Rutgers University
Math 151

Section 2.2: Calculating Limits - Worksheet Solutions

1. Evaluate the following limits. If a limit does not exist, explain why.

   (a) $\lim_{x\to 3} \dfrac{x^2 - x - 6}{9x^{-2} - 1}$.

   Solution.

$$\begin{aligned}
\lim_{x\to 3} \frac{x^2 - x - 6}{9x^{-2} - 1} &= \lim_{x\to 3} \frac{(x-3)(x+2)}{\frac{9}{x^2} - 1} \\
&= \lim_{x\to 3} \frac{(x-3)(x+2)}{\frac{9-x^2}{x^2}} \\
&= \lim_{x\to 3} \frac{(x-3)(x+2)x^2}{(3-x)(3+x)} \\
&= \lim_{x\to 3} -\frac{(x+2)x^2}{3+x} \\
&= -\frac{(3+2)3^2}{3+3} \\
&= \boxed{-\frac{15}{2}}.
\end{aligned}$$

   (b) $\lim_{t\to 2} \dfrac{\sqrt{t^2+12} - 2t}{2-t}$.

   Solution.

$$\begin{aligned}
\lim_{t\to 2} \frac{\sqrt{t^2+12} - 2t}{2-t} &= \lim_{t\to 2} \frac{\sqrt{t^2+12} - 2t}{2-t} \cdot \frac{\sqrt{t^2+12} + 2t}{\sqrt{t^2+12} + 2t} \\
&= \lim_{t\to 2} \frac{\left(\sqrt{t^2+12}\right)^2 - (2t)^2}{(2-t)\left(\sqrt{t^2+12} + 2t\right)} \\
&= \lim_{t\to 2} \frac{t^2 + 12 - 4t^2}{(2-t)\left(\sqrt{t^2+12} + 2t\right)} \\
&= \lim_{t\to 2} \frac{12 - 3t^2}{(2-t)\left(\sqrt{t^2+12} + 2t\right)} \\
&= \lim_{t\to 2} \frac{3(4 - t^2)}{(2-t)\left(\sqrt{t^2+12} + 2t\right)} \\
&= \lim_{t\to 2} \frac{3(2-t)(2+t)}{(2-t)\left(\sqrt{t^2+12} + 2t\right)} \\
&= \lim_{t\to 2} \frac{3(2+t)}{\sqrt{t^2+12} + 2t}
\end{aligned}$$

$$= \frac{3(2+2)}{\sqrt{2^2+12}+4}$$
$$= \boxed{\frac{3}{2}}.$$

(c) $\lim_{y\to 0} y\cot(5y)$.

Solution

$$\begin{aligned}
\lim_{y\to 0} y\cot(5y) &= \lim_{y\to 0} \frac{y\cos(5y)}{\sin(5y)} \cdot \frac{5y}{5y} \\
&= \lim_{y\to 0} \frac{y\cos(5y)}{5y} \cdot \frac{5y}{\sin(5y)} \\
&= \left(\lim_{y\to 0} \frac{\cos(5y)}{5}\right)\left(\lim_{y\to 0} \frac{5y}{\sin(5y)}\right) \\
&= \frac{\cos(0)}{5} \cdot 1 \\
&= \boxed{\frac{1}{5}}.
\end{aligned}$$

(d) $\lim_{x\to 0} \dfrac{(x-2)^3+8-12x}{x^2}$.

Solution. Recall that $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$. Using this, we get

$$\begin{aligned}
\lim_{x\to 0} \frac{(x-2)^3+8-12x}{x^2} &= \lim_{x\to 0} \frac{x^3-6x^2+12x-8+8-12x}{x^2} \\
&= \lim_{x\to 0} \frac{x^3-6x^2}{x^2} \\
&= \lim_{x\to 0} x-6 \\
&= \boxed{-6}.
\end{aligned}$$

(e) $\lim_{u\to 4} \dfrac{u-4}{\sqrt{2u+1}-\sqrt{u+5}}$.

