= \lim_{x\to\infty} \frac{x}{3^x} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{1}{\ln(3)3^x} \\
&= 0.
\end{aligned}$$

Since $\rho < 1$, we conclude that $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^{n^2}} \text{ converges absolutely}}$.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{5\cos(n) - 2}{n^2}$

Solution. We test for absolute convergence with the DCT. We have $-1 \leqslant \cos(n) \leqslant 1$, so $-7 \leqslant 5\cos(n) - 2 \leqslant 3$. Therefore, $0 \leqslant |5\cos(n) - 2| \leqslant 7$. So we have

$$0 \leqslant \left|\frac{5\cos(n) - 2}{n^2}\right| \frac{7}{n^2},$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges as a p-series with $p = 2 > 1$. It follows that $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{5\cos(n) - 2}{n^2} \text{ converges absolutely}}$.

Advanced:

(j) $\displaystyle\sum_{n=3}^{\infty} \left(\frac{n+3}{n}\right)^{n^2}$

Solution. We use the Root Test. We have

$$\rho = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\frac{n+3}{n}\right)^n = \lim_{n\to\infty} e^{n\ln\left(1+\frac{3}{n}\right)}.$$

We now compute the limit of the exponent using L'Hôpital's Rule:

$$\begin{aligned}
\lim_{n\to\infty} n\ln\left(1+\frac{3}{n}\right) &= \lim_{x\to\infty} \frac{\ln(1+3/x)}{\frac{1}{x}} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \frac{-\frac{3}{x^2}\cdot\frac{1}{1+3/x}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{3}{1+3/x} \\
&= 3.
\end{aligned}$$

So

$$\rho = \lim_{n\to\infty} e^{n\ln\left(1+\frac{3}{n}\right)} = e^3.$$

Since $\rho > 1$, we conclude that $\boxed{\sum_{n=3}^{\infty}\left(\frac{n+3}{n}\right)^{n^2} \text{ diverges}}$.

(k) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3\cos(n)}{n^4+1}$

Solution. We use the LCT with the reference series $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{n^4} = \sum_{n=1}^{\infty}\frac{1}{n^2}$, which converges as a p-series with $p = 2 > 1$. We have

$$\begin{aligned}
L &= \lim_{n\to\infty}\frac{a_n}{b_n} \\
&= \lim_{n\to\infty}\frac{\frac{2n^2+3\cos(n)}{n^4+1}}{\frac{n^2}{n^4}} \\
&= \lim_{n\to\infty}\frac{2+\frac{3\cos(n)}{n^2}}{1+\frac{1}{n^4}}.
\end{aligned}$$

To finish computing this limit, we must use the Squeeze Theorem. Since $-1 \leqslant \cos(n) \leqslant 1$, we have

$$-\frac{3}{n^2} \leqslant \frac{3\cos(n)}{n^2} \leqslant \frac{3}{n^2}.$$

Since $\displaystyle\lim_{n\to\infty} -\frac{3}{n^2} = \lim_{n\to\infty}\frac{3}{n^2} = 0$, we conclude that $\displaystyle\lim_{n\to\infty}\frac{3\cos(n)}{n^2} = 0$. Therefore, our limit for the LCT becomes

$$L = \lim_{n\to\infty}\frac{2+\frac{3\cos(n)}{n^2}}{1+\frac{1}{n^4}} = \frac{2+0}{1+0} = 2.$$

Since $0 < L < 0$, the two series have the same behavior. So $\boxed{\sum_{n=1}^{\infty}\frac{2n^2+3\cos(n)}{n^4+1} \text{ converges (absolutely)}}$.

(l) $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln(n)}$

Solution. This series is alternating. We use the AST with $a_n = \dfrac{1}{n\ln(n)}$.

- $\dfrac{1}{n\ln(n)} \geqslant 0$ for all $n > 0$.
- $\left\{\dfrac{1}{n\ln(n)}\right\}$ is decreasing as the reciprocal of a positive increasing sequence.
- $\displaystyle\lim_{n\to\infty} \frac{1}{n\ln(n)} = 0$.

So the AST applies and we deduce that $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)}$ converges.

To determine if the convergence is absolute or conditional, we look at

$$\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{n\ln(n)}\right| = \sum_{n=2}^{\infty} \frac{1}{n\ln(n)}.$$

For this series, we can apply the Integral Test with the function $f(x) = \dfrac{1}{x\ln(x)}$. This function is positive, continuous and decreasing on the interval $[2,\infty)$. Therefore, the Integral Test applies and we can determine whether the series converges or diverges by computing the corresponding improper integral. We have

$$\begin{aligned}\int_2^{\infty} \frac{dx}{x\ln(x)} &= \lim_{b\to\infty} \int_2^b \frac{dx}{x\ln(x)} \\ &= \lim_{b\to\infty} \left[\ln|\ln(x)|\right]_2^b \\ &= \lim_{b\to\infty} \left(\ln|\ln(b)| - \ln|\ln(2)|\right) \\ &= \infty.\end{aligned}$$

Since $\displaystyle\int_2^{\infty} \frac{dx}{x\ln(x)}$ diverges, we conclude that $\displaystyle\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{n\ln(n)}\right|$ diverges.

