Rutgers University
Math 151

## Section 2.4: One-Sided Limits - Worksheet Solutions

1. Evaluate the following limits. If a limit does not exist, explain why.

(a) $\displaystyle\lim_{x\to 3^-} \frac{x^2-4x+3}{|x-3|}$.

*Solution.* When $x \to 3^-$, we have $x < 3$, so $x - 3 < 0$ and $|x-3| = -(x-3)$. So

$$\begin{aligned}\lim_{x\to 3^-} \frac{x^2-4x+3}{|x-3|} &= \lim_{x\to 3^-} \frac{(x-3)(x-1)}{-(x-3)} \\ &= \lim_{x\to 3^-} -(x-1) \\ &= -(3-1) \\ &= \boxed{-2}.\end{aligned}$$

(b) $\displaystyle\lim_{x\to 3} \frac{x^2-4x+3}{|x-3|}$.

*Solution.* We have already computed the left limit in the previous question. Let us compute the right limit. This time, when $x \to 3^+$, we have $x > 3$, so $x - 3 > 0$ and $|x-3| = x-3$. It follows that

$$\begin{aligned}\lim_{x\to 3^+} \frac{x^2-4x+3}{|x-3|} &= \lim_{x\to 3^+} \frac{(x-3)(x-1)}{x-3} \\ &= \lim_{x\to 3^+} (x-1) \\ &= (3-1) \\ &= 2.\end{aligned}$$

Since $\displaystyle\lim_{x\to 3^-} \frac{x^2-4x+3}{|x-3|} \neq \lim_{x\to 3^+} \frac{x^2-4x+3}{|x-3|}$, we conclude that $\boxed{\displaystyle\lim_{x\to 3} \frac{x^2-4x+3}{|x-3|} \text{ does not exist}}$.

(c) $\displaystyle\lim_{h\to 0^-} \frac{1-(1-|h|)^3}{h}$.

*Solution.* When $h \to 0^-$, we have $h < 0$, so $|h| = -h$. Therefore

$$\begin{aligned}\lim_{h\to 0^-} \frac{1-(1-|h|)^3}{h} &= \lim_{h\to 0^-} \frac{1-(1+h)^3}{h} \\ &= \lim_{h\to 0^-} \frac{1-(1+3h+3h^2+h^3)}{h} \\ &= \lim_{h\to 0^-} \frac{-3h-3h^2-h^3}{h}\end{aligned}$$

$$\begin{aligned} &= \lim_{h\to 0^-} -3-3h-h^2 \\ &= \boxed{-3}. \end{aligned}$$

(d) $\displaystyle\lim_{t\to 1} \frac{t^3-2t^2+t}{|t-1|}$.

*Solution.* Let us compute the left limit and the right limit. For the left limit, we have $t<1$, so $|t-1| = -(t-1)$. So

$$\begin{aligned} \lim_{t\to 1^-} \frac{t^3-2t^2+t}{|t-1|} &= \lim_{t\to 1^-} \frac{t(t-1)^2}{-(t-1)} \\ &= \lim_{t\to 1^-} -t(t-1) \\ &= -1(1-1) \\ &= 0. \end{aligned}$$

For the right limit, we have $t>1$ so $|t-1| = t-1$. Therefore

$$\begin{aligned} \lim_{t\to 1^+} \frac{t^3-2t^2+t}{|t-1|} &= \lim_{t\to 1^+} \frac{t(t-1)^2}{t-1} \\ &= \lim_{t\to 1^-} t(t-1) \\ &= 1(1-1) \\ &= 0. \end{aligned}$$

Since $\displaystyle\lim_{t\to 1^-} \frac{t^3-2t^2+t}{|t-1|} = \lim_{t\to 1^+} \frac{t^3-2t^2+t}{|t-1|} = 0$, it follows that $\boxed{\displaystyle\lim_{t\to 1^-} \frac{t^3-2t^2+t}{|t-1|} = 0}$.

