Section 2.1

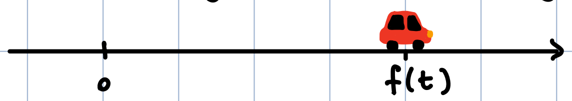
Introduction to Limits

Learning Goals

Learning Goal	*Homework Problems*
2.1.1 Compute the average rate of change of a function over an interval.	1-6.
2.1.2 Use the secant lines of a curve to find the line tangent to that curve at a point.	7-18.
2.1.3 Solve applications involving rates of change.	19-26.

Motivation : average velocity vs instantaneous velocity

Suppose you are driving on a straight road.



$s(t)$ = position at the time t

$v(t)$ = instantaneous velocity

(what you read on speedometer)

How to calculate the instantaneous velocity ?

Let's do it with the example $s(t) = 7t^2$ at $t = 1$.

→ Start with the average velocity on a small interval around $t = 1$.

Average velocity / rate of change on $[t_1, t_2]$:

$$av = \frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

$$= \frac{s(t_1 + h) - s(t_1)}{h} \quad \text{if } t_2 = t_1 + h.$$

For our example $s(t) = 7t^2$

- on $[1, 1.1]$ $(h = 0.1)$: $\frac{\Delta s}{\Delta t} = \frac{7 \cdot (1.1)^2 - 7 \cdot 1^2}{1.1 - 1} = \boxed{14.7}$
- on $[0.9, 1]$ $(h = -0.1)$: $\frac{\Delta s}{\Delta t} = \frac{7 \cdot 1^2 - 7 \cdot (0.9)^2}{1 - 0.9} = \boxed{13.3}$

approximations of $v(1)$

Over smaller and smaller intervals, we get a better estimate of $v(1)$.

→ Then, find what value the average velocity tends to as the length of the interval $\Delta t = h$ approaches 0.

Call $av(h)$ the average velocity between $t=1$ and $t = 1+h$. Explicitly:

$$av(h) = \frac{\Delta s}{\Delta t} = \frac{s(1+h)-s(1)}{h} = \frac{7(1+h)^2-7}{h}.$$

Above, we computed $av(0.1) = 14.7$ and $av(-0.1) = 13.3$. Let's look at a table of values to find the tendency of $av(h)$ as h gets closer to 0.

h	$av(h)$
0.1	14.7
0.01	14.07
0.001	14.007
-0.1	13.3
-0.01	13.93
-0.001	13.993

As h gets closer to 0, $av(h)$ gets closer to 14.

$$\boxed{v(1) = \lim_{h \to 0} av(h) = 14}.$$

How can we make a formal calculation without using a table of values.

↳ We start by simplifying the average rate of change $av(h)$ for $h \neq 0$, and then take the limit as h approaches 0.

$$av(h) = \frac{s(1+h)-s(1)}{h} = \frac{7(1+h)^2-7}{h}$$

cannot take $h = 0$ in this expression: simplify algebraically first

$$= \frac{7(1+2h+h^2)-7}{h} = \frac{14h+7h^2}{h} = 14+7h \quad \text{if} \quad h \neq 0.$$

Now we can calculate the limit as $h \to 0$ to find $v(1)$:

$$v(1) = \lim_{h\to 0} \frac{s(1+h)-s(1)}{h} = \lim_{h\to 0} (14+7h) = 14+7\cdot 0 = \boxed{14}$$

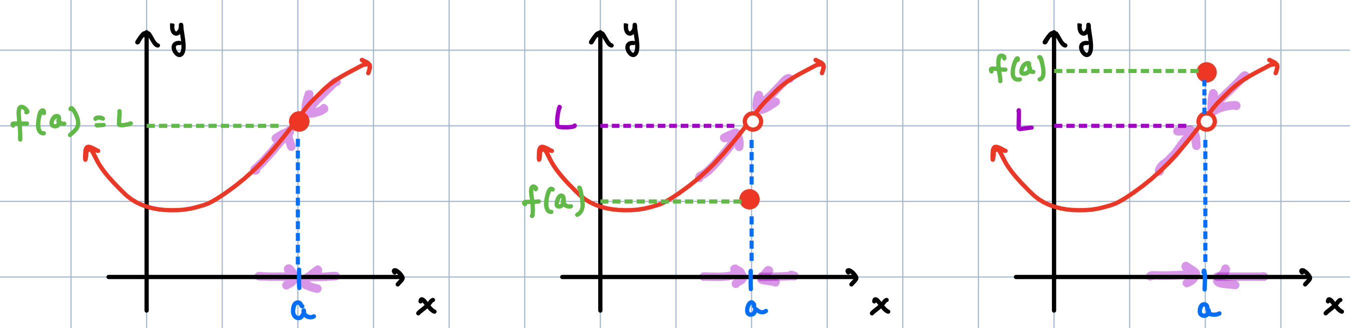
Formal definition of limits:

$\boxed{\lim_{x\to a} f(x) = L}$ means that $f(x)$ gets arbitrarily close to L when x gets close to a (but $x \neq a$).

