Rutgers University
Math 151

Section 2.1: Introduction to Limits - Worksheet Solutions

1. Calculate the average rate of change of the following functions on the given intervals.

   (a) $f(x) = 2\ln(5x+1)$ on the interval $[0,3]$.

   Solution.

$$\begin{aligned}\frac{\Delta f}{\Delta x} &= \frac{f(3)-f(0)}{3-0}\\ &= \frac{2\ln(16)-2\ln(1)}{3}\\ &= \boxed{\frac{8\ln(2)}{3}}.\end{aligned}$$

   (b) $f(x) = \sin(4x)$ on the interval $\left[\frac{\pi}{24}, \frac{\pi}{12}\right]$.

   Solution.

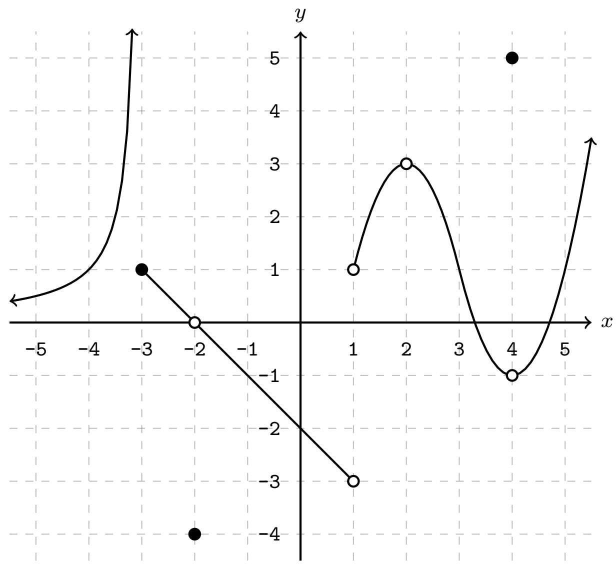
$$\begin{aligned}\frac{\Delta f}{\Delta x} &= \frac{f\left(\frac{\pi}{12}\right)-f\left(\frac{\pi}{24}\right)}{\frac{\pi}{12}-\frac{\pi}{24}}\\ &= \frac{\sin\left(\frac{\pi}{3}\right)-\sin\left(\frac{\pi}{6}\right)}{\frac{\pi}{24}}\\ &= \frac{\frac{\sqrt{3}}{2}-\frac{1}{2}}{\frac{\pi}{24}}\\ &= \boxed{\frac{12\left(\sqrt{3}-1\right)}{\pi}}.\end{aligned}$$

   (c) $f(x) = \arctan(3x)$ on the interval $\left[-\frac{1}{3}, \frac{1}{3}\right]$.

   Solution.

$$\begin{aligned}\frac{\Delta f}{\Delta x} &= \frac{f\left(\frac{1}{3}\right)-f\left(-\frac{1}{3}\right)}{\frac{1}{3}-\left(-\frac{1}{3}\right)}\\ &= \frac{\arctan(1)-\arctan(-1)}{\frac{2}{3}}\\ &= \frac{\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)}{\frac{2}{3}}\\ &= \boxed{\frac{3\pi}{4}}.\end{aligned}$$

2. The graph of the function $y = f(x)$ is given below.



Evaluate $f(a)$ and $\lim_{x \to a} f(x)$ for the following values of a, or say if the quantity does not exist.

(a) $a = -3$

Solution. $\boxed{f(-3) = 1}$ and $\boxed{\lim_{x \to -3} f(x) \text{ does not exist}}$.

(b) $a = -2$

Solution. $\boxed{f(-2) = -4}$ and $\boxed{\lim_{x \to -2} f(x) = 0}$.

(c) $a = 1$

Solution. $\boxed{f(1) \text{ is undefined}}$ and $\boxed{\lim_{x \to 1} f(x) \text{ does not exist}}$.

(d) $a = 2$

Solution. $\boxed{f(2) \text{ is undefined}}$ and $\boxed{\lim_{x \to 2} f(x) = 3}$.

(e) $a = 4$

Solution. $\boxed{f(4) = 5}$ and $\boxed{\lim_{x \to 5} f(x) = -1}$.

