Section 2.2

Calculating Limits

Learning Goals

Learning Goal	*Homework Problems*
2.2.1 Find the limits of a function and evaluate the function using a graph of the function.	1-4, 83, 84.
2.2.2 Understand and be able to explain concepts related to the existence of limits.	5-10, 78-82, 85-90.
2.2.3 Compute the limits of polynomial and rational functions. Find the limits of difference quotients in preparation for computing derivatives.	5, 6, 11-42, 67-70.
2.2.4 Compute basic limits of trigonometric functions.	43-50.
2.2.5 Find limits using the rules of limits.	51-56, 79-82.
2.2.6 Evaluate the limit of average rates of change.	57-62.
2.2.7 Solve applications of the Sandwich Theorem.	63-66, 78.
2.2.8 Estimate limits using tables.	67-76.

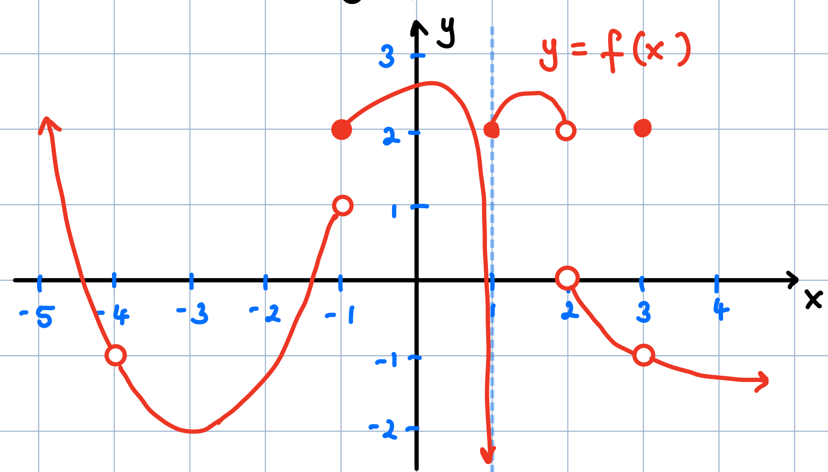
Recall :

$$\boxed{\lim_{x \to a} f(x) = L}$$ means that $f(x)$ gets arbitrarily close to L when x gets close to a.

In this section, we are going to learn techniques to compute limits.

i) **Graphically:** $\lim_{x \to a} f(x) = L$ means that the y-values on the graph $y = f(x)$ approach L as the x-values approach a, $x \neq a$.

Example : use the graph to evaluate the following.



$\lim_{x \to -4} f(x) = -1$, $f(-4)$ DNE.	$\lim_{x \to 1} f(x)$ DNE, $f(1) = 2$
$\lim_{x \to -3} f(x) = -2$, $f(-3) = -2$.	$\lim_{x \to 2} f(x)$ DNE, $f(2)$ DNE
$\lim_{x \to -1} f(x)$ DNE, $f(-1) = 2$	$\lim_{x \to 3} f(x) = -1$, $f(3) = 2$.

ii) Substitution: for "common functions" on their domain

$$\lim_{x \to a} f(x) = f(a).$$

Common functions: rational (e.g. $\frac{2x^2+4}{x^3-8}$)
algebraic (e.g. $\sqrt[3]{\frac{1}{x}+2x}$)
exponential (e.g $7e^{4x}$)
logarithmic (e.g. $\log_2(3-5x)$)
trigonometric (cos, sin, etc.)
inverse trigonometric (arcsin, arctan etc.)

and their combinations (algebraic and compositions).

Examples: 1) $\lim\limits_{x \to 1} \frac{\ln(1+xe^{x^2-1})}{2+\sin(\pi x)} = \frac{\ln(1+1e^{1^2-1})}{2+\sin(\pi)} = \boxed{\frac{\ln(2)}{2}}$.

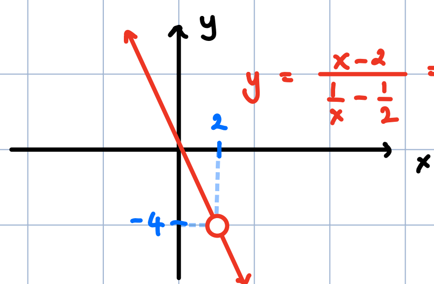
2) $\lim\limits_{x \to -3} \arctan\left(1+\ln(x+4)\right) = \arctan\left(1+\ln(1)\right) = \arctan(1) = \boxed{\frac{\pi}{4}}$.

iii) Algebra: if substitution gives $\frac{0}{0}$, need to use algebra to rewrite the expression before evaluating.

