Section 2.4

One - Sided Limits.

Learning Goals

Learning Goal	*Homework Problems*
2.4.1 Use graphs of functions to find one-sided limits.	1-10, 21, 22.
2.4.2 Compute one-sided limits algebraically.	11-20.
2.4.3 Find the limits of trigonometric functions using the limit of $\sin\theta / \theta$.	23-46.
2.4.4 Understand the relationship between one-sided and two-sided limits. Answer conceptual question about one-sided limits.	47-50.

One-sided limits: in the definition of $\lim_{x \to a} f(x)$, x is allowed to approach a on both sides (left and right). We can look at limits when x approaches a from one side only.

Limit from the right: $\lim_{x \to a^+} f(x)$ x goes to a and $x > a$.

$\lim_{x \to a^+} f(x) = L$ means $f(x)$ gets arbitrarily close to L when x approaches a with $x > a$

Limit from the left: $\lim_{x \to a^-} f(x)$ x goes to a and $x < a$.

$\lim_{x \to a^-} f(x) = L$ means $f(x)$ gets arbitrarily close to L when x approaches a with $x < a$

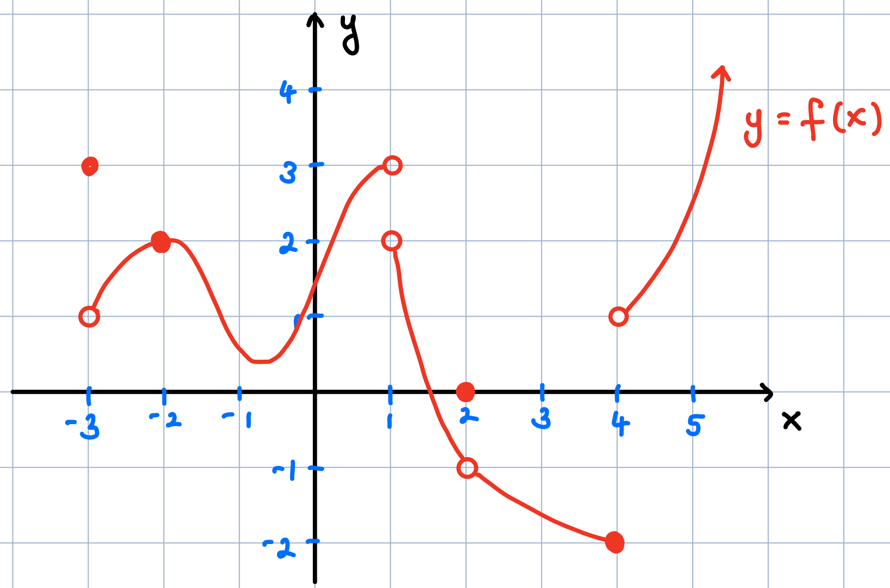
Relationship between one-sided and two-sided limits:

$$\boxed{\lim_{x \to a} f(x) = L \iff \left(\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L \right)}$$

More explicitly, this means that:

- If left and right limits are different, the two-sided limit does not exist (DNE).
- If left and right limits are equal, the two-sided limit exists and is equal to both one-sided limits.

Examples: 1) Use the graph of f below to find $\lim_{x\to a^-} f(x)$, $\lim_{x\to a^+} f(x)$, $\lim_{x\to a} f(x)$ and $f(a)$ for $a = -3, -2, 1, 2, 4$.



a	$\lim_{x\to a^-} f(x)$	$\lim_{x\to a^+} f(x)$	$\lim_{x\to a} f(x)$	$f(a)$
-3	DNE	1	DNE	3
-2	2	2	2	2
1	3	2	DNE	DNE
2	-1	-1	-1	0
4	-2	1	DNE	-2

2) Calculate $\lim_{x\to 7} \frac{|x-7|}{x^2-49}$ or explain why it does not exist.

Recall that $|x-7| = \begin{cases} x-7 & \text{if } x>7 \\ -(x-7) & \text{if } x<7 \end{cases}$

Since we have a different expression to the left and right of 7, we use one-sided limits.