↳ "the limit of $f(x)$ as x goes to a is L"

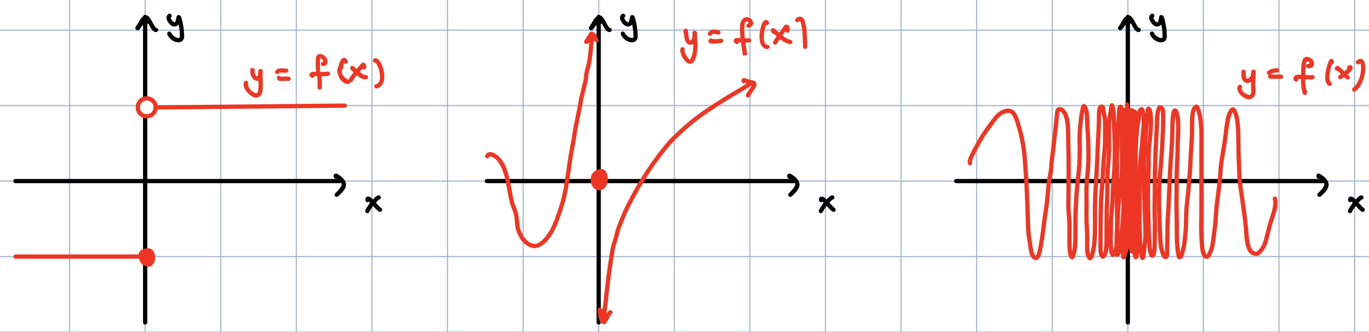
Important remarks:

- $\lim_{x\to a} f(x)$ does not depend on the value of f at a, only on the tendency of $f(x)$ as x gets close to a.



In all 3 cases, $\lim_{x\to a} f(x) = L$ even though $f(a)$ is different each time. In fact, $f(a)$ does not even need to be defined for $\lim_{x\to a} f(x)$ to exist!

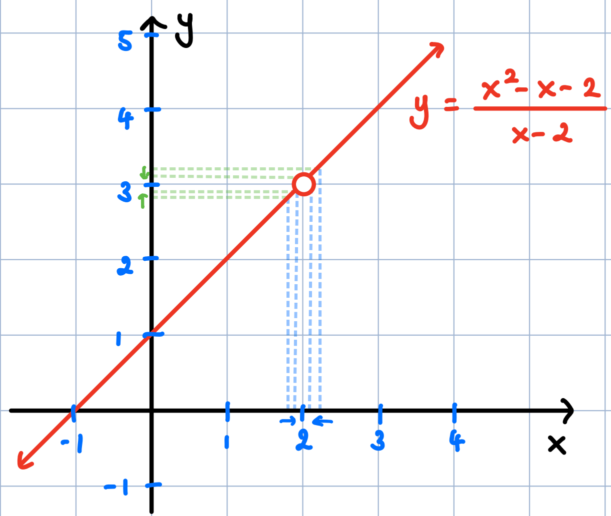
- Limits may or may not exist.



Jump　　　Vertical Asymptote　　　Infinite oscillation

In all 3 cases, $\lim\limits_{x \to 0} f(x)$ Does Not Exist (DNE).

Examples: 1) Calculate $\lim\limits_{x \to 2} \dfrac{x^2 - x - 2}{x - 2}$.



$$\frac{x^2 - x - 2}{x - 2} = \frac{(x+1)(x-2)}{x-2} = \begin{cases} x+1 & \text{if } x \neq 2 \\ \text{undef} & \text{if } x = 2 \end{cases}$$

When $x \to 2$, $y \to 3$ so

$$\boxed{\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = 3}$$

Formal computation:

$$\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2}$$

cannot take $x = 2$, simplify first.

$$= \lim_{\substack{x \to 2 \\ x \neq 2}} \frac{(x+1)\cancel{(x-2)}}{\cancel{x-2}}$$

$$= \lim_{x \to 2} x + 1$$

$$= \boxed{3.}$$

2) Estimate $\lim_{x \to 8} \frac{x-8}{\sqrt[3]{x}-2}$ using a) a table of values,
b) a graph.