⚠ $\frac{0}{0}$ does not necessarily mean the limit DNE.

Examples: 1) $\lim\limits_{x \to 2} \frac{x-2}{\frac{1}{x}-\frac{1}{2}}$ substitution gives $\frac{0}{0}$: more work needed.

$$= \lim_{x \to 2} \frac{x-2}{\frac{2-x}{2x}} = \lim_{\substack{x \to 2 \\ x \neq 2}} -\cancel{(x-2)}\frac{2x}{\cancel{2-x}} = \lim_{x \to 2} -2x = \boxed{-4}$$



$$y = \frac{x-2}{\frac{1}{x}-\frac{1}{2}} = \begin{cases} -2x, & x \neq 2 \\ \text{undef.}, & x = 2 \end{cases}$$

2) $\lim_{x \to 1} \dfrac{\sqrt{x} - 1}{x^2 - 1}$ substitution gives $\frac{0}{0}$: more work needed.

$(a-b)(a+b) = a^2 - b^2$

$= \lim_{x \to 1} \dfrac{\sqrt{x} - 1}{(x-1)(x+1)} \cdot \dfrac{\sqrt{x}+1}{\sqrt{x}+1}$ multiply top and bottom by conjugate

$$= \lim_{x \to 1} \frac{\overset{a^2-b^2}{\cancel{x-1}}}{\cancel{(x-1)}(x+1)(\sqrt{x}+1)} = \lim_{x \to 1} \frac{1}{(x+1)(\sqrt{x}+1)} = \frac{1}{(1+1)(\sqrt{1}+1)} = \boxed{\frac{1}{4}}.$$

3) $\lim_{x \to 3} \dfrac{x-3}{\sqrt{x^2+16} - 5}$ substitution gives $\frac{0}{0}$: more work needed.

$= \lim_{x \to 3} \dfrac{x-3}{\sqrt{x^2+16} - 5} \cdot \dfrac{\sqrt{x^2+16} + 5}{\sqrt{x^2+16} + 5}$ multiply top and bottom by conjugate

$(a - b) \cdot (a + b) = a^2 - b^2$

$$= \lim_{x \to 3} \frac{(x-3)(\sqrt{x^2+16} + 5)}{x^2+16-25} = \lim_{x \to 3} \frac{(x-3)(\sqrt{x^2+16} + 5)}{x^2 - 9}$$

$$= \lim_{x \to 3} \frac{\cancel{(x-3)}(\sqrt{x^2+16} + 5)}{\cancel{(x-3)}(x+3)} = \lim_{x \to 3} \frac{\sqrt{x^2+16} + 5}{x+3} = \frac{\sqrt{3^2+16} + 5}{3+3} = \frac{10}{6} = \boxed{\frac{5}{3}}.$$

iv) Limit laws : if $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ then :

$$\lim_{x \to a} f(x) + g(x) = L + M \qquad \lim_{x \to a} kf(x) = kL$$

$$\lim_{x \to a} f(x) - g(x) = L - M \qquad \lim_{x \to a} f(x)^n = L^n$$

$$\lim_{x \to a} f(x)g(x) = LM \qquad \lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$$

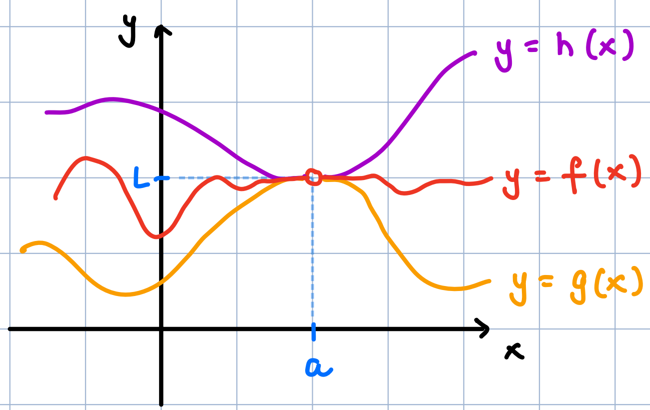
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M} \quad (M \neq 0)$$