- Right limit:

$$\lim_{x\to 7^+} \frac{|x-7|}{x^2-49} = \lim_{x\to 7^+} \frac{\cancel{x-7}}{(\cancel{x-7})(x+7)} = \lim_{x\to 7^+} \frac{1}{x+7} = \frac{1}{7+7} = \frac{1}{14}.$$

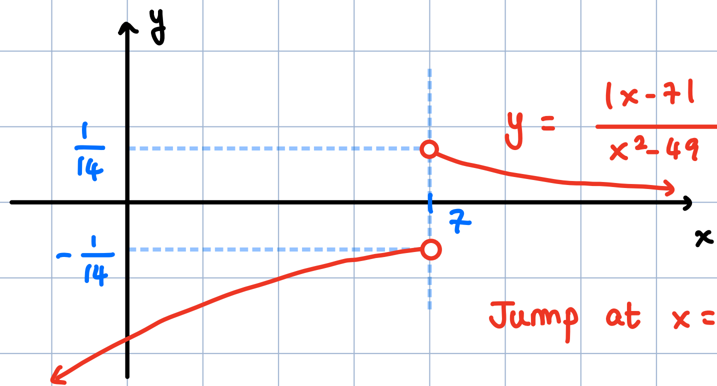
$x>7$ so $|x-7| = x-7$

- Left limit

$$\lim_{x\to 7^-}\frac{|x-7|}{x^2-49}=\lim_{x\to 7^-}\frac{-\cancel{(x-7)}}{\cancel{(x-7)}(x+7)}=\lim_{x\to 7^-}\frac{-1}{x+7}=\frac{-1}{7+7}=-\frac{1}{14}.$$

$x<7$ so $|x-7|=-(x-7)$

Since $\lim_{x\to 7^-}\frac{|x-7|}{x^2-49}\neq\lim_{x\to 7^+}\frac{|x-7|}{x^2-49}$, $\boxed{\lim_{x\to 7}\frac{|x-7|}{x^2-49}\ \text{DNE}}$



$$y=\frac{|x-7|}{x^2-49}=\begin{cases}\frac{1}{x+7}, & x>7\\ -\frac{1}{x+7}, & x<7\end{cases}$$

Jump at $x=7$.

3) For $f(x)=\begin{cases}\frac{\sin(6x)}{x} & x<0\\ -2 & x=0\\ \frac{x}{\sqrt{x+9}-3} & x>0\end{cases}$

Calculate $\lim_{x\to 0}f(x)$ or explain why it does not exist.

Since we have a different expression to the left and right of 0, we use one-sided limits.

- Right limit :

$$\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}\frac{x}{\sqrt{x+9}-3}\cdot\frac{\sqrt{x+9}+3}{\sqrt{x+9}+3}=\lim_{x\to 0^+}\frac{x(\sqrt{x+9}+3)}{x+9-9}$$

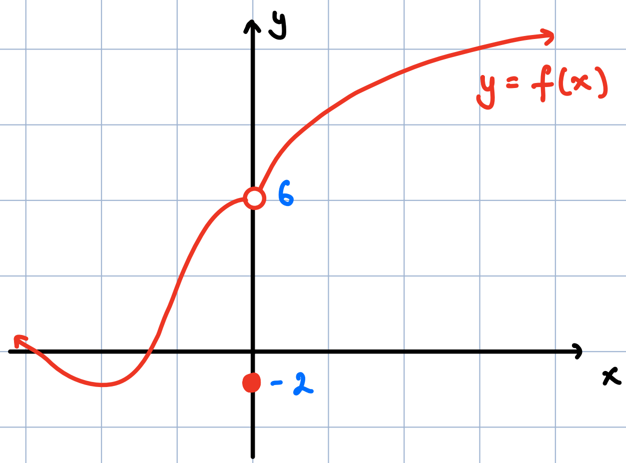
$$=\lim_{x\to 0^+}\frac{\cancel{x}(\sqrt{x+9}+3)}{\cancel{x}}=\lim_{x\to 0^+}(\sqrt{x+9}+3)=\sqrt{9}+3=6.$$

- Left limit

$$\lim_{x\to0^-} f(x) = \lim_{x\to0^-} \frac{\sin(6x)}{x}\cdot\frac{6x}{6x} = \lim_{x\to0^-}\frac{\sin(6x)}{6x}\cdot\frac{6\cancel{x}}{\cancel{x}}$$

$$= \Big(\underbrace{\lim_{x\to0^-}\frac{\sin(6x)}{6x}}_{=1}\Big)\Big(\underbrace{\lim_{x\to0^-} 6}_{=6}\Big) = 6.$$

Since $\lim_{x\to0^-} f(x) = \lim_{x\to0^+} f(x) = 6$, $\boxed{\lim_{x\to0} f(x) = 6}$.