Therefore, $\boxed{\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)} \text{ converges conditionally}}$.

(m) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2 e^n}{(2n)!}$

Solution. We use the Ration Test. We have

$$\begin{aligned}\rho \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty} \left|\frac{((n+1)!)^2 e^{n+1}}{(2(n+1))!} \cdot \frac{(2n)!}{(n!)^2 e^n}\right| \\ &= \lim_{n\to\infty} \frac{((n+1)!)^2 e^{n+1} (2n)!}{(2n+2)!(n!)^2} \\ &= \lim_{n\to\infty} \frac{(n+1)^2 e}{(2n+1)(2n+2)} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \\ &= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^2 e}{\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)} \\ &= \frac{e}{4}.\end{aligned}$$

Since $\rho < 1$, we conclude that $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2 e^n}{(2n)!} \text{ converges absolutely}}$.

(n) $\displaystyle\sum_{n=3}^{\infty} \left(\tan\left(\frac{\pi}{n}\right) - \tan\left(\frac{\pi}{n+2}\right)\right)$

Solution. Writing out the first few terms of the partial sum $S_N = \displaystyle\sum_{n=3}^{N} \left(\tan\left(\frac{\pi}{n}\right) - \tan\left(\frac{\pi}{n+2}\right)\right)$ reveals a telescoping series. More precisely, observe that

$$\begin{aligned} S_N &= \left(\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{5}\right)\right) + \left(\tan\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{6}\right)\right) + \left(\tan\left(\frac{\pi}{5}\right) - \tan\left(\frac{\pi}{7}\right)\right) + \cdots \\ &\quad \cdots + \left(\tan\left(\frac{\pi}{N-1}\right) - \tan\left(\frac{\pi}{N+1}\right)\right) + \left(\tan\left(\frac{\pi}{N}\right) - \tan\left(\frac{\pi}{N+2}\right)\right) \\ &= \tan\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{N+1}\right) - \tan\left(\frac{\pi}{N+2}\right). \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{n=3}^{\infty} \left(\tan\left(\frac{\pi}{n}\right) - \tan\left(\frac{\pi}{n+2}\right)\right) &= \lim_{N\to\infty} S_N \\ &= \lim_{N\to\infty} \left(\tan\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{N+1}\right) - \tan\left(\frac{\pi}{N+2}\right)\right) \\ &= \tan\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{4}\right) - \tan(0) - \tan(0) \\ &= \boxed{\sqrt{3}+1}. \end{aligned}$$

(o) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)^2}{n}$

Solution. Note that the first term of the series is 0. We use the DCT. Since ln is an increasing function, we have

$$\frac{\ln(n)^2}{n} \geqslant \frac{\ln(2)^2}{n} \geqslant 0$$

for all $n \geqslant 2$. The series $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(2)^2}{n} = \ln(2)^2 \sum_{n=1}^{\infty} \frac{1}{n}$ diverges as a p-series with $p = 1$. Therefore, $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)^2}{n} \text{ diverges}}$.

3. Consider the series $S = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(3n+11)^2}$.

   (a) *(Basic)* Show that this series meets the conditions of the Alternating Series Estimation Theorem.

   Solution. We have

- $\dfrac{1}{(3n+11)^2} \geqslant 0$.
- $\left\{\dfrac{1}{(3n+11)^2}\right\}$ is decreasing since $\dfrac{1}{(3(n+1)+11)^2} < \dfrac{1}{(3n+11)^2}$.
- $\lim\limits_{n\to\infty} \dfrac{1}{(3n+11)^2} = 0$.

Therefore, $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(3n+11)^2} \text{ meets the conditions of the ASET}}$.

(b) *(Advanced)* Find the smallest integer N for which the partial sum $S_N = \displaystyle\sum_{n=1}^{N} \frac{(-1)^n}{(3n+11)^2}$ approximates the sum of the series S with an error of at most 0.0001.

Solution. The ASET tells us that the error is bounded by a_{N+1}. Therefore, we will want

$$
\begin{aligned}
&a_{N+1} < 0.0001\\
&\frac{1}{(3(N+1)+11)^2} < 0.0001\\
&(3N+14)^2 > 10000\\
&3N+14 > 100\\
&3N > 86\\
&N > \frac{86}{3}.
\end{aligned}
$$

Therefore the smallest integer N giving us the desired level of accuracy is $\boxed{N = 29}$.