Example if $\lim_{x \to 4} \frac{f(x)+7}{x} = 6$, find $L = \lim_{x \to 4} f(x)$.

$$\lim_{x \to 4} \frac{f(x)+7}{x} = \frac{\left(\lim_{x\to 4} f(x)\right) + \left(\lim_{x \to 4} 7\right)}{\lim_{x \to 4} x} = \frac{L+7}{4} = 6, \text{ so } L+7 = 24$$

$$\boxed{L = 17}$$

v) Squeeze / Sandwich Theorem :



If $g(x) \leq f(x) \leq h(x)$

and $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$,

then $\boxed{\lim_{x \to a} f(x) = L.}$

Examples : 1) Suppose that $1 - \frac{x^2}{2} \leq f(x) \leq 1 + \frac{x^2}{2}$.

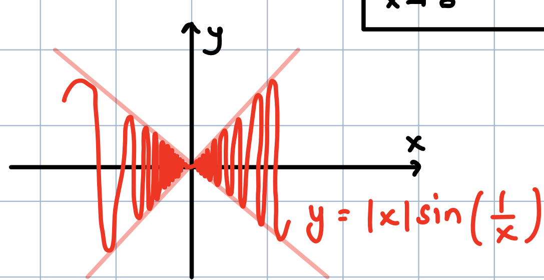
Since $\lim_{x \to 0} 1 - \frac{x^2}{2} = \lim_{x \to 0} 1 + \frac{x^2}{2} = 1$, we have $\boxed{\lim_{x \to 0} f(x) = 1}$ by the Squeeze Theorem.

2) Find $\lim_{x \to 0} |x| \sin\left(\frac{1}{x}\right)$.

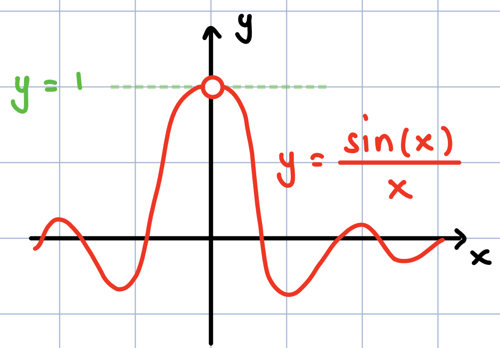
Since $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$, we have $-|x| \leq |x| \sin\left(\frac{1}{x}\right) \leq |x|$.

Also, $\lim_{x \to 0} -|x| = \lim_{x \to 0} |x| = 0$.

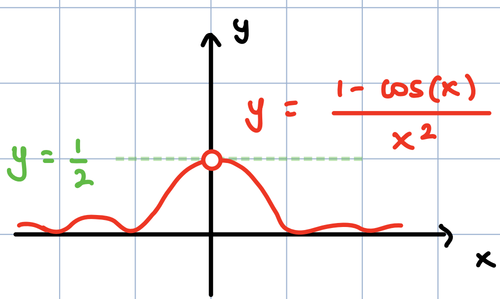
So by the Squeeze Theorem, $\boxed{\lim_{x \to 0} |x| \sin\left(\frac{1}{x}\right) = 0}$



vi) Special trig limits



$$\boxed{\lim_{x\to 0} \frac{\sin(x)}{x} = 1 = \lim_{x\to 0} \frac{x}{\sin(x)}}$$



$$\boxed{\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}}$$

Examples : evaluate 1) $\lim\limits_{x\to 0} \frac{5x}{\sin(2x)}$, 2) $\lim\limits_{x\to 0} \frac{\cos(3x)-1}{x}$

and 3) $\lim\limits_{x\to 0} \frac{\tan(7x)^2}{x\sin(3x)}$.