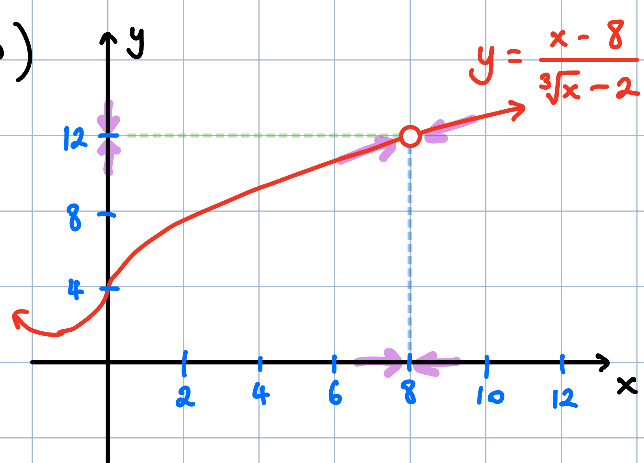
a)

x	$\frac{x-8}{\sqrt[3]{x}-2}$
8.1	12.0498...
7.9	11.9498...
7.999	11.9994...
8.001	12.0004
7.9999	11.99995...
8.00001	12.000005...

As x gets closer to 8, $\frac{x-8}{\sqrt[3]{x}-2}$ gets closer to 12.

$$\Rightarrow \boxed{\lim_{x \to 8} \frac{x-8}{\sqrt[3]{x}-2} = 12}$$

b)



$$\boxed{\lim_{x \to 8} \frac{x-8}{\sqrt[3]{x}-2} = 12}$$

3) Suppose that the position of an object moving on an axis is given by $s(t) = 5 - \frac{8}{t}$. Find :
a) the average velocity on $[1,2]$.
b) the instantaneous velocity at $t=2$, $v(2)$.

a) $av = \frac{\Delta s}{\Delta t} = \frac{s(2)-s(1)}{2-1}$

$s(2) = 5 - \frac{8}{2} = 1$
$s(1) = 5 - \frac{8}{1} = -3$

$$= \frac{1-(-3)}{2-1} = \boxed{4}.$$

b) To find $v(2)$ we first find and simplify the average velocity $av(h)$ between $t=2$ and $t=2+h$, and then we take the limit as $h \to 0$.

$$av(h) = \frac{\Delta s}{\Delta t} = \frac{s(2+h) - s(2)}{h} = \frac{\left(5 - \frac{8}{2+h}\right) - 1}{h}$$

$$= \frac{4 - \frac{8}{2+h}}{h} = \frac{\frac{4(2+h) - 8}{2+h}}{h} = \frac{8 + 4h - 8}{h(2+h)} = \frac{4\cancel{h}}{\cancel{h}(2+h)}$$

$$= \frac{4}{2+h} \quad \text{if } h \neq 0.$$

So $v(2) = \lim_{h \to 0} av(h) = \lim_{h \to 0} \frac{4}{2+h} = \frac{4}{2+0} = \boxed{2}$.

4) Find the instantaneous rate of change of $f(x) = \sqrt{x-3}$ at $x = 7$.

First, we find and simplify the average rate of change $arc(h)$ between $x=7$ and $x=7+h$.

$$arc(h) = \frac{\Delta f}{\Delta x} = \frac{f(7+h) - f(7)}{h} = \frac{\sqrt{7+h-3} - \sqrt{7-3}}{h}$$

cannot take $h=0$, simplify first.

$$= \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}$$

multiply by conjugate

$$= \frac{(\sqrt{4+h})^2 - 2^2}{h(\sqrt{4+h}+2)} = \frac{4+h-4}{h(\sqrt{4+h}+2)} = \frac{\cancel{h}}{\cancel{h}(\sqrt{4+h}+2)} = \frac{1}{\sqrt{4+h}+2}$$

Then, we take the limit as $h \to 0$ to find the instantaneous rate of change at $x = 7$:

$$\lim_{h\to 0} \text{arc}(h) = \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2} = \frac{1}{\sqrt{4+0}+2} = \boxed{\frac{1}{4}}.$$

5) Find the instantaneous rate of change of $f(x) = \frac{6}{3x-4}$ at $x = 2$.

First, we find and simplify the average rate of change $\text{arc}(h)$ between $x=2$ and $x=2+h$.

$$\text{arc}(h) = \frac{\Delta f}{\Delta x} = \frac{f(2+h)-f(2)}{h} = \frac{\frac{6}{3(2+h)-4} - 3}{h} = \frac{\frac{6}{2+3h} - 3}{h}$$

$$= \frac{6-3(2+3h)}{h(2+3h)} = \frac{6-6-9h}{h(2+3h)} = \frac{-9\cancel{h}}{\cancel{h}(2+3h)} = \frac{-9}{2+3h}$$

Then, we take the limit as $h \to 0$ to find the instantaneous rate of change at $x = 2$:

$$\lim_{h\to 0} \text{arc}(h) = \lim_{h\to 0} \frac{-9}{2+3h} = \frac{-9}{2+3\cdot 0} = \boxed{-\frac{9}{2}}.$$