(e) $\displaystyle\lim_{x\to -2} f(x)$ where $f(x) = \begin{cases} 3x+8 & \text{if } x<-2 \\ 8 & \text{if } x=-2 \\ \dfrac{x+2}{\sqrt{x+3}-1} & \text{if } x>-2 \end{cases}$.

*Solution.* We have

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} 3x+8 = 3(-2)+8 = 2$$

and

$$\begin{aligned} \lim_{x\to -2^+} f(x) &= \lim_{x\to -2^+} \frac{x+2}{\sqrt{x+3}-1} \cdot \frac{\sqrt{x+3}+1}{\sqrt{x+3}+1} \\ &= \lim_{x\to -2^+} \frac{(x+2)(\sqrt{x+3}+1)}{(\sqrt{x+3})^2-1^2} \\ &= \lim_{x\to -2^+} \frac{(x+2)(\sqrt{x+3}+1)}{x-2} \\ &= \lim_{x\to -2^+} \sqrt{x+3}+1 \\ &= 2. \end{aligned}$$

Since $\displaystyle\lim_{x\to -2^-} f(x) = \lim_{x\to -2^+} f(x) = 2$, it follows that $\boxed{\displaystyle\lim_{x\to -2} f(x) = 2}$.

(f) $\lim_{x\to 0} f(x)$ where $f(x) = \begin{cases} \dfrac{\sin(3x)}{x} & \text{if } x < 0 \\ 2e^{\cos(x)-1} & \text{if } x \geqslant 0 \end{cases}$.

*Solution.* We have

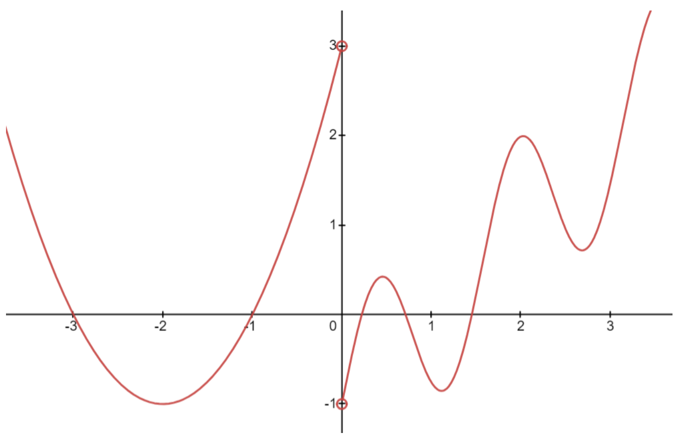
$$\begin{aligned}\lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} \frac{\sin(3x)}{x}\cdot\frac{3x}{3x}\\ &= \lim_{x\to 0^-} \frac{\sin(3x)}{3x}\cdot\frac{3x}{x}\\ &= \left(\lim_{x\to 0^-}\frac{\sin(3x)}{3x}\right)\left(\lim_{x\to 0^-} 3\right)\\ &= 1\cdot 3\\ &= 3\end{aligned}$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2e^{\cos(x)-1} = 2e^{\cos(0)-1} = 2e^{1-1} = 2e^0 = 2.$$

Since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$, we conclude that $\boxed{\lim_{x\to 0} f(x) \text{ does not exist}}$.

2. Consider the function $f(x) = \dfrac{\tan(8x)}{|2x|}$ and suppose that the graph of another function $g$ is given below.




(a) Find $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ or explain why it does not exist.

*Solution.* We have

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin(8x)}{-2x\cos(8x)}\cdot\frac{8x}{8x} = \left(\lim_{x\to 0^-}\frac{\sin(8x)}{8x}\right)\left(\lim_{x\to 0^-}\frac{8x}{-2x\cos(8x)}\right) = -\frac{4}{\cos(0)} = -4$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sin(8x)}{2x\cos(8x)}\cdot\frac{8x}{8x} = \left(\lim_{x\to 0^+}\frac{\sin(8x)}{8x}\right)\left(\lim_{x\to 0^+}\frac{8x}{2x\cos(8x)}\right) = \frac{4}{\cos(0)} = 4.$$

Since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$, we conclude that $\boxed{\lim_{x\to 0} f(x) \text{ does not exist}}$.