Solution.

$$\begin{aligned}
\lim_{u\to 4} \frac{u-4}{\sqrt{2u+1}-\sqrt{u+5}} &= \lim_{u\to 4} \frac{u-4}{\sqrt{2u+1}-\sqrt{u+5}} \cdot \frac{\sqrt{2u+1}+\sqrt{u+5}}{\sqrt{2u+1}+\sqrt{u+5}} \\
&= \lim_{u\to 4} \frac{(u-4)\left(\sqrt{2u+1}+\sqrt{u+5}\right)}{(\sqrt{2u+1})^2-(\sqrt{u+5})^2} \\
&= \lim_{u\to 4} \frac{(u-4)\left(\sqrt{2u+1}+\sqrt{u+5}\right)}{2u+1-(u+5)} \\
&= \lim_{u\to 4} \frac{(u-4)\left(\sqrt{2u+1}+\sqrt{u+5}\right)}{u-4}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{u\to 4} \sqrt{2u+1} + \sqrt{u+5} \\
&= \sqrt{2\cdot 4+1} + \sqrt{4+5} \\
&= \boxed{6}.
\end{aligned}
$$

(f) $\displaystyle \lim_{x\to 0} \frac{\sin^2(4x)}{x\sin(3x)}$.

Solution.

$$
\begin{aligned}
\lim_{x\to 0} \frac{\sin^2(4x)}{x\sin(3x)} &= \lim_{x\to 0} \frac{\sin^2(4x)}{x\sin(3x)} \cdot \frac{(4x)^2}{(4x)^2} \cdot \frac{3x}{3x} \\
&= \lim_{x\to 0} \left(\frac{\sin(4x)}{4x}\right)^2 \cdot \frac{3x}{\sin(3x)} \cdot \frac{(4x)^2}{x(3x)} \\
&= \left(\lim_{x\to 0} \frac{\sin(4x)}{4x}\right)^2 \left(\lim_{x\to 0} \frac{3x}{\sin(3x)}\right) \left(\lim_{x\to 0} \frac{16}{3}\right) \\
&= 1^2 \cdot 1 \cdot \frac{16}{3} \\
&= \boxed{\frac{16}{3}}.
\end{aligned}
$$

(g) $\displaystyle \lim_{x\to 1} \frac{\sqrt{4x^2+7} - \sqrt{x+10}}{x^2+4x-5}$

Solution. Direct substitution gives $\frac{0}{0}$, so we rewrite the expression by rationalizing the numerator and canceling out common factors.

$$
\begin{aligned}
\lim_{x\to 1} \frac{\sqrt{4x^2+7} - \sqrt{x+10}}{x^2+4x-5} &= \lim_{x\to 1} \frac{\sqrt{4x^2+7} - \sqrt{x+10}}{(x-1)(x+5)} \cdot \frac{\sqrt{4x^2+7} + \sqrt{x+10}}{\sqrt{4x^2+7} + \sqrt{x+10}} \\
&= \lim_{x\to 1} \frac{4x^2+7-(x+10)}{(x-1)(x+5)(\sqrt{4x^2+7} + \sqrt{x+10})} \\
&= \lim_{x\to 1} \frac{4x^2-x-3}{(x-1)(x+5)(\sqrt{4x^2+7} + \sqrt{x+10})} \\
&= \lim_{x\to 1} \frac{(x-1)(4x+3)}{(x-1)(x+5)(\sqrt{4x^2+7} + \sqrt{x+10})} \\
&= \lim_{x\to 1} \frac{4x+3}{(x+5)\left(\sqrt{4x^2+7} + \sqrt{x+10}\right)} \\
&= \boxed{\frac{7}{12\sqrt{11}}}.
\end{aligned}
$$

(h) $\displaystyle \lim_{h\to 0} \frac{\frac{6}{3+7h} - 2}{h}$

Solution. Direct substitution gives $\frac{0}{0}$, so we rewrite the expression as a simple fraction and cancel the common factors.