3. The following table of values are given for the functions $f(x)$ and $g(x)$. Use these to estimate $\lim_{x \to 3} f(x)$ and $\lim_{x \to 3} g(x)$ or say if a limit does not exist.

x	2.9	3.01	2.999	3.0001	2.99999
$f(x)$	4.15	3.95	4.05	3.9993	4.0005
$g(x)$	7.98	1.001	7.997	1.0002	7.99992

Solution. $\boxed{\lim_{x\to 3} f(x) = 4}$ and $\boxed{\lim_{x\to 3} g(x) \text{ does not exist}}$.

4. Using a limit of average rates of change, find the instantaneous rate of change of the following functions at the given value of x.

   (a) $f(x) = x^2 - 3x + 7$ at $x = 0$.

   Solution.

$$\begin{aligned}\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} &= \lim_{h\to 0} \frac{h^2 - 3h + 7 - 7}{h} \\ &= \lim_{h\to 0} \frac{h^2 - 3h}{h} \\ &= \lim_{h\to 0} h - 3 \\ &= \boxed{-3}.\end{aligned}$$

   (b) $f(x) = \dfrac{x}{5-x}$ at $x = -1$.

   Solution.

$$\begin{aligned}\lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} &= \lim_{h\to 0} \frac{\frac{-1+h}{5-(-1+h)} - \left(-\frac{1}{6}\right)}{h} \\ &= \lim_{h\to 0} \frac{\frac{-1+h}{6-h} + \frac{1}{6}}{h} \\ &= \lim_{h\to 0} \frac{6(-1+h) + 6 - h}{6h(6-h)} \\ &= \lim_{h\to 0} \frac{5h}{6h(6-h)} \\ &= \lim_{h\to 0} \frac{5}{6(6-h)} \\ &= \boxed{\frac{5}{36}}.\end{aligned}$$

   (c) **[Advanced]** $f(x) = \dfrac{1}{\sqrt{2x+1}}$ at $x = 4$.

   Solution.

$$\begin{aligned}\lim_{h\to 0} \frac{f(4+h) - f(4)}{h} &= \lim_{h\to 0} \frac{\frac{1}{\sqrt{9+2h}} - \frac{1}{3}}{h} \\ &= \lim_{h\to 0} \frac{3 - \sqrt{9+2h}}{3h\sqrt{9+2h}} \cdot \frac{3 + \sqrt{9+2h}}{3 + \sqrt{9+2h}}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{9-(9+2h)}{3h\sqrt{9+2h}\left(3+\sqrt{9+2h}\right)} \\
&= \lim_{h\to 0} \frac{-2h}{3h\sqrt{9+2h}\left(3+\sqrt{9+2h}\right)} \\
&= \lim_{h\to 0} \frac{-2}{3\sqrt{9+2h}\left(3+\sqrt{9+2h}\right)} \\
&= \frac{-2}{3\sqrt{9+0}\left(3+\sqrt{9+0}\right)} \\
&= \boxed{-\frac{1}{27}}.
\end{aligned}
$$

5. The position of an object moving along an axis is given by the function $s(t) = 6\sqrt{x+1}$.

   (a) Find the average velocity of the object between $t = 0$ and $t = 15$.

   Solution.

$$
\begin{aligned}
\frac{\Delta s}{\Delta t} &= \frac{s(15)-s(0)}{15-0} \\
&= \frac{6\sqrt{15+1}-6\sqrt{0+1}}{15} \\
&= \frac{24-6}{15} \\
&= \frac{18}{15} \\
&= \boxed{\frac{6}{5}}.
\end{aligned}
$$

   (b) Find the position and instantaneous velocity of the object at $t = 3$.

   Solution. At $t = 3$, the position is

$$
s(3) = 6\sqrt{3+1} = \boxed{12}.
$$

   The velocity is

$$
\begin{aligned}
v(3) &= \lim_{h\to 0} \frac{s(3+h)-s(3)}{h} \\
&= \lim_{h\to 0} \frac{6\sqrt{4+h}-12}{h} \\
&= \lim_{h\to 0} 6\frac{\sqrt{4+h}-2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} \\
&= \lim_{h\to 0} 6\frac{4+h-4}{h\left(\sqrt{4+h}+2\right)} \\
&= \lim_{h\to 0} 6\frac{h}{h\left(\sqrt{4+h}+2\right)} \\
&= \lim_{h\to 0} 6\frac{1}{\sqrt{4+h}+2}
\end{aligned}
$$

$$
\begin{aligned}
&= 6\frac{1}{\sqrt{4+0}+2} \\
&= \boxed{\frac{3}{2}}.
\end{aligned}
$$