1) $\lim\limits_{x\to 0} \frac{5x}{\sin(2x)} \cdot \frac{2x}{2x}$ multiply by $\frac{2x}{2x}$ to get $\frac{2x}{\sin(2x)}$.

$$= \lim_{x\to 0} \frac{5\cancel{x}}{2\cancel{x}} \cdot \frac{2x}{\sin(2x)} = \underbrace{\left(\lim_{x\to 0} \frac{5}{2}\right)}_{\frac{5}{2}} \underbrace{\left(\lim_{x\to 0} \frac{2x}{\sin(2x)}\right)}_{=1} = \boxed{\frac{5}{2}}$$

2) $\lim\limits_{x\to 0} \frac{\cos(3x)-1}{x} \cdot \frac{(3x)^2}{(3x)^2}$ multiply by $\frac{(3x)^2}{(3x)^2}$ to get $\frac{\cos(3x)-1}{(3x)^2}$

$$= \lim_{x\to 0} \frac{\cos(3x)-1}{(3x)^2} \cdot \frac{(3x)^2}{x} = \left(\lim_{x\to 0} \frac{\cos(3x)-1}{(3x)^2}\right)\left(\lim_{x\to 0} \frac{9x^2}{x}\right)$$

$$= \underbrace{\left(-\lim_{x\to 0} \frac{1-\cos(3x)}{(3x)^2}\right)}_{\frac{1}{2}} \underbrace{\left(\lim_{x\to 0} 9x\right)}_{0} = -\frac{1}{2} \cdot 0 = \boxed{0}.$$

3) $\lim_{x\to 0} \frac{\tan(7x)^2}{x\sin(3x)} \cdot \frac{(7x)^2}{(7x)^2} \cdot \frac{3x}{3x}$

$= \lim_{x\to 0} \frac{1}{\cos(7x)^2} \cdot \frac{\sin(7x)^2}{(7x)^2} \cdot \frac{3x}{\sin(3x)} \cdot \frac{(7x)^2}{x(3x)}$.

$= \underbrace{\left(\lim_{x\to 0} \frac{1}{\cos(7x)^2}\right)}_{\frac{1}{\cos(0)}=1} \underbrace{\left(\lim_{x\to 0} \frac{\sin(7x)}{7x}\right)^2}_{1^2=1} \underbrace{\left(\lim_{x\to 0} \frac{3x}{\sin(3x)}\right)}_{1} \underbrace{\left(\lim_{x\to 0} \frac{49\cancel{x^2}}{3\cancel{x^2}}\right)}_{\frac{49}{3}}$

$= \boxed{\frac{49}{3}}$.

Practice problems: evaluate the following limits.

1) $\lim_{x\to 5} \frac{\sqrt{3x+1}-4}{\sqrt{x+4}-3}$

2) $\lim_{x\to -1} \frac{(x+2)^{-1}+x^{-1}}{x^2+x}$

3) $\lim_{h\to 0} \frac{h}{\frac{1}{\sqrt{4+h}}-\frac{1}{2}}$

4) $\lim_{x\to 0} x^2\cos\left(\frac{1}{x}\right)$.

5) $\lim_{x\to 0} \frac{\tan(4x)x}{\sin(5x)^2}$

6) $\lim_{x\to 0} \frac{\sec(7x)-1}{x^2}$.

Solutions: 1) $\lim_{x\to 5} \frac{\sqrt{3x+1}-4}{\sqrt{x+4}-3}$ substitution gives $\frac{\sqrt{16}-4}{\sqrt{9}-3} = \frac{0}{0}$: more work

$= \lim_{x\to 5} \frac{\sqrt{3x+1}-4}{\sqrt{x+4}-3} \cdot \frac{(\sqrt{3x+1}+4)(\sqrt{x+4}+3)}{(\sqrt{3x+1}+4)(\sqrt{x+4}+3)}$ multiply top and bottom by both conjugates.

$= \lim_{x\to 5} \frac{(3x+1-16)(\sqrt{x+4}+3)}{(x+4-9)(\sqrt{3x+1}+4)} = \lim_{x\to 5} \frac{(3x-15)(\sqrt{x+4}+3)}{(x-5)(\sqrt{3x+1}+4)}$

$= \lim_{x\to 5} \frac{3\cancel{(x-5)}(\sqrt{x+4}+3)}{\cancel{(x-5)}(\sqrt{3x+1}+4)} = \lim_{x\to 5} \frac{3(\sqrt{x+4}+3)}{\sqrt{3x+1}+4} = \frac{3\cdot 6}{8} = \boxed{\frac{9}{4}}$.