$$\begin{aligned}
\lim_{h\to 0} \frac{\frac{6}{3+7h}-2}{h} &= \lim_{h\to 0} \frac{\frac{6-2(3+7h)}{3+7h}}{h} \\
&= \lim_{h\to 0} \frac{-14h}{h(3+7h)} \\
&= \lim_{h\to 0} \frac{-14}{3+7h} \\
&= \boxed{-\frac{14}{3}}.
\end{aligned}$$

(i) $\displaystyle\lim_{x\to 0} \frac{x\sin(5x)}{\tan^2(3x)}$

Solution.

$$\begin{aligned}
\lim_{x\to 0} \frac{x\sin(5x)}{\tan^2(3x)} &= \lim_{x\to 0} \frac{x\sin(5x)\cos^2(3x)}{\sin^2(3x)} \cdot \frac{5x}{5x} \cdot \frac{(3x)^2}{(3x)^2} \\
&= \lim_{x\to 0} \frac{\sin(5x)}{5x} \cdot \left(\frac{3x}{\sin(3x)}\right)^2 \cdot \frac{x(5x)\cos^2(3x)}{(3x)^2} \\
&= \left(\lim_{x\to 0} \frac{\sin(5x)}{5x}\right) \cdot \left(\lim_{x\to 0} \frac{3x}{\sin(3x)}\right)^2 \cdot \left(\lim_{x\to 0} \frac{5\cos^2(3x)}{9}\right) \\
&= 1 \cdot 1^2 \cdot \frac{5\cos^2(0)}{9} \\
&= \boxed{\frac{5}{9}}.
\end{aligned}$$

[Advanced]

(j) $\displaystyle\lim_{\theta\to 0} \frac{\sin(3\theta)^2}{\cos(5\theta)-1}$.

Solution.

$$\begin{aligned}
\lim_{\theta\to 0} \frac{\sin(3\theta)^2}{\cos(5\theta)-1} &= \lim_{\theta\to 0} \frac{\sin(3\theta)^2}{\cos(5\theta)-1} \cdot \frac{(3\theta)^2}{(3\theta)^2} \cdot \frac{(5\theta)^2}{(5\theta)^2} \\
&= \lim_{\theta\to 0} \left(\frac{\sin(3\theta)}{3\theta}\right)^2 \cdot \frac{(5\theta)^2}{\cos(5\theta)-1} \cdot \frac{(3\theta)^2}{(5\theta)^2} \\
&= \left(\lim_{\theta\to 0} \frac{\sin(3\theta)}{3\theta}\right)^2 \left(\lim_{\theta\to 0} -\frac{(5\theta)^2}{1-\cos(5\theta)}\right) \left(\lim_{\theta\to 0} \frac{9}{25}\right) \\
&= 1^2\,(-2)\,\frac{9}{25} \\
&= \boxed{-\frac{18}{25}}.
\end{aligned}$$

(k) $\lim_{x \to 0} x \sin(\ln |x|)$.

Solution. We use the Squeeze Theorem. For any $x \neq 0$, we have

$$-1 \leqslant \sin(\ln |x|) \leqslant 1,$$

so

$$-|x| \leqslant x \sin(\ln |x|) \leqslant |x|.$$

Since $\lim_{x \to 0} -|x| = \lim_{x \to 0} |x| = 0$, we conclude that

$$\boxed{\lim_{x \to 0} x \sin(\ln |x|) = 0}.$$

(l) $\lim_{h \to 1} \dfrac{\sqrt[3]{h} - 1}{h - 1}$.

Solution. We would like to rationalize the numerator to be able to simplify the h in the denominator. To this end, we will want to use the identity