By inspection of the graph, we see that

$$\lim_{x\to 0^-} g(x) = 3, \qquad \lim_{x\to 0^+} g(x) = -1.$$

Since $\lim_{x\to 0^-} g(x) \neq \lim_{x\to 0^+} g(x)$, we conclude that $\boxed{\lim_{x\to 0} g(x) \text{ does not exist}}$.

(b) Find $\lim_{x\to 0} |f(x)|$ and $\lim_{x\to 0} |g(x)|$ or explain why it does not exist.

*Solution.* We have

$$\lim_{x\to 0^-} |f(x)| = \left| \lim_{x\to 0^-} f(x) \right| = |-4| = 4,$$

and

$$\lim_{x\to 0^+} |f(x)| = \left| \lim_{x\to 0^+} f(x) \right| = |4| = 4.$$

Since $\lim_{x\to 0^-} |f(x)| = \lim_{x\to 0^+} |f(x)| = 4$, we conclude that $\boxed{\lim_{x\to 0} |f(x)| = 4}$.

We have

$$\lim_{x\to 0^-} |g(x)| = \left| \lim_{x\to 0^-} g(x) \right| = |3| = 3,$$

and

$$\lim_{x\to 0^+} |g(x)| = \left| \lim_{x\to 0^+} g(x) \right| = |-1| = 1.$$

Since $\lim_{x\to 0^-} |g(x)| \neq \lim_{x\to 0^+} |g(x)|$, we conclude that $\boxed{\lim_{x\to 0} |g(x)| \text{ does not exist}}$.

(c) Find $\lim_{x\to 0} f(x) + 2g(x)$ or explain why it does not exist.

*Solution.* We have

$$\lim_{x\to 0^-} f(x) + 2g(x) = \left( \lim_{x\to 0^-} f(x) \right) + 2\left( \lim_{x\to 0^-} g(x) \right) = -4 + 2 \cdot 3 = 2$$

and

$$\lim_{x\to 0^+} f(x) + 2g(x) = \left( \lim_{x\to 0^+} f(x) \right) + 2\left( \lim_{x\to 0^+} g(x) \right) = 4 + 2 \cdot (-1) = 2.$$

Since $\lim_{x\to 0^-} f(x) + 2g(x) = \lim_{x\to 0^+} f(x) + 2g(x) = 2$, we conclude that $\boxed{\lim_{x\to 0} f(x) + 2g(x) = 2}$.

(d) [**Advanced**] Find the value of the constant $a$ for which $\lim_{x\to 0} \dfrac{g(x)}{f(x)+a}$ exists. For this value of $a$, find the value of the limit.

*Solution.* We have

$$\lim_{x\to 0^-} \frac{g(x)}{f(x)+a} = \frac{\lim_{x\to 0^-} g(x)}{\left( \lim_{x\to 0^-} f(x) \right) + \left( \lim_{x\to 0^-} a \right)} = \frac{3}{-4+a},$$

and

$$\lim_{x\to 0^+} \frac{g(x)}{f(x)+a} = \frac{\lim_{x\to 0^+} g(x)}{\left(\lim_{x\to 0^+} f(x)\right) + \left(\lim_{x\to 0^+} a\right)} = \frac{-1}{4+a}.$$

The two-sided limit exists when both one-sided limits are equal. This gives the condition $\frac{3}{-4+a} = \frac{-1}{4+a}$, that is $3(4+a) = -(-4+a)$, or $12+3a = 4-a$. Solving this for $a$ gives $\boxed{a=-2}$. Then we have

$$\boxed{\lim_{x\to 0} \frac{g(x)}{f(x)-2} = \frac{-1}{4-2} = -\frac{1}{2}}.$$