2) $$\lim_{x\to -1} \frac{(x+2)^{-1}+x^{-1}}{x^2+x}$$ substitution gives $\frac{0}{0}$: more work needed.

$$= \lim_{x\to -1} \frac{\frac{1}{x+2}+\frac{1}{x}}{x(x+1)} = \lim_{x\to -1} \frac{\frac{x+x+2}{(x+2)x}}{x(x+1)} = \lim_{x\to -1} \frac{2x+2}{x^2(x+1)(x+2)}$$

$$= \lim_{x\to -1} \frac{2}{x^2(x+2)} = \frac{2}{(-1)^2(-1+2)} = \boxed{2}.$$

3) $$\lim_{h\to 0} \frac{h}{\frac{1}{\sqrt{4+h}}-\frac{1}{2}} = \lim_{h\to 0} \frac{h}{\frac{2-\sqrt{4+h}}{2\sqrt{4+h}}} = \lim_{h\to 0} \frac{2h\sqrt{4+h}}{2-\sqrt{4+h}} \cdot \frac{2+\sqrt{4+h}}{2+\sqrt{4+h}}$$

$$= \lim_{h\to 0} \frac{2h\sqrt{4+h}\,(2+\sqrt{4+h})}{4-(4+h)} = \lim_{h\to 0} \frac{2\cancel{h}\sqrt{4+h}\,(2+\sqrt{4+h})}{-\cancel{h}}$$

$$= \lim_{h\to 0} -2\sqrt{4+h}\,(2+\sqrt{4+h}) = -2\sqrt{4+0}\,(2+\sqrt{4+0}) = \boxed{-16}$$

4) $$\lim_{x\to 0} x^2\cos\left(\frac{1}{x}\right).$$

Since $-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$, we have $-x^2 \leq x^2\cos\left(\frac{1}{x}\right) \leq x^2$.

Also $\lim_{x\to 0}(-x^2) = \lim_{x\to 0} x^2 = 0.$

So by the Squeeze Theorem, $\boxed{\lim_{x\to 0} x^2\cos\left(\frac{1}{x}\right) = 0}$.

5) $$\lim_{x\to 0} \frac{\tan(4x)x}{\sin(5x)^2} = \lim_{x\to 0} \frac{\sin(4x)\,x}{\cos(4x)\sin(5x)\sin(5x)} \cdot \frac{4x}{4x} \cdot \frac{5x}{5x} \cdot \frac{5x}{5x}$$

$$= \lim_{x\to 0} \frac{1}{\cos(4x)} \cdot \frac{\sin(4x)}{4x} \cdot \frac{5x}{\sin(5x)} \cdot \frac{5x}{\sin(5x)} \cdot \frac{\cancel{x}\cdot 4\cancel{x}}{5\cancel{x}\cdot 5\cancel{x}}$$

$$= \left(\lim_{x\to 0} \frac{1}{\cos(4x)}\right) \underbrace{\left(\lim_{x\to 0} \frac{\sin(4x)}{4x}\right)}_{1} \underbrace{\left(\lim_{x\to 0} \frac{5x}{\sin(5x)}\right)}_{1} \underbrace{\left(\lim_{x\to 0} \frac{5x}{\sin(5x)}\right)}_{1} \left(\lim_{x\to 0} \frac{4}{25}\right)$$

special trig limits

$$= \frac{1}{\cos(0)} \cdot 1 \cdot 1 \cdot 1 \cdot \frac{4}{25} = \boxed{\frac{4}{25}}.$$

6) $\lim_{x \to 0} \frac{\sec(7x)-1}{x^2} = \lim_{x \to 0} \frac{\frac{1}{\cos(7x)} - 1}{x^2} = \lim_{x \to 0} \frac{1-\cos(7x)}{x^2 \cos(7x)} \cdot \frac{(7x)^2}{(7x)^2}$

$= \lim_{x \to 0} \frac{1-\cos(7x)}{(7x)^2} \cdot \frac{49x^2}{x^2 \cos(7x)}$

$= \underbrace{\left(\lim_{x \to 0} \frac{1-\cos(7x)}{(7x)^2} \right)}_{\frac{1}{2}} \left(\lim_{x \to 0} \frac{49}{\cos(7x)} \right) = \frac{1}{2} \cdot \frac{49}{\cos(0)} = \boxed{\frac{49}{2}}.$