4) Find the values of the constant a for which the function

$$f(x) = \begin{cases} \cos(\pi x) + x & x < 2 \\ -1 & x = 2 \\ ax + 11 & x > 2 \end{cases}$$ has a limit at 2.

We need to ensure that $\lim_{x\to2^-} f(x) = \lim_{x\to2^+} f(x)$.

$$\lim_{x\to2^-} f(x) = \lim_{x\to2^-} \cos(\pi x) + x = \cos(2\pi) + 2 = 3$$
$$\lim_{x\to2^+} f(x) = \lim_{x\to2^+} ax + 11 = 2a + 11$$

So we need $2a + 11 = 3 \Rightarrow 2a = -8$

$$\boxed{a = -4}$$

5) Evaluate $\lim_{x \to 0} \frac{\sin(3x^2)}{|x|}$ or explain why it does not exist.

- Right limit :

$$\lim_{x \to 0^+} \frac{\sin(3x^2)}{|x|} = \lim_{x \to 0^+} \frac{\sin(3x^2)}{x} \cdot \frac{3x^2}{3x^2} = \lim_{x \to 0^+} \frac{\sin(3x^2)}{3x^2} \cdot \frac{3x^{\cancel{2}}}{\cancel{x}}$$

$x > 0$ so $|x| = x$

$$= \underbrace{\left(\lim_{x \to 0^+} \frac{\sin(3x^2)}{3x^2}\right)}_{1} \underbrace{\left(\lim_{x \to 0^+} 3x\right)}_{0} = 1 \cdot 0 = 0$$

- Left limit :

$$\lim_{x \to 0^-} \frac{\sin(3x^2)}{|x|} = \lim_{x \to 0^-} \frac{\sin(3x^2)}{-x} \cdot \frac{3x^2}{3x^2} = \lim_{x \to 0^-} \frac{\sin(3x^2)}{3x^2} \cdot \frac{3x^{\cancel{2}}}{-\cancel{x}}$$

$x < 0$ so $|x| = -x$

$$= \underbrace{\left(\lim_{x \to 0^-} \frac{\sin(3x^2)}{3x^2}\right)}_{1} \underbrace{\left(\lim_{x \to 0^-} -3x\right)}_{0} = 1 \cdot 0 = 0$$

Since $\lim_{x \to 0^+} \frac{\sin(3x^2)}{|x|} = \lim_{x \to 0^+} \frac{\sin(3x^2)}{|x|} = 0$, $\boxed{\lim_{x \to 0} \frac{\sin(3x^2)}{|x|} = 0}$

6) For $f(x) = \begin{cases} \frac{2x+6}{3x^{-1}+1} & x < -3 \\ -x + k & -3 < x \leq 1 \\ 4k\arctan(x) & x > 1 \end{cases}$

find the values of k for which f has a limit at a) $x = -3$ and b) $x = 1$.

a) $\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} \frac{2x+6}{3x^{-1}+1} = \lim_{x \to -3^-} \frac{2(x+3)}{\frac{3}{x}+1} = \lim_{x \to -3^-} \frac{2x\cancel{(x+3)}}{\cancel{3+x}} = -6$

$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} -x + k = 3 + k.$

We want $3+k=-6 \Rightarrow \boxed{k=-9}$

b) $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} -x+k = -1+k$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 4k\arctan(x) = 4k\arctan(1) = 4\frac{k\pi}{4} = k\pi$

We want $-1+k = k\pi \Rightarrow k(\pi-1) = -1$

$$\boxed{k = -\frac{1}{\pi-1}}$$