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

with $a = \sqrt[3]{h}$ and $b = 1$. Therefore, we will multiply the numerator and denominator by $a^2 + ab + b^2 = \sqrt[3]{h^2} + \sqrt[3]{h} + 1$. This gives

$$\begin{aligned}
\lim_{h \to 1} \frac{\sqrt[3]{h} - 1}{h - 1} &= \lim_{h \to 1} \frac{\sqrt[3]{h} - 1}{h - 1} \cdot \frac{\sqrt[3]{h^2} + \sqrt[3]{h} + 1}{\sqrt[3]{h^2} + \sqrt[3]{h} + 1} \\
&= \lim_{h \to 1} \frac{(\sqrt[3]{h})^3 - 1^3}{(h - 1)\left(\sqrt[3]{h^2} + \sqrt[3]{h} + 1\right)} \\
&= \lim_{h \to 1} \frac{h - 1}{(h - 1)\left(\sqrt[3]{h^2} + \sqrt[3]{h} + 1\right)} \\
&= \lim_{h \to 1} \frac{h - 1}{(h - 1)\left(\sqrt[3]{h^2} + \sqrt[3]{h} + 1\right)} \\
&= \lim_{h \to 1} \frac{1}{\sqrt[3]{h^2} + \sqrt[3]{h} + 1} \\
&= \frac{1}{\sqrt[3]{1^2} + \sqrt[3]{1} + 1} \\
&= \boxed{\frac{1}{3}}.
\end{aligned}$$

2. Suppose that f is a function such that for any number x, we have

$$x - 8 \leqslant f(x) \leqslant x^2 - 3x - 4.$$

For which values of a can you determine $\lim_{x \to a} f(x)$? For these values of a, evaluate $\lim_{x \to a} f(x)$.

Solution. We have $\lim_{x\to a} x - 8 = a - 8$ and $\lim_{x\to a} x^2 - 3x - 4 = a^2 - 3a - 4$. The Squeeze Theorem will guarantee that $\lim_{x\to a} f(x)$ exists when $a - 8 = a^2 - 3a - 4$, that is $a^2 - 4a + 4 = 0$. This equation gives $(a-2)^2$, or $\boxed{a = 2}$. For $a = 2$ we will have

$$\lim_{x\to 2} f(x) = 2 - 8 = \boxed{-6}.$$

3. **[Advanced]** Suppose that f is a function such that

$$\lim_{x\to 0} \frac{f(x)}{\sin(3x)} = 2.$$

Evaluate the following limits.

(a) $\lim_{x\to 0} f(x)$.

Solution.

$$\begin{aligned}\lim_{x\to 0} f(x) &= \lim_{x\to 0} f(x) \cdot \frac{\sin(3x)}{\sin(3x)} \\ &= \left(\lim_{x\to 0} \frac{f(x)}{\sin(3x)}\right)\left(\lim_{x\to 0} \sin(3x)\right) \\ &= 2 \cdot \sin(0) \\ &= 0.\end{aligned}$$

(b) $\lim_{x\to 0} \frac{f(x)}{x}$.

Solution.

$$\begin{aligned}\lim_{x\to 0} \frac{f(x)}{x} &= \lim_{x\to 0} \frac{f(x)}{x} \cdot \frac{\sin(3x)}{\sin(3x)} \cdot \frac{3x}{3x} \\ &= \left(\lim_{x\to 0} \frac{f(x)}{\sin(3x)}\right)\left(\lim_{x\to 0} \frac{\sin(3x)}{3x}\right)\left(\lim_{x\to 0} \frac{3x}{x}\right) \\ &= 2 \cdot 1 \cdot 3 \\ &= \boxed{6}.\end{aligned}$$

(c) $\lim_{x\to 0} \frac{1-\cos(x)}{f(2x)^2}$.

Solution.

$$\begin{aligned}\lim_{x\to 0} \frac{1-\cos(x)}{f(2x)^2} &= \lim_{x\to 0} \frac{1-\cos(x)}{f(2x)^2} \cdot \frac{x^2}{x^2} \cdot \frac{(2x)^2}{(2x)^2} \\ &= \left(\lim_{x\to 0} \frac{1-\cos(x)}{x^2}\right)\left(\lim_{x\to 0} \frac{2x}{f(2x)}\right)^2\left(\lim_{x\to 0} \frac{x^2}{(2x)^2}\right) \\ &= \frac{1}{2} \cdot \left(\frac{1}{6}\right)^2 \cdot \frac{1}{4} \\ &= \boxed{\frac{1}{288}}.\end{